# INTERNATIONAL SCHOOL FOR ADVANCED STUDIES <br> Trieste 

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## INTRODUCTION TO

## ALGEBRAIC TOPOLOGY AND

## ALGEBRAIC GEOMETRY

La filosofia è scritta in questo grandissimo libro che continuamente ci sta aperto innanzi a gli occhi (io dico l'universo), ma non si può intendere se prima non si impara a intender la lingua, e conoscer $i$ caratteri, ne' quali è scritto. Egli è scritto in lingua matematica, e i caratteri son triangoli, cerchi, ed altre figure geometriche, senza i quali mezi è impossibile a intenderne umanamente parola; senza questi è un aggirarsi vanamente per un oscuro laberinto.

Galileo Galilei (from "Il Saggiatore")

## Preface

These notes assemble the contents of the introductory courses I have been giving at SISSA since 1995/96. Originally the course was intended as introduction to (complex) algebraic geometry for students with an education in theoretical physics, to help them to master the basic algebraic geometric tools necessary for doing research in algebraically integrable systems and in the geometry of quantum field theory and string theory. This motivation still transpires from the chapters in the second part of these notes.

The first part on the contrary is a brief but rather systematic introduction to two topics, singular homology (Chapter 2) and sheaf theory, including their cohomology (Chapter 3). Chapter 1 assembles some basics fact in homological algebra and develops the first rudiments of de Rham cohomology, with the aim of providing an example to the various abstract constructions.

Chapter 4 is an introduction to spectral sequences, a rather intricate but very powerful computation tool. The examples provided here are from sheaf theory but this computational techniques is also very useful in algebraic topology.

I thank all my colleagues and students, in Trieste and Genova and other locations, who have helped me to clarify some issues related to these notes, or have pointed out mistakes. In this connection special thanks are due to Fabio Pioli. Most of Chapter 3 is an adaptation of material taken from [2]. I thank my friends and collaborators Claudio Bartocci and Daniel Hernández Ruipérez for granting permission to use that material. I thank Lothar Göttsche for useful suggestions and for pointing out an error and the students of the 2002/2003 course for their interest and constant feedback.

Genova, 4 December 2004

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## Part 1

## Algebraic Topology

## CHAPTER 1

## Introductory material

The aim of the first part of these notes is to introduce the student to the basics of algebraic topology, especially the singular homology of topological spaces. The future developments we have in mind are the applications to algebraic geometry, but also students interested in modern theoretical physics may find here useful material (e.g., the theory of spectral sequences).

As its name suggests, the basic idea in algebraic topology is to translate problems in topology into algebraic ones, hopefully easier to deal with.

In this chapter we give some very basic notions in homological algebra and then introduce the fundamental group of a topological space. De Rham cohomology is introduced as a first example of a cohomology theory, and is homotopic invariance is proved.

## 1. Elements of homological algebra

1.1. Exact sequences of modules. Let $R$ be a ring, and let $M, M^{\prime}, M^{\prime \prime}$ be $R$-modules. We say that two $R$-module morphisms $i: M^{\prime} \rightarrow M, p: M \rightarrow M^{\prime \prime}$ form an exact sequence of $R$-modules, and write

$$
0 \rightarrow M^{\prime} \xrightarrow{i} M \xrightarrow{p} M^{\prime \prime} \rightarrow 0,
$$

if $i$ is injective, $p$ is surjective, and $\operatorname{ker} p=\operatorname{Im} i$.
A morphism of exact sequences is a commutative diagram

of $R$-module morphisms whose rows are exact.
1.2. Differential complexes. Let $R$ be a ring, and $M$ an $R$-module.

Definition 1.1. A differential on $M$ is a morphism $d: M \rightarrow M$ of $R$-modules such that $d^{2} \equiv d \circ d=0$. The pair $(M, d)$ is called a differential module.

The elements of the spaces $M, Z(M, d) \equiv \operatorname{ker} d$ and $B(M, d) \equiv \operatorname{Im} d$ are called cochains, cocycles and coboundaries of $(M, d)$, respectively. The condition $d^{2}=0$ implies
that $B(M, d) \subset Z(M, d)$, and the $R$-module

$$
H(M, d)=Z(M, d) / B(M, d)
$$

is called the cohomology group of the differential module ( $M, d$ ). We shall often write $Z(M), B(M)$ and $H(M)$, omitting the differential $d$ when there is no risk of confusion.

Let ( $M, d$ ) and ( $M^{\prime}, d^{\prime}$ ) be differential $R$-modules.
Definition 1.2. A morphism of differential modules is a morphism $f: M \rightarrow M^{\prime}$ of $R$-modules which commutes with the differentials, $f \circ d^{\prime}=d \circ f$.

A morphism of differential modules maps cocycles to cocycles and coboundaries to coboundaries, thus inducing a morphism $H(f): H(M) \rightarrow H\left(M^{\prime}\right)$.

Proposition 1.3. Let $0 \rightarrow M^{\prime} \xrightarrow{i} M \xrightarrow{p} M^{\prime \prime} \rightarrow 0$ be an exact sequence of differential $R$-modules. There exists a morphism $\delta: H\left(M^{\prime \prime}\right) \rightarrow H\left(M^{\prime}\right)$ (called connecting morphism) and an exact triangle of cohomology


Proof. The construction of $\delta$ is as follows: let $\xi^{\prime \prime} \in H\left(M^{\prime \prime}\right)$ and let $m^{\prime \prime}$ be a cocycle whose class is $\xi^{\prime \prime}$. If $m$ is an element of $M$ such that $p(m)=m^{\prime \prime}$, we have $p(d(m))=d\left(m^{\prime \prime}\right)=0$ and then $d(m)=i\left(m^{\prime}\right)$ for some $m^{\prime} \in M^{\prime}$ which is a cocycle. Now, the cocycle $m^{\prime}$ defines a cohomology class $\delta\left(\xi^{\prime \prime}\right)$ in $H\left(M^{\prime}\right)$, which is independent of the choices we have made, thus defining a morphism $\delta: H\left(M^{\prime \prime}\right) \rightarrow H\left(M^{\prime}\right)$. One proves by direct computation that the triangle is exact.

The above results can be translated to the setting of complexes of $R$-modules.
Definition 1.4. A complex of $R$-modules is a differential $R$-module $\left(M^{\bullet}, d\right)$ which is $\mathbb{Z}$-graded, $M^{\bullet}=\bigoplus_{n \in \mathbb{Z}} M^{n}$, and whose differential fulfills $d\left(M^{n}\right) \subset M^{n+1}$ for every $n \in \mathbb{Z}$.

We shall usually write a complex of $R$-modules in the more pictorial form

$$
\ldots \xrightarrow{d_{n-2}} M^{n-1} \xrightarrow{d_{n-1}} M^{n} \xrightarrow{d_{n}} M^{n+1} \xrightarrow{d_{n+1}} \ldots
$$

For a complex $M^{\bullet}$ the cocycle and coboundary modules and the cohomology group split as direct sums of terms $Z^{n}\left(M^{\bullet}\right)=\operatorname{ker} d_{n}, B^{n}\left(M^{\bullet}\right)=\operatorname{Im} d_{n-1}$ and $H^{n}\left(M^{\bullet}\right)=$ $Z^{n}\left(M^{\bullet}\right) / B^{n}\left(M^{\bullet}\right)$ respectively. The groups $H^{n}\left(M^{\bullet}\right)$ are called the cohomology groups of the complex $M^{\bullet}$.

Definition 1.5. A morphism of complexes of $R$-modules $f: N^{\bullet} \rightarrow M^{\bullet}$ is a collection of morphisms $\left\{f_{n}: N^{n} \rightarrow M^{n} \mid n \in \mathbb{Z}\right\}$, such that the following diagram commutes:


For complexes, Proposition 1.3 takes the following form:
Proposition 1.6. Let $0 \rightarrow N^{\bullet} \xrightarrow{i} M^{\bullet} \xrightarrow{p} P^{\bullet} \rightarrow 0$ be an exact sequence of complexes of $R$-modules. There exist connecting morphisms $\delta_{n}: H^{n}\left(P^{\bullet}\right) \rightarrow H^{n+1}\left(N^{\bullet}\right)$ and a long exact sequence of cohomology

$$
\begin{aligned}
\ldots \xrightarrow{\delta_{n-1}} H^{n}\left(N^{\bullet}\right) \xrightarrow{H(i)} & H^{n}\left(M^{\bullet}\right) \\
& \xrightarrow{H(p)} H^{n}\left(P^{\bullet}\right) \xrightarrow{\delta_{n}} \\
& H^{n+1}\left(N^{\bullet}\right) \xrightarrow{H(i)} H^{n+1}\left(M^{\bullet}\right) \xrightarrow{H(p)} H^{n+1}\left(P^{\bullet}\right) \xrightarrow{\delta_{n+1}} \ldots
\end{aligned}
$$

Proof. The connecting morphism $\delta: H^{\bullet}\left(P^{\bullet}\right) \rightarrow H^{\bullet}\left(N^{\bullet}\right)$ defined in Proposition 1.3 splits into morphisms $\delta_{n}: H^{n}\left(P^{\bullet}\right) \rightarrow H^{n+1}\left(N^{\bullet}\right)$ (indeed the connecting morphism increases the degree by one) and the long exact sequence of the statement is obtained by developing the exact triangle of cohomology introduced in Proposition 1.3.
1.3. Homotopies. Different (i.e., nonisomorphic) complexes may nevertheless have isomorphic cohomologies. A sufficient conditions for this to hold is that the two complexes are homotopic. While this condition is not necessary, in practice the (by far) commonest way to prove the isomorphism between two cohomologies is to exhibit a homototopy between the corresponding complexes.

Definition 1.7. Given two complexes of $R$-modules, $\left(M^{\bullet}, d\right)$ and $\left(N^{\bullet}, d^{\prime}\right)$, and two morphisms of complexes, $f, g: M^{\bullet} \rightarrow N^{\bullet}$, a homotopy between $f$ and $g$ is a morphism $K: N^{\bullet} \rightarrow M^{\bullet-1}$ (i.e., for every $k$, a morphism $K: N^{k} \rightarrow M^{k-1}$ ) such that $d^{\prime} \circ K+K \circ$ $d=f-g$.

The situation is depicted in the following commutative diagram.


Proposition 1.8. If there is a homotopy between $f$ and $g$, then $H(f)=H(g)$, namely, homotopic morphisms induce the same morphism in cohomology.

Proof. Let $\xi=[m] \in H^{k}\left(M^{\bullet}, d\right)$. Then

$$
H(f)(\xi)=[f(m)]=[g(m)]+\left[d^{\prime}(K(m))\right]+[K(d m)]=[g(m)]=H(g)(\xi)
$$

since $d m=0,\left[d^{\prime}(K(m))\right]=0$.
Definition 1.9. Two complexes of $R$-modules, $\left(M^{\bullet}, d\right)$ and $\left(N^{\bullet}, d^{\boldsymbol{\top}}\right)$, are said to be homotopically equivalent (or homotopic) if there exist morphisms $f: M^{\bullet} \rightarrow N^{\bullet}$, $g: N^{\bullet} \rightarrow M^{\bullet}$, such that:
$f \circ g: N^{\bullet} \rightarrow N^{\bullet}$ is homotopic to the identity map $\operatorname{id}_{N}$;
$g \circ f: M^{\bullet} \rightarrow M^{\bullet}$ is homotopic to the identity map $\mathrm{id}_{M}$.
Corollary 1.10. Two homotopic complexes have isomorphic cohomologies.

Proof. We use the notation of the previous Definition. One has

$$
\begin{aligned}
& H(f) \circ H(g)=H(f \circ g)=H\left(\mathrm{id}_{N}\right)=\operatorname{id}_{H^{(N)}} \\
& H(g) \circ H(f)=H(g \circ f)=H\left(\operatorname{id}_{M}\right)=\operatorname{id}_{\left.H^{( } M\right)}
\end{aligned}
$$

so that both $H(f)$ and $H(g)$ are isomorphism.
Definition 1.11. A homotopy of a complex of $R$-modules $\left(M^{\bullet}, d\right)$ is a homotopy between the identity morphism on $M$, and the zero morphism; more explicitly, it is a morphism $K: M^{\bullet} \rightarrow M^{\bullet-1}$ such that $d \circ K+K \circ d=\mathrm{id}_{M}$.

Proposition 1.12. If a complex of $R$-modules $\left(M^{\bullet}, d\right)$ admits a homotopy, then it is exact (i.e., all its cohomology groups vanish; one also says that the complex is acyclic).

Proof. One could use the previous definitions and results to yield a proof, but it is easier to note that if $m \in M^{k}$ is a cocycle (so that $d m=0$ ), then

$$
d(K(m))=m-K(d m)=m
$$

so that $m$ is also a coboundary.
Remark 1.13. More generally, one can state that if a homotopy $K: M^{k} \rightarrow M^{k-1}$ exists for $k \geq k_{0}$, then $H^{k}(M, d)=0$ for $k \geq k_{0}$. In the case of complexes bounded below zero (i.e., $M=\oplus_{k \in \mathbb{N}} M^{k}$ ) often a homotopy is defined only for $k \geq 1$, and it may happen that $H^{0}(M, d) \neq 0$. Examples of such situations will be given later in this chapter.

Remark 1.14. One might as well define a homotopy by requiring $d^{\prime} \circ K-K \circ d=\ldots$; the reader may easily check that this change of sign is immaterial.

## 2. De Rham cohomology

As an example of a cohomology theory we may consider the de Rham cohomology of a differentiable manifold $X$. Let $\Omega^{k}(X)$ be the vector space of differential $k$-forms on $X$, and let $d: \Omega^{k}(X) \rightarrow \Omega^{k+1}(X)$ be the exterior differential. Then $\left(\Omega^{\bullet}(X), d\right)$ is a differential complex of $\mathbb{R}$-vector spaces (the de Rham complex), whose cohomology groups are denoted $H_{d R}^{k}(X)$ and are called the de Rham cohomology groups of $X$. Since $\Omega^{k}(X)=0$ for $k>n$ and $k<0$, the groups $H_{d R}^{k}(X)$ vanish for $k>n$ and $k<0$. Moreover, since $\operatorname{ker}\left[d: \Omega^{0}(X) \rightarrow \Omega^{1}(X)\right]$ is formed by the locally constant functions on $X$, we have $H_{d R}^{0}(X)=\mathbb{R}^{C}$, where $C$ is the number of connected components of $X$.

If $f: X \rightarrow Y$ is a smooth morphism of differentiable manifolds, the pullback morphism $f^{*}: \Omega^{k}(Y) \rightarrow \Omega^{k}(X)$ commutes with the exterior differential, thus giving rise to a morphism of differential complexes $\left.\left(\Omega^{\bullet}(Y), d\right) \rightarrow\left(\Omega^{\bullet}(X), d\right)\right)$; the corresponding morphism $H(f): H_{d R}^{\bullet}(Y) \rightarrow H_{d R}^{\bullet}(X)$ is usually denoted $f^{\sharp}$.

We may easily compute the cohomology of the Euclidean spaces $\mathbb{R}^{n}$. Of course one has $H_{d R}^{0}\left(\mathbb{R}^{n}\right)=\operatorname{ker}\left[d: C^{\infty}\left(\mathbb{R}^{n}\right) \rightarrow \Omega^{1}\left(\mathbb{R}^{n}\right)\right]=\mathbb{R}$.

Proposition 1.1. (Poincaré lemma) $H_{d R}^{k}\left(\mathbb{R}^{n}\right)=0$ for $k>0$.
Proof. We define a linear operator $K: \Omega^{k}\left(\mathbb{R}^{n}\right) \rightarrow \Omega^{k-1}\left(\mathbb{R}^{n}\right)$ by letting, for any $k$-form $\omega \in \Omega^{k}\left(\mathbb{R}^{n}\right), k \geq 1$, and all $x \in \mathbb{R}^{n}$,

$$
(K \omega)(x)=k\left[\int_{0}^{1} t^{k-1} \omega_{i_{1} i_{2} \ldots i_{k}}(t x) d t\right] x^{i_{1}} d x^{i_{2}} \wedge \cdots \wedge d x^{i_{k}}
$$

One easily shows that $d K+K d=\mathrm{Id}$; this means that $K$ is a homotopy of the de Rham complex of $\mathbb{R}^{n}$ defined for $k \geq 1$, so that, according to Proposition 1.12 and Remark 1.13, all cohomology groups vanish in positive degree. Explicitly, if $\omega$ is closed, we have $\omega=d K \omega$, so that $\omega$ is exact.

Exercise 1.2. Realize the circle $S^{1}$ as the unit circle in $\mathbb{R}^{2}$. Show that the integration of 1-forms on $S^{1}$ yields an isomorphism $H_{d R}^{1}\left(S^{1}\right) \simeq \mathbb{R}$. This argument can be quite easily generalized to show that, if $X$ is a connected, compact and orientable $n$-dimensional manifold, then $H_{d R}^{n}(X) \simeq \mathbb{R}$.

If a manifold is a cartesian product, $X=X_{1} \times X_{2}$, there is a way to compute the de Rham cohomology of $X$ out of the de Rham cohomology of $X_{1}$ and $X_{2}$ (Künneth theorem, cf. [3]). For later use, we prove here a very particular case. This will serve also as an example of the notion of homotopy between complexes.

Proposition 1.3. If $X$ is a differentiable manifold, then $H_{d R}^{k}(X \times \mathbb{R})$ $\simeq H_{d R}^{k}(X)$ for all $k \geq 0$.

Proof. Let $t$ a coordinate on $\mathbb{R}$. Denoting by $p_{1}, p_{2}$ the projections of $X \times \mathbb{R}$ onto its two factors, every $k$-form $\omega$ on $X \times \mathbb{R}$ can be written as

$$
\begin{equation*}
\omega=f p_{1}^{*} \omega_{1}+g p_{1}^{*} \omega_{2} \wedge p_{2}^{*} d t \tag{1.1}
\end{equation*}
$$

where $\omega_{1} \in \Omega^{k}(X), \omega_{2} \in \Omega^{k-1}(X)$, and $f, g$ are functions on $X \times \mathbb{R} .^{1}$ Let $s: X \rightarrow X \times \mathbb{R}$ be the section $s(x)=(x, 0)$. One has $p_{1} \circ s=\mathrm{id}_{X}$ (i.e., $s$ is indeed a section of $p_{1}$ ), hence $s^{*} \circ p_{1}^{*}: \Omega^{\bullet}(X) \rightarrow \Omega^{\bullet}(X)$ is the identity. We also have a morphism $p_{1}^{*} \circ s^{*}: \Omega^{\bullet}(X \times \mathbb{R}) \rightarrow$ $\Omega^{\bullet}(X \times \mathbb{R})$. This is not the identity (as a matter of fact one, has $\left.p_{1}^{*} \circ s^{*}(\omega)=f(x, 0) p_{1}^{*} \omega_{1}\right)$. However, this morphism is homotopic to $\operatorname{id}_{\Omega \bullet(X \times \mathbb{R})}$, while $\operatorname{id}_{\Omega \bullet(X)}$ is definitely homotopic to itself, so that the complexes $\Omega^{\bullet}(X)$ and $\Omega^{\bullet}(X \times \mathbb{R})$ are homotopic, thus proving our claim as a consequence of Corollary 1.10. So we only need to exhibit a homotopy between $p_{1}^{*} \circ s^{*}$ and $\operatorname{id}_{\Omega} \bullet(X \times \mathbb{R})$.

This homotopy $K: \Omega^{\bullet}(X \times \mathbb{R}) \rightarrow \Omega^{\bullet-1}(X \times \mathbb{R})$ is defined as (with reference to equation (1.1))

$$
K(\omega)=(-1)^{k}\left[\int_{0}^{t} g(x, s) d s\right] p_{2}^{*} \omega_{2}
$$

The proof that $K$ is a homotopy is an elementary direct computation, ${ }^{2}$ after which one gets

$$
d \circ K+K \circ d=\operatorname{id}_{\Omega \bullet(X \times \mathbb{R})}-p_{1}^{*} \circ s^{*}
$$

In particular we obtain that the morphisms

$$
p_{1}^{\sharp}: H_{d R}^{\bullet}(X) \rightarrow H_{d R}^{\bullet}(X \times \mathbb{R}), \quad s^{\sharp}: H_{d R}^{\bullet}(X \times \mathbb{R}) \rightarrow H_{d R}^{\bullet}(X \times)
$$

are isomorphisms.
REMARK 1.4. If we take $X=\mathbb{R}^{n}$ and make induction on $n$ we get another proof of Poincaré lemma.

EXERCISE 1.5. By a similar argument one proves that for all $k>0$

$$
H_{d R}^{k}\left(X \times S^{1}\right) \simeq H_{d R}^{k}(X) \oplus H_{d R}^{k-1}(X)
$$

Now we give an example of a long cohomology exact sequence within de Rham's theory. Let $X$ be a differentiable manifold, and $Y$ a closed submanifold. Let $r_{k}: \Omega^{k}(X) \rightarrow$ $\Omega^{k}(Y)$ be the restriction morphism; this is surjective. Since the exterior differential commutes with the restriction, after letting $\Omega^{k}(X, Y)=\operatorname{ker} r_{k}$ a differential $d^{\prime}: \Omega^{k}(X, Y) \rightarrow$

[^0]$\Omega^{k+1}(X, Y)$ is defined. We have therefore an exact sequence of differential modules, in a such a way that the diagram

commutes. The complex $\left(\Omega^{\bullet}(X, Y), d^{\prime}\right)$ is called the relative de Rham complex, ${ }^{3}$ and its cohomology groups by $H_{d R}^{k}(X, Y)$ are called the relative de Rham cohomology groups. One has a long cohomology exact sequence
\[

$$
\begin{aligned}
0 & \rightarrow H_{d R}^{0}(X, Y) \rightarrow H_{d R}^{0}(X) \rightarrow H_{d R}^{0}(Y) \stackrel{\delta}{\rightarrow} H_{d R}^{1}(X, Y) \\
& \rightarrow H_{d R}^{1}(X) \rightarrow H_{d R}^{1}(Y) \xrightarrow{\delta} H_{d R}^{2}(X, Y) \rightarrow \ldots
\end{aligned}
$$
\]

ExERCISE 1.6. 1. Prove that the space $\operatorname{ker} d^{\prime}: \Omega^{k}(X, Y) \rightarrow \Omega^{k+1}(X, Y)$ is for all $k \geq 0$ the kernel of $r_{k}$ restricted to $Z^{k}(X)$, i.e., is the space of closed $k$-forms on $X$ which vanish on $Y$. As a consequence $H_{d R}^{0}(X, Y)=0$ whenever $X$ and $Y$ are connected.
2. Let $n=\operatorname{dim} X$ and $\operatorname{dim} Y \leq n-1$. Prove that $H_{d R}^{n}(X, Y) \rightarrow H_{d R}^{n}(X)$ surjects, and that $H_{d R}^{k}(X, Y)=0$ for $k \geq n+1$. Make an example where $\operatorname{dim} X=\operatorname{dim} Y$ and check if the previous facts still hold true.

Example 1.7. Given the standard embedding of $S^{1}$ into $\mathbb{R}^{2}$, we compute the relative cohomology $H_{d R}^{\bullet}\left(\mathbb{R}^{2}, S^{1}\right)$. We have the long exact sequence

$$
\begin{aligned}
0 & \rightarrow H_{d R}^{0}\left(\mathbb{R}^{2}, S^{1}\right) \rightarrow H_{d R}^{0}\left(\mathbb{R}^{2}\right) \rightarrow H_{d R}^{0}\left(S^{1}\right) \stackrel{\delta}{\rightarrow} H_{d R}^{1}\left(\mathbb{R}^{2}, S^{1}\right) \\
& \rightarrow H_{d R}^{1}\left(\mathbb{R}^{2}\right) \rightarrow H_{d R}^{1}\left(S^{1}\right) \xrightarrow{\delta} H_{d R}^{2}\left(\mathbb{R}^{2}, S^{1}\right) \rightarrow H_{d R}^{2}\left(\mathbb{R}^{2}\right) \rightarrow 0
\end{aligned}
$$

As in the previous exercise, we have $H_{d R}^{k}\left(\mathbb{R}^{2}, S^{1}\right)=0$ for $k \geq 3$. Since $H_{d R}^{0}\left(\mathbb{R}^{2}\right) \simeq \mathbb{R}$, $H_{d R}^{1}\left(\mathbb{R}^{2}\right)=H_{d R}^{2}\left(\mathbb{R}^{2}\right)=0, H_{d R}^{0}\left(S^{1}\right) \simeq H_{d R}^{1}\left(S^{1}\right) \simeq \mathbb{R}$, we obtain the exact sequences

$$
\begin{aligned}
0 \rightarrow H_{d R}^{0}\left(\mathbb{R}^{2}, S^{1}\right) & \rightarrow \mathbb{R} \stackrel{r}{\rightarrow} \mathbb{R} \rightarrow H_{d R}^{1}\left(\mathbb{R}^{2}, S^{1}\right) \rightarrow 0 \\
0 & \rightarrow \mathbb{R} \rightarrow H_{d R}^{2}\left(\mathbb{R}^{2}, S^{1}\right) \rightarrow 0
\end{aligned}
$$

where the morphism $r$ is an isomorphism. Therefore from the first sequence we get $H_{d R}^{0}\left(\mathbb{R}^{2}, S^{1}\right)=0$ (as we already noticed) and $H_{d R}^{1}\left(\mathbb{R}^{2}, S^{1}\right)=0$. From the second we obtain $H_{d R}^{2}\left(\mathbb{R}^{2}, S^{1}\right) \simeq \mathbb{R}$.

From this example we may abstract the fact that whenever $X$ and $Y$ are connected, then $H_{d R}^{0}(X, Y)=0$.

ExERCISE 1.8. Consider a submanifold $Y$ of $\mathbb{R}^{2}$ formed by two disjoint embedded copies of $S^{1}$. Compute $H_{d R}^{\bullet}\left(\mathbb{R}^{2}, Y\right)$.

[^1]
## 3. Mayer-Vietoris sequence in de Rham cohomology

The Mayer-Vietoris sequence is another example of long cohomology exact sequence associated with de Rham cohomology, and is very useful for making computations. Assume that a differentiable manifold $X$ is the union of two open subset $U, V$. For every $k, 0 \leq k \leq n=\operatorname{dim} X$ we have the sequence of morphisms

$$
\begin{equation*}
0 \rightarrow \Omega^{k}(X) \xrightarrow{i} \Omega^{k}(U) \oplus \Omega^{k}(V) \xrightarrow{p} \Omega^{k}(U \cap V) \rightarrow 0 \tag{1.2}
\end{equation*}
$$

where

$$
i(\omega)=\left(\omega_{\mid U}, \omega_{\mid V}\right), \quad p\left(\left(\omega_{1}, \omega_{2}\right)\right)=\omega_{1 \mid U \cap V}-\omega_{2 \mid U \cap V} .
$$

One easily checks that $i$ is injective and that $\operatorname{ker} p=\operatorname{Im} i$. The surjectivity of $p$ is somehow less trivial, and to prove it we need a partition of unity argument. From elementary differential geometry we recall that a partition of unity subordinated to the cover $\{U, V\}$ of $X$ is a pair of smooth functions $f_{1}, f_{2}: X \rightarrow \mathbb{R}$ such that

$$
\operatorname{supp}\left(f_{1}\right) \subset U, \quad \operatorname{supp}\left(f_{2}\right) \subset V, \quad f_{1}+f_{2}=1
$$

Given $\tau \in \Omega^{k}(U \cap V)$, let

$$
\omega_{1}=f_{2} \tau, \quad \omega_{2}=-f_{1} \tau
$$

These $k$-form are defined on $U$ and $V$, respectively. Then $p\left(\left(\omega_{1}, \omega_{2}\right)\right)=\tau$. Thus the sequence (1.2) is exact. Since the exterior differential $d$ commutes with restrictions, we obtain a long cohomology exact sequence
(1.3)

$$
\begin{align*}
0 \rightarrow H_{d R}^{0}(X) \rightarrow H_{d R}^{0}(U) \oplus H_{d R}^{0}(V) \rightarrow H_{d R}^{0}(U \cap V) \stackrel{\delta}{\rightarrow} & H_{d R}^{1}(X) \rightarrow  \tag{1.3}\\
& \rightarrow H_{d R}^{1}(U) \oplus H_{d R}^{1}(V) \rightarrow H_{d R}^{1}(U \cap V) \xrightarrow{\delta} H_{d R}^{2}(X) \rightarrow \ldots
\end{align*}
$$

This is the Mayer-Vietoris sequence. The argument may be generalized to a union of several open sets. ${ }^{4}$

Exercise 1.1. Use the Mayer-Vietoris sequence (1.3) to compute the de Rham cohomology of the circle $S^{1}$.

Example 1.2. We use the Mayer-Vietoris sequence (1.3) to compute the de Rham cohomology of the sphere $S^{2}$ (as a matter of fact we already know the 0th and 2nd group, but not the first). Using suitable stereographic projections, we can assume that $U$ and $V$ are diffeomorphic to $\mathbb{R}^{2}$, while $U \cap V \simeq S^{1} \times \mathbb{R}$. Since $S^{1} \times \mathbb{R}$ is homotopic to $S^{1}$, it has the same de Rham cohomology. Hence the sequence (1.3) becomes

$$
\begin{gathered}
0 \rightarrow H_{d R}^{0}\left(S^{2}\right) \rightarrow \mathbb{R} \oplus \mathbb{R} \rightarrow \mathbb{R} \rightarrow H_{d R}^{1}\left(S^{2}\right) \rightarrow 0 \\
0 \rightarrow \mathbb{R} \rightarrow H_{d R}^{2}\left(S^{2}\right) \rightarrow 0
\end{gathered}
$$

From the first sequence, since $H_{d R}^{0}\left(S^{2}\right) \simeq \mathbb{R}$, the map $H_{d R}^{0}\left(S^{2}\right) \rightarrow \mathbb{R} \oplus \mathbb{R}$ is injective, and then we get $H_{d R}^{1}\left(S^{2}\right)=0$; from the second sequence, $H_{d R}^{2}\left(S^{2}\right) \simeq \mathbb{R}$.

[^2]EXERCISE 1.3. Use induction to show that if $n \geq 3$, then $H_{d R}^{k}\left(S^{n}\right) \simeq \mathbb{R}$ for $k=0, n$, $H_{d R}^{k}\left(S^{n}\right)=0$ otherwise.

ExErcise 1.4. Consider $X=S^{2}$ and $Y=S^{1}$, embedded as an equator in $S^{2}$. Compute the relative de Rham cohomology $H_{d R}^{\bullet}\left(S^{2}, S^{1}\right)$.

## 4. Elementary homotopy theory

4.1. Homotopy of paths. Let $X$ be a topological space. We denote by $I$ the closed interval $[0,1]$. A path in $X$ is a continuous map $\gamma: I \rightarrow X$. We say that $X$ is pathwise connected if given any two points $x_{1}, x_{2} \in X$ there is a path $\gamma$ such that $\gamma(0)=x_{1}, \gamma(1)=x_{2}$.

A homotopy $\Gamma$ between two paths $\gamma_{1}, \gamma_{2}$ is a continuous map $\Gamma: I \times I \rightarrow X$ such that

$$
\Gamma(t, 0)=\gamma_{1}(t), \quad \Gamma(t, 1)=\gamma_{2}(t)
$$

If the two paths have the same end points (i.e. $\gamma_{1}(0)=\gamma_{2}(0)=x_{1}, \gamma_{1}(1)=\gamma_{2}(1)=x_{2}$ ), we may introduce the stronger notion of homotopy with fixed end points by requiring additionally that $\Gamma(0, s)=x_{1}, \Gamma(1, s)=x_{2}$ for all $s \in I$.

Let us fix a base point $x_{0} \in X$. A loop based at $x_{0}$ is a path such that $\gamma(0)=\gamma(1)=$ $x_{0}$. Let us denote $\mathcal{L}\left(x_{0}\right)$ th set of loops based at $x_{0}$. One can define a composition between elements of $\mathcal{L}\left(x_{0}\right)$ by letting

$$
\left(\gamma_{2} \cdot \gamma_{1}\right)(t)= \begin{cases}\gamma_{1}(2 t), & 0 \leq t \leq \frac{1}{2} \\ \gamma_{2}(2 t-1), & \frac{1}{2} \leq t \leq 1\end{cases}
$$

This does not make $\mathcal{L}\left(x_{0}\right)$ into a group, since the composition is not associative (composing in a different order yields different parametrizations).

Proposition 1.1. If $x_{1}, x_{2} \in X$ and there is a path connecting $x_{1}$ with $x_{2}$, then $\mathcal{L}\left(x_{1}\right) \simeq \mathcal{L}\left(x_{2}\right)$.

Proof. Let $c$ be such a path, and let $\gamma_{1} \in \mathcal{L}\left(x_{1}\right)$. We define $\gamma_{2} \in \mathcal{L}\left(x_{2}\right)$ by letting

$$
\gamma_{2}(t)= \begin{cases}c(1-3 t), & 0 \leq t \leq \frac{1}{3} \\ \gamma_{1}(3 t-1), & \frac{1}{3} \leq t \leq \frac{2}{3} \\ c(3 t-2), & \frac{2}{3} \leq t \leq 1\end{cases}
$$

This establishes the isomorphism.
4.2. The fundamental group. Again with reference with a base point $x_{0}$, we consider in $\mathcal{L}\left(x_{0}\right)$ an equivalence relation by decreeing that $\gamma_{1} \sim \gamma_{2}$ if there is a homotopy with fixed end points between $\gamma_{1}$ and $\gamma_{2}$. The composition law in $\mathcal{L}_{x_{0}}$ descends to a group structure in the quotient

$$
\pi_{1}\left(X, x_{0}\right)=\mathcal{L}\left(x_{0}\right) / \sim
$$

$\pi_{1}\left(X, x_{0}\right)$ is the fundamental group of $X$ with base point $x_{0}$; in general it is nonabelian, as we shall see in examples. As a consequence of Proposition 1.1, if $x_{1}, x_{2} \in X$ and there is a path connecting $x_{1}$ with $x_{2}$, then $\pi_{1}\left(X, x_{1}\right) \simeq \pi_{1}\left(X, x_{2}\right)$. In particular, if $X$ is pathwise connected the fundamental group $\pi_{1}\left(X, x_{0}\right)$ is independent of $x_{0}$ up to isomorphism; in this situation, one uses the notation $\pi_{1}(X)$.

Definition 1.2. $X$ is said to be simply connected if it is pathwise connected and $\pi_{1}(X)=\{e\}$.

The simplest example of a simply connected space is the one-point space $\{*\}$.
Since the definition of the fundamental group involves the choice of a base point, to describe the behaviour of the fundamental group we need to introduce a notion of map which takes the base point into account. Thus, we say that a pointed space $\left(X, x_{0}\right)$ is a pair formed by a topological space $X$ with a chosen point $x_{0}$. A map of pointed spaces $f:\left(X, x_{0}\right) \rightarrow\left(Y, y_{0}\right)$ is a continuous map $f: X \rightarrow Y$ such that $f\left(x_{0}\right)=y_{0}$. It is easy to show that a map of pointed spaces induces a group homomorphism $f_{*}: \pi\left(X, x_{0}\right) \rightarrow$ $\pi_{1}\left(Y, y_{0}\right)$.
4.3. Homotopy of maps. Given two topological spaces $X, Y$, a homotopy between two continuous maps $f, g: X \rightarrow Y$ is a map $F: X \times I \rightarrow Y$ such that $F(x, 0)=f(x)$, $F(x, 1)=g(x)$ for all $x \in X$. One then says that $f$ and $g$ are homotopic.

Definition 1.3. One says that two topological spaces $X, Y$ are homotopically equivalent if there are continuous maps $f: X \rightarrow Y, g: Y \rightarrow X$ such that $g \circ f$ is homotopic to $i d_{X}$, and $f \circ g$ is homotopic to $i d_{Y}$. The map $f, g$ are said to be homotopical equivalences,

Of course, homeomorphic spaces are homotopically equivalent.
Example 1.4. For any manifold $X$, take $Y=X \times \mathbb{R}, f(x)=(x, 0), g$ the projection onto $X$. Then $F: X \times I \rightarrow X, F(x, t)=x$ is a homotopy between $g \circ f$ and $\mathrm{id}_{X}$, while $G: X \times \mathbb{R} \times I \rightarrow X \times \mathbb{R}, G(x, s, t)=(x, s t)$ is a homotopy between $f \circ g$ and id ${ }_{Y}$. So $X$ and $X \times \mathbb{R}$ are homotopically equivalent. The reader should be able to concoct many similar examples.

Given two pointed spaces $\left(X, x_{0}\right),\left(Y, y_{0}\right)$, we say they are homotopically equivalent if there exist maps of pointed spaces $f:\left(X, x_{0}\right) \rightarrow\left(Y, y_{0}\right), g:\left(Y, y_{0}\right) \rightarrow\left(X, x_{0}\right)$ that make the topological spaces $X, Y$ homotopically equivalent.

Proposition 1.5. Let $f:\left(X, x_{0}\right) \rightarrow\left(Y, y_{0}\right)$ be a homotopical equivalence. Then $f_{*}: \pi_{*}\left(X, x_{0}\right) \rightarrow\left(Y, y_{0}\right)$ is an isomorphism.

Proof. Let $g:\left(Y, y_{0}\right) \rightarrow\left(X, x_{0}\right)$ be a map that realizes the homotopical equivalence, and denote by $F$ a homotopy between $g \circ f$ and $\mathrm{id}_{X}$. Let $\gamma$ be a loop based at $x_{0}$.

Then $g \circ f \circ \gamma$ is again a loop based at $x_{0}$, and the map

$$
\Gamma: I \times I \rightarrow X, \quad \Gamma(s, t)=F(\gamma(s), t)
$$

is a homotopy between $\gamma$ and $g \circ f \circ \gamma$, so that $\gamma=g \circ f \circ \gamma$ in $\pi_{1}\left(X, x_{0}\right)$. Hence, $g_{*} \circ f_{*}=\operatorname{id}_{\pi_{1}\left(X, x_{0}\right)}$. In the same way one proves that $f_{*} \circ g_{*}=\operatorname{id}_{\pi_{1}\left(Y, y_{0}\right)}$, so that the claim follows.

Corollary 1.6. If two pathwise connected spaces $X$ and $Y$ are homotopic, then their fundamental groups are isomorphic.

Definition 1.7. A topological space is said to be contractible if it is homotopically equivalent to the one-point space $\{*\}$.

A contractible space is simply connected.

ExERCISE 1.8. 1. Show that $\mathbb{R}^{n}$ is contractible, hence simply connected. 2. Compute the fundamental groups of the following spaces: the punctured plane ( $\mathbb{R}^{2}$ minus a point); $\mathbb{R}^{3}$ minus a line; $\mathbb{R}^{n}$ minus a $(n-2)$-plane (for $n \geq 3$ ).
4.4. Homotopic invariance of de Rham cohomology. We may now prove the invariance of de Rham cohomology under homotopy.

Lemma 1.9. Let $X, Y$ be differentiable manifolds, and let $f, g: X \rightarrow Y$ be two homotopic smooth maps. Then the morphisms they induce in cohomology coincide, $f^{\sharp}=g^{\sharp}$.

Proof. We choose a homotopy between $f$ and $g$ in the form of a $\operatorname{smooth}^{5}$ map $F: X \times \mathbb{R} \rightarrow Y$ such that

$$
F(x, t)=f(x) \quad \text { if } \quad t \leq 0, \quad F(x, t)=g(x) \quad \text { if } \quad t \geq 1
$$

We define sections $s_{0}, s_{1}: X \rightarrow X \times \mathbb{R}$ by letting $s_{0}(x)=(x, 0), s_{1}(x)=(x, 1)$. Then $f=F \circ s_{0}, g=F \circ s_{1}$, so $f^{\sharp}=s_{0}^{\sharp} \circ F^{\sharp}$ and $g^{\sharp}=s_{1}^{\sharp} \circ F^{\sharp}$. Let $p_{1}: X \times \mathbb{R} \rightarrow X$, $p_{2}: X \times \mathbb{R} \rightarrow \mathbb{R}$ be the projections. Then $s_{0}^{\sharp} \circ p_{1}^{\sharp}=s_{1}^{\sharp} \circ p_{1}^{\sharp}=\mathrm{Id}$. By Proposition $1.3 p_{1}^{\sharp}$ is an isomorphism. Then $s_{0}^{\sharp}=s_{1}^{\sharp}$, and $f^{\sharp}=F^{\sharp}=g^{\sharp}$.

Proposition 1.10. Let $X$ and $Y$ be homotopic differentiable manifolds. Then $H_{d R}^{k}(X) \simeq H_{d R}^{k}(Y)$ for all $k \geq 0$.

Proof. If $f, g$ are two smooth maps realizing the homotopy, then $f^{\sharp} \circ g^{\sharp}=g^{\sharp} \circ f^{\sharp}=$ Id, so that both $f^{\sharp}$ and $g^{\sharp}$ are isomorphisms.

[^3]4.5. The van Kampen theorem. The computation of the fundamental group of a topological space is often unsuspectedly complicated. An important tool for such computations is the van Kampen theorem, which we state without proof. This theorem allows one, under some conditions, to compute the fundamental group of an union $U \cup V$ if one knows the fundamental groups of $U, V$ and $U \cap V$. As a prerequisite we need the notion of amalgamated product of two groups. Let $G, G_{1}, G_{2}$ be groups, with fixed morphisms $f_{1}: G \rightarrow G_{1}, f_{2}: G \rightarrow G_{2}$. Let $F$ the free group generated by $G_{1} \amalg G_{2}$ and denote by . the product in this group. ${ }^{6}$ Let $R$ be the normal subgroup generated by elements of the form ${ }^{7}$
\[

$$
\begin{gathered}
(x y) \cdot y^{-1} \cdot x^{-1} \quad \text { with } x, y \text { both in } G_{1} \text { or } G_{2} \\
f_{1}(g) \cdot f_{2}(g)^{-1} \quad \text { for } g \in G .
\end{gathered}
$$
\]

Then one defines the amalgamated product $G_{1} *_{G} G_{2}$ as $F / R$. There are natural maps $g_{1}: G_{1} \rightarrow G_{1} *_{G} G_{2}, g_{2}: G_{2} \rightarrow G_{1} *_{G} G_{2}$ obtained by composing the inclusions with the projection $F \rightarrow F / R$, and one has $g_{1} \circ f_{1}=g_{2} \circ f_{2}$. Intuitively, one could say that $G_{1} *_{G} G_{2}$ is the smallest subgroup generated by $G_{1}$ and $G_{2}$ with the identification of $f_{1}(g)$ and $f_{2}(g)$ for all $g \in G$.

Exercise 1.11. (1) Prove that if $G_{1}=G_{2}=\{e\}$, and $G$ is any group, then $G_{1} *_{G} G_{2}=\{e\}$.
(2) Let $G$ be the group with three generators $a, b, c$, satisfying the relation $a b=c b a$. Let $\mathbb{Z} \rightarrow G$ be the homomorphism induced by $1 \mapsto c$. Prove that $G *_{\mathbb{Z}} G$ is isomorphic to a group with four generators $m, n, p, q$, satisfying the relation $m n m^{-1} n^{-1} p q p^{-1} q^{-1}=e$.

Suppose now that a pathwise connected space $X$ is the union of two pathwise connected open subsets $U, V$, and that $U \cap V$ is pathwise connected. There are morphisms $\pi_{1}(U \cap V) \rightarrow \pi_{1}(U), \pi_{1}(U \cap V) \rightarrow \pi_{1}(V)$ induced by the inclusions.

Proposition 1.12. $\pi_{1}(X) \simeq \pi_{1}(U) *_{\pi_{1}(U \cap V)} \pi_{1}(V)$.
This is a simplified form of van Kampen's theorem, for a full statement see [6].
Example 1.13. We compute the fundamental group of a figure 8. Think of the figure 8 as the union of two circles $X$ in $\mathbb{R}^{2}$ which touch in one pount. Let $p_{1}, p_{2}$ be points in the two respective circles, different from the common point, and take $U=X-\left\{p_{1}\right\}$, $V=X-\left\{p_{2}\right\}$. Then $\pi_{1}(U) \simeq \pi_{1}(V) \simeq \mathbb{Z}$, while $U \cap V$ is simply connected. It follows that $\pi_{1}(X)$ is a free group with two generators. The two generators do not commute; this can also be checked "experimentally" if you think of winding a string along the

[^4]figure 8 in a proper way... More generally, the fundamental group of the corolla with $n$ petals ( $n$ copies of $S^{1}$ all touching in a single point) is a free group with $n$ generators.

Exercise 1.14. Prove that for any $n \geq 2$ the sphere $S^{n}$ is simply connected. Deduce that for $n \geq 3, \mathbb{R}^{n}$ minus a point is simply connected.

ExErcise 1.15. Compute the fundamental group of $\mathbb{R}^{2}$ with $n$ punctures.
4.6. Other ways to compute fundamental groups. Again, we state some results without proof.

Proposition 1.16. If $G$ is a simply connected topological group, and $H$ is a normal discrete subgroup, then $\pi_{1}(G / H) \simeq H$.

Since $S^{1} \simeq \mathbb{R} / \mathbb{Z}$, we have thus proved that

$$
\pi_{1}\left(S^{1}\right) \simeq \mathbb{Z}
$$

In the same way we compute the fundamental group of the $n$-dimensional torus

$$
T^{n}=S^{1} \times \cdots \times S^{1}(n \text { times }) \simeq \mathbb{R}^{n} / \mathbb{Z}^{n}
$$

obtaining $\pi_{1}\left(T^{n}\right) \simeq \mathbb{Z}^{n}$.
EXERCISE 1.17. Compute the fundamental group of a 2-dimensional punctured torus (a torus minus a point). Use van Kampen's theorem to compute the fundamental group of a Riemann surface of genus 2 (a compact, orientable, connected 2-dimensional differentiable manifold of genus 2, i.e., "with two handles"). Generalize your result to any genus.

ExERCISE 1.18. Prove that, given two pointed topological spaces $\left(X, x_{0}\right),\left(Y, y_{0}\right)$, then

$$
\pi_{1}\left(X \times Y,\left(x_{0}, y_{0}\right)\right) \simeq \pi_{1}\left(X, x_{0}\right) \times \pi_{1}\left(Y, y_{0}\right)
$$

This gives us another way to compute the fundamental group of the $n$-dimensional torus $T^{n}$ (once we know $\pi_{1}\left(S^{1}\right)$ ).

ExERCISE 1.19. Prove that the manifolds $S^{3}$ and $S^{2} \times S^{1}$ are not homeomorphic.
EXERCISE 1.20. Let $X$ be the space obtained by removing a line from $\mathbb{R}^{2}$, and a circle linking the line. Prove that $\pi_{1}(X) \simeq \mathbb{Z} \oplus \mathbb{Z}$. Prove the stronger result that $X$ is homotopic to the 2 -torus.

## CHAPTER 2

## Singular homology theory

## 1. Singular homology

In this Chapter we develop some elements of the homology theory of topological spaces. There are many different homology theories (simplicial, cellular, singular, ČechAlexander, ...) even though these theories coincide when the topological space they are applied to is reasonably well-behaved. Singular homology has the disadvantage of appearing quite abstract at a first contact, but in exchange of this we have the fact that it applies to any topological space, its functorial properties are evident, it requires very little combinatorial arguments, it relates to homotopy in a clear way, and once the basic properties of the theory have been proved, the computation of the homology groups is not difficult.
1.1. Definitions. The basic blocks of singular homology are the continuous maps from standard subspaces of Euclidean spaces to the topological space one considers. We shall denote by $P_{0}, P_{1}, \ldots, P_{n}$ the points in $\mathbb{R}^{n}$
$P_{0}=0, \quad P_{i}=(0,0, \ldots, 0,1,0, \ldots, 0) \quad$ (with just one 1 in the $i$ th position).
The convex hull of these points is denoted by $\Delta_{n}$ and is called the standard $n$-simplex. Alternatively, one can describe $\Delta_{k}$ as the set of points in $\mathbb{R}^{n}$ such that

$$
x_{i} \geq 0, \quad i=1, \ldots, n, \quad \sum_{i=1}^{n} x_{i} \leq 1
$$

The boundary of $\Delta_{n}$ is formed by $n+1$ faces $F_{n}^{i}(i=0,1, \ldots, n)$ which are images of the standard $(n-1)$-simplex by affine maps $\mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n}$. These faces may be labelled by the vertex of the simplex which is opposite to them: so, $F_{n}^{i}$ is the face opposite to $P_{i}$.

Given a topological space $X$, a singular $n$-simplex in $X$ is a continuous map $\sigma: \Delta_{n} \rightarrow$ $X$. The restriction of $\sigma$ to any of the faces of $\Delta_{n}$ defines a singular $(n-1)$-simplex $\sigma_{i}=\sigma_{\mid F_{n}^{i}}\left(\right.$ or $\sigma \circ F_{n}^{i}$ if we regard $F_{n}^{i}$ as a singular $(n-1)$-simplex $)$.

If $Q_{0}, \ldots, Q_{k}$ are $k+1$ points in $\mathbb{R}^{n}$, there is a unique affine map $\mathbb{R}^{k} \rightarrow \mathbb{R}^{n}$ mapping $P_{0}, \ldots, P_{k}$ to the $Q$ 's. This affine map yields a singular $k$-simplex in $\mathbb{R}^{n}$ that we denote $<Q_{0}, \ldots, Q_{k}>$. If $Q_{i}=P_{i}$ for $0 \leq i \leq k$, then the affine map is the identity on $\mathbb{R}^{k}$, and we denote the resulting singular $k$-simplex by $\delta_{k}$. The standard $n$-simplex $\Delta_{n}$ may so
also be denoted $\left\langle P_{0}, \ldots, P_{n}\right\rangle$, and the face $F_{n}^{i}$ of $\Delta_{n}$ is the singular ( $n-1$ )-simplex $<P_{0}, \ldots, \hat{P}_{i}, \ldots, P_{n}>$, where the hat denotes omission.

Choose now a commutative unital ring $R$. We denote by $S_{k}(X, R)$ the free group generated over $R$ by the singular $k$-simplexes in $X$. So an element in $S_{k}(X, R)$ is a "formal" finite linear combination (called a singular chain)

$$
\sigma=\sum_{j} a_{j} \sigma_{j}
$$

with $a_{j} \in R$, and the $\sigma_{j}$ are singular $k$-simplexes. Thus, $S_{k}(X, R)$ is an $R$-module, and, via the inclusion $\mathbb{Z} \rightarrow R$ given by the identity in $R$, an abelian group. For $k \geq 1$ we define a morphism $\partial: S_{k}(X, R) \rightarrow S_{k-1}(X, R)$ by letting

$$
\partial \sigma=\sum_{i=0}^{k}(-1)^{i} \sigma \circ F_{k}^{i}
$$

for a singular $k$-simplex $\sigma$ and exteding by $R$-linearity. For $k=0$ we define $\partial \sigma=0$.
Example 2.1. If $Q_{0}, \ldots, Q_{k}$ are $k+1$ points in $\mathbb{R}^{n}$, one has

$$
\partial<Q_{0}, \ldots, Q_{k}>=\sum_{i=0}^{k}(-1)^{i}<Q_{0}, \ldots, \hat{Q}_{i}, \ldots, Q_{k}>
$$

Proposition 2.2. $\partial^{2}=0$.
Proof. Let $\sigma$ be a singular $k$-simplex.

$$
\begin{aligned}
\partial^{2} \sigma & =\sum_{i=0}^{k}(-1)^{i} \partial\left(\sigma \circ F_{k}^{i}\right)=\sum_{i=0}^{k}(-1)^{i} \sum_{j=0}^{k-1}(-1)^{j} \sigma \circ F_{k}^{i} \circ F_{k-1}^{j} \\
& =\sum_{j<i=1}^{k}(-1)^{i+j} \sigma \circ F_{k}^{j} \circ F_{k-1}^{i-1}+\sum_{0=i \leq j}^{k-1}(-1)^{i+j} \sigma \circ F_{k}^{i} \circ F_{k-1}^{j}
\end{aligned}
$$

Resumming the first sum by letting $i=j, j=i-1$ the last two terms cancel.
So $\left(S_{\bullet}(X, R), \partial\right)$ is a (homology) graded differential module. Its homology groups $H_{k}(X, R)$ are the singular homology groups of $X$ with coefficients in $R$. We shall use the following notation and terminology:
$Z_{k}(X, R)=\operatorname{ker} \partial: S_{k}(X, R) \rightarrow S_{k-1}(X, R)$ (the module of $k$-cycles);
$B_{k}(X, R)=\operatorname{Im} \partial: S_{k+1}(X, R) \rightarrow S_{k}(X, R)$ (the module of $k$-boundaries);
therefore, $H_{k}(X, R)=Z_{k}(X, R) / B_{k}(X, R)$. Notice that $Z_{0}(X, R) \equiv S_{0}(X, R)$.

### 1.2. Basic properties.

Proposition 2.3. If $X$ is the union of pathwise connected components $X_{j}$, then $H_{k}(X, R) \simeq \oplus_{j} H_{k}\left(X_{j}, R\right)$ for all $k \geq 0$.

Proof. Any singular $k$-simplex must map $\Delta_{k}$ inside a pathwise connected components (if two points of $\Delta_{k}$ would map to points lying in different components, that would yield path connecting the two points).

Proposition 2.4. If $X$ is pathwise connected, then $H_{0}(X, R) \simeq R$.
Proof. This follows from the fact that a 0 -cycle $c=\sum_{j} a_{j} x_{j}$ is a boundary if and only if $\sum_{j} a_{j}=0$. Indeed, if $c$ is a boundary, then $c=\partial\left(\sum_{j} b_{j} \gamma_{j}\right)$ for some paths $\gamma_{j}$, so that $c=\sum_{j} b_{j}\left(\gamma_{j}(1)-\gamma_{j}(0)\right)$, and the coefficients sum up to zero. On the other hand, if $\sum_{j} a_{j}=0$, choose a base point $x_{0} \in X$. Then one can write

$$
c=\sum_{j} a_{j} x_{j}=\sum_{j} a_{j} x_{j}-\left(\sum_{j} a_{j}\right) x_{0}=\sum_{j} a_{j}\left(x_{j}-x_{0}\right)=\partial \sum_{j} a_{j} \gamma_{j}
$$

if $\gamma_{j}$ is a path joining $x_{0}$ to $x_{j}$.
This means that $B_{0}(X, R)$ is the kernel of the surjective map $Z_{0}(X, R)=S_{0}(X, R) \rightarrow$ $R$ given by $\sum_{j} a_{j} x_{j} \mapsto \sum_{j} a_{j}$, so that $H_{0}(X, R)=Z_{0}(X, R) / B_{0}(X, R) \simeq R$.

Let $f: X \rightarrow Y$ be a continuous map of topological spaces. If $\sigma$ is a singular $k$-simplex in $X$, then $f \circ \sigma$ is a singular $k$-simplex in $Y$. This yields a morphism $S_{k}(f): S_{k}(X, R) \rightarrow$ $S_{k}(Y, R)$ for every $k \geq 0$. It is immediate to prove that $S_{k}(f) \circ \partial=\partial \circ S_{k+1}(f)$ :

$$
S_{k}(f)(\partial \sigma)=f \circ \sum_{i=0}^{k+1}(-1)^{i} \sigma \circ F_{k+1}^{i}=\partial(f \circ \sigma)=\partial\left(S_{k}(f)(\sigma)\right) .
$$

This implies that $f$ induces a morphism $H_{k}(X, R) \rightarrow H_{k}(Y, R)$, that we denote $f_{b}$. It is also easy to check that, if $g: Y \rightarrow W$ is another continous map, then $S_{k}(g \circ f)=$ $S_{k}(g) \circ S_{k}(f)$, and $(g \circ f)_{b}=g_{b} \circ f_{b}$.

### 1.3. Homotopic invariance.

Proposition 2.5. If $f, g: X \rightarrow Y$ are homotopic map, the induced maps in homology coincide.

It should be by now clear that this yields as an immediate consequence the homotopic invariance of the singular homology.

Corollary 2.6. If two topological spaces are homotopically equivalent, their singular homologies are isomorphic.

To prove Proposition 2.5 we build, for every $k \geq 0$ and any topological space $X$, a morphism (called the prism operator) $P: S_{k}(X) \rightarrow S_{k+1}(X \times I)$ (here $I$ denotes again the unit closed interval in $\mathbb{R}$ ). We define the morphism $P$ in two steps.

Step 1. The first step consists in definining a singular $(k+1)$-chain $\pi_{k+1}$ in the topological space $\Delta_{k} \times I$ by subdiving the polyhedron $\Delta_{k} \times I \subset \mathbb{R}^{k+1}$ (a "prysm"


Figure 1. The prism $\pi_{2}$ over $\Delta_{1}$
over the standard symplesx $\Delta_{k}$ ) into a number of $\operatorname{singular}(k+1)$-simplexes, and summing them with suitable signs. The polyhedron $\Delta_{k} \times I \subset \mathbb{R}^{k+1}$ has $2(k+1)$ vertices $A_{0}, \ldots, A_{k}, B_{0}, \ldots, B_{k}$, given by $A_{i}=\left(P_{i}, 0\right), B_{i}=\left(P_{i}, 1\right)$. We define

$$
\pi_{k+1}=\sum_{i=0}^{k}(-1)^{i}<A_{0}, \ldots, A_{i}, B_{i}, \ldots, B_{k}>
$$

For instance, for $k=1$ we have

$$
\pi_{2}=<A_{0}, B_{0}, B_{1}>-<A_{0}, A_{1}, B_{1}>.
$$

Step 2. If $\sigma$ is a singular $k$-simplex in a topological space $X$, then $\sigma \times$ id is a continous map $\Delta_{k} \times I \rightarrow X \times I$. Therefore it makes sense to define the singular $(k+1)$-chain $P(\sigma)$ in $X$ as

$$
\begin{equation*}
P(\sigma)=S_{k+1}(\sigma \times \mathrm{id})\left(\pi_{k+1}\right) \tag{2.1}
\end{equation*}
$$

The definition of the prism operator implies its functoriality:
Proposition 2.7. If $f: X \rightarrow Y$ is a continuous map, the diagram

commutes.
Proof. It is just a matter of computation.

$$
\begin{aligned}
S_{k+1}(f \times \mathrm{id}) \circ P(\sigma) & =S_{k+1}(f \times \mathrm{id}) \circ S_{k+1}(\sigma \times \mathrm{id})\left(\pi_{k+1}\right) \\
& =S_{k+1}(f \circ \sigma \times \mathrm{id})\left(\pi_{k+1}\right)=P\left(S_{k}(f)\right) .
\end{aligned}
$$

The relevant property of the prism operator is proved in the next Lemma.

Lemma 2.8. Let $\lambda_{0}, \lambda_{i}: X \rightarrow X \times I$ be the maps $\lambda_{0}(x)=(x, 0), \lambda_{1}(x)=(x, 1)$. Then

$$
\begin{equation*}
\partial \circ P+P \circ \partial=S_{k}\left(\lambda_{1}\right)-S_{k}\left(\lambda_{0}\right) \tag{2.2}
\end{equation*}
$$

as maps $S_{k}(X) \rightarrow S_{k}(X \times I)$.
Proof. Let $\delta_{k}: \Delta_{k} \rightarrow \Delta_{k}$ be the identity map regarded as singular $k$-simplex in $\Delta_{k}$. Notice that $P\left(\delta_{k}\right)=\pi_{k+1}$.

We first check the identity (2.2) for $X=\Delta_{k}$, applying both sides of (2.2) to $\delta_{k}$. The right side yields

$$
<B_{0}, \ldots, B_{k}>-<A_{0}, \ldots A_{k}>
$$

We compute now the action of the left side of $(2.2)$ on $\delta_{k}$.

$$
\begin{aligned}
\partial P\left(\delta_{k}\right) & =\sum_{i=0}^{k}(-1)^{i} \partial<A_{0}, \ldots, A_{i}, B_{i}, \ldots, B_{k}> \\
& =\sum_{j \leq i=0}^{k}(-1)^{i+j}<A_{0}, \ldots, \hat{A}_{j}, \ldots A_{i}, B_{i}, \ldots, B_{k}> \\
& +\sum_{i \leq j=0}^{k}(-1)^{i+j+1}<A_{0}, \ldots A_{i}, B_{i}, \ldots, \hat{B}_{j}, \ldots B_{k}>
\end{aligned}
$$

All terms with $i=j$ cancel with the exception of $\left.\left\langle B_{0}, \ldots, B_{k}\right\rangle-<A_{0}, \ldots A_{k}\right\rangle$. So we have

$$
\begin{aligned}
\partial P\left(\delta_{k}\right) & =<B_{0}, \ldots, B_{k}>-<A_{0}, \ldots A_{k}> \\
& +\sum_{j<i=1}^{k}(-1)^{i+j}<A_{0}, \ldots, \hat{A}_{j}, \ldots A_{i}, B_{i}, \ldots, B_{k}> \\
& -\sum_{i<j=1}^{k}(-1)^{i+j}<A_{0}, \ldots A_{i}, B_{i}, \ldots, \hat{B}_{j}, \ldots B_{k}>.
\end{aligned}
$$

On the other hand, one has

$$
\partial \delta_{k}=\sum_{j=0}^{k}(-1)^{j}<P_{0}, \ldots, \hat{P}_{j}, \ldots, P_{k}>
$$

Since

$$
\begin{aligned}
P\left(<P_{0}, \ldots, \hat{P}_{j}, \ldots, P_{k}>\right) & =\sum_{i<j}(-1)^{i}<A_{0}, \ldots, A_{i}, B_{i}, \ldots, \hat{B}_{j}, \ldots, B_{k}> \\
& -\sum_{i>j}(-1)^{i}<A_{0}, \ldots, \hat{A}_{j}, \ldots, A_{i}, B_{i}, \ldots, B_{k}>
\end{aligned}
$$

we obtain the equation (2.2) (note that exchanging the indices $i, j$ changes the sign).

We must now prove that if equation (2.2) holds when both sides are applied to $\delta_{k}$, then it holds in general. One has indeed

$$
\begin{aligned}
\partial P(\sigma)= & \partial S_{k+1}(\sigma \times \mathrm{id})\left(P\left(\delta_{k}\right)\right)=S_{k}(\sigma \times \mathrm{id})\left(\partial P\left(\delta_{k}\right)\right) \\
P(\partial \sigma) & =P \partial\left(S_{k}(\sigma)\left(\delta_{k}\right)\right) \\
& =P\left(S_{k-1}(\sigma)\left(\partial \delta_{k}\right)\right)=S_{k}(\sigma \times \mathrm{id})\left(P\left(\partial \delta_{k}\right)\right)
\end{aligned}
$$

so that

$$
\begin{aligned}
\partial P(\sigma)+P(\partial \sigma) & \left.=S_{k+1}(\sigma \times \mathrm{id})\left(\partial P\left(\delta_{k}\right)\right)+P\left(\partial \delta_{k}\right)\right) \\
& =S_{k+1}(\sigma \times \mathrm{id})\left(S_{k}\left(\bar{\lambda}_{1}\right)-S_{k}\left(\bar{\lambda}_{0}\right)\right)=S_{k}\left(\lambda_{1}\right)-S_{k}\left(\lambda_{0}\right)
\end{aligned}
$$

where $\bar{\lambda}_{0}, \bar{\lambda}_{1}$ are the obvious maps $\Delta_{k} \rightarrow \Delta_{k} \times I$.
Equation (2.2) states that $P$ is a hotomopy (in the sense of homological algebra) between the maps $\lambda_{0}$ and $\lambda_{1}$, so that one has $\left(\lambda_{1}\right)_{b}=\left(\lambda_{2}\right)_{b}$ in homology.

Proof of Proposition 2.5. Let $F$ be a hotomopy between the maps $f$ and $g$. Then, $f=F \circ \lambda_{0}, g=F \circ \lambda_{1}$, so that

$$
f_{b}=\left(F \circ \lambda_{0}\right)_{b}=F_{b} \circ\left(\lambda_{0}\right)_{b}=F_{b} \circ\left(\lambda_{1}\right)_{b}=\left(F \circ \lambda_{1}\right)_{b}=g_{b} .
$$

Corollary 2.9. If $X$ is a contractible space then

$$
H_{0}(X, R) \simeq R, \quad H_{k}(X, R)=0 \quad \text { for } \quad k>0 .
$$

1.4. Relation between the first fundamental group and homology. A loop $\gamma$ in $X$ may be regarded as a closed singular 1 -simplex. If we fix a point $x_{0} \in X$, we have a set-theoretic map $\chi: \mathcal{L}\left(x_{0}\right) \rightarrow S_{1}(X, \mathbb{Z})$. The following result tells us that $\chi$ descends to a group homomorphism $\chi: \pi_{1}\left(X, x_{0}\right) \rightarrow H_{1}(X, \mathbb{Z})$.

Proposition 2.10. If two loops $\gamma_{1}, \gamma_{2}$ are homotopic, then they are homologous as singular 1-simplexes. Moreover, given two loops at $x_{0}, \gamma_{1}, \gamma_{2}$, then $\chi\left(\gamma_{2} \circ \gamma_{1}\right)=$ $\chi\left(\gamma_{1}\right)+\chi\left(\gamma_{2}\right)$ in $H_{1}(X, \mathbb{Z})$.

Proof. Choose a homotopy with fixed endpoints between $\gamma_{1}$ and $\gamma_{2}$, i.e., a map $\Gamma: I \times I \rightarrow X$ such that

$$
\Gamma(t, 0)=\gamma_{1}(t), \quad \Gamma(t, 1)=\gamma_{2}(t), \quad \Gamma(0, s)=\Gamma(1, s)=x_{0} \text { for all } s \in I
$$

Define the loops $\gamma_{3}(t)=\Gamma(1, t), \gamma_{4}(t)=\Gamma(0, t), \gamma_{5}(t)=\Gamma(t, t)$. Both loops $\gamma_{3}$ and $\gamma_{4}$ are actually the constant loop at $x_{0}$. Consider the points $P_{0}, P_{1}, P_{2}, Q=(1,1)$ in $\mathbb{R}^{2}$, and define the singular 2-simplex

$$
\sigma=\Gamma \circ<P_{0}, P_{1}, Q>-\Gamma \circ<P_{0}, P_{2}, Q>
$$



Figure 2
(cf. Figure 2). We then have

$$
\begin{aligned}
\partial \sigma & =\Gamma \circ<P_{1}, Q>-\Gamma \circ<P_{0}, Q>+\Gamma \circ<P_{0}, P_{1}> \\
& -\Gamma \circ<P_{2}, Q>+\Gamma \circ<P_{0}, Q>-\Gamma \circ<P_{0}, P_{2}> \\
& =\gamma_{3}-\gamma_{5}+\gamma_{1}-\gamma_{2}+\gamma_{5}+\gamma_{4}=\gamma_{1}-\gamma_{2} .
\end{aligned}
$$

This proves that $\chi\left(\gamma_{1}\right)$ and $\chi\left(\gamma_{2}\right)$ are homologous. To prove the second claim we need to define a singular 2-simplex $\sigma$ such that

$$
\partial \sigma=\gamma_{1}+\gamma_{2}-\gamma_{2} \cdot \gamma_{1}
$$

Consider the point $T=\left(0, \frac{1}{2}\right)$ in the standard 2 -simplex $\Delta_{2}$ and the segment $\Sigma$ joining $T$ with $P_{1}$ (cf. Figure 3). If $Q \in \Delta_{2}$ lies on or below $\Sigma$, consider the line joining $P_{0}$ with $Q$, parametrize it with a parameter $t$ such that $t=0$ in $P_{0}$ and $t=1$ in the intersection of the line with $\Sigma$, and set $\sigma(Q)=\gamma_{1}(t)$. Analogously, if $Q$ lies above or on $\Sigma$, consider the line joining $P_{2}$ with $Q$, parametrize it with a parameter $t$ such that $t=1$ in $P_{2}$ and $t=0$ in the intersection of the line with $\Sigma$, and set $\sigma(Q)=\gamma_{2}(t)$. This defines a singular 2-simplex $\sigma: \Delta_{2} \rightarrow \mathrm{X}$, and one has

$$
\begin{aligned}
\partial \sigma & =\sigma \circ<P_{1}, P_{2}>-\sigma \circ<P_{0}, P_{2}>+\sigma \circ<P_{0}, P_{1}> \\
& =\gamma_{2}-\gamma_{2} \cdot \gamma_{1}+\gamma_{1}
\end{aligned}
$$

We recall from basic group theory the notion of commutator subgroup. Let $G$ be any group, and let $C(G)$ be the subgroup generated by elements of the form $g h g^{-1} h^{-1}$, $g, h \in G$. The subgroup $C(G)$ is obviously normal in $G$; the quotient group $G / C(G)$ is abelian. We call it the abelianization of $G$. It turns out that the first homology group of a space with integer coefficients is the abelianization of the fundamental group.

Proposition 2.11. If $X$ is pathwise connected, the morphism $\chi: \pi_{1}\left(X, x_{0}\right) \rightarrow$ $H_{1}(X, \mathbb{Z})$ is surjective, and its kernel is the commutator subgroup of $\pi_{1}\left(X, x_{0}\right)$.


Figure 3
Proof. Let $c=\sum_{j} a_{j} \sigma_{j}$ be a 1 -cycle. So we have

$$
0=\partial c=\sum_{j} a_{i}\left(\sigma_{j}(1)-\sigma_{j}(0)\right) .
$$

In this linear combination of points with coefficients in $\mathbb{Z}$ some of the points may coincide; the sum of the coefficients corresponding to the same point must vanish. Choose a base point $x_{0} \in X$ and for every $j$ choose a path $\alpha_{j}$ from $x_{0}$ to $\sigma_{j}(0)$ and a path $\beta_{j}$ from $x_{0}$ to $\sigma_{j}(1)$, in such a way that they depend on the endpoints and not on the indexing (e.g, if $\sigma_{j}(0)=\sigma_{k}(0)$, choose $\alpha_{j}=\alpha_{k}$ ). Then we have

$$
\sum_{j} a_{j}\left(\beta_{j}-\alpha_{j}\right)=0 .
$$

Now if we set $\bar{\sigma}_{j}=\alpha_{j}+\sigma_{j}-\beta_{j}$ we have $c=\sum_{j} a_{j} \bar{\sigma}_{j}$. Let $\gamma_{j}$ be the loop $\beta^{-1} \cdot \sigma_{j} \cdot \alpha$; then,

$$
\chi\left(\left[\Pi_{j} \gamma_{j}^{a_{j}}\right]\right)=[c]
$$

so that $\chi$ is surjective.
To prove the second claim we need to show that the commutator subgroup of $\pi_{1}\left(X, x_{0}\right)$ coincides with $\operatorname{ker} \chi$. We first notice that since $H_{1}(X, \mathbb{Z})$ is abelian, the commutator subgroup is necessarily contained in ker $\chi$. To prove the opposite inclusion, let $\gamma$ be a loop that in homology is a 1-boundary, i.e., $\gamma=\partial \sum_{j} a_{j} \sigma_{j}$. So we may write

$$
\begin{equation*}
\sigma_{j}=\gamma_{0 j}-\gamma_{1 j}+\gamma_{2 j} \tag{2.3}
\end{equation*}
$$

for some paths $\gamma_{k j}, k=0,1,2$. Choose paths (cf. Figure 4)

$$
\begin{aligned}
& \alpha_{0 j} \text { from } x_{0} \text { to } \gamma_{1 j}(0)=\gamma_{2 j}(0)=P_{0} \\
& \alpha_{1 j} \text { from } x_{0} \text { to } \gamma_{2 j}(1)=\gamma_{0 j}(0)=P_{1} \\
& \alpha_{2 j} \text { from } x_{0} \text { to } \gamma_{1 j}(1)=\gamma_{0 j}(1)=P_{2}
\end{aligned}
$$

and consider the loops

$$
\beta_{0 j}=\alpha_{0 j}^{-1} \cdot \gamma_{1 j}^{-1} \cdot \alpha_{2 j}, \quad \beta_{1 j}=\alpha_{2 j}^{-1} \cdot \gamma_{0 j} \cdot \alpha_{1 j}, \quad \beta_{2 j}=\alpha_{1 j}^{-1} \cdot \gamma_{2 j} \cdot \alpha_{0 j} .
$$



Figure 4
Note that the loops

$$
\beta_{j}=\beta_{0 j} \cdot \beta_{1 j} \cdot \beta_{2 j}=\alpha_{0 j}^{-1} \cdot \gamma_{1 j}^{-1} \cdot \gamma_{0 j} \cdot \gamma_{2 j} \cdot \alpha_{0 j}
$$

are homotopic to the constant loop at $x_{0}$ (since the image of a singular 2-simplex is contractible). As a consequence one has the equality in $\pi_{1}\left(X, x_{0}\right)$

$$
\Pi_{j}\left[\beta_{j}\right]^{a_{j}}=e .
$$

This implies that the image of $\Pi_{j}\left[\beta_{j}\right]^{a_{j}}$ in $\pi_{1}\left(X, x_{0}\right) / C\left(\pi_{1}\left(X, x_{0}\right)\right)$ is the identity. On the other hand from (2.3) we see that $\gamma$ coincides, up to reordering of terms, with $\Pi_{j} \beta_{j}^{a_{j}}$, so that the image of the class of $\gamma$ in $\pi_{1}\left(X, x_{0}\right) / C\left(\pi_{1}\left(X, x_{0}\right)\right)$ is the identity as well. This means that $\gamma$ lies in the commutator subgroup.

So whenever in the examples in Chapter 1 the fundamental groups we computed turned out to be abelian, we were also computing the group $H_{1}(X, \mathbb{Z})$. In particular,

Corollary 2.12. $H_{1}(X, \mathbb{Z})=0$ if $X$ is simply connected.
Exercise 2.13. Compute $H_{1}(X, \mathbb{Z})$ when $X$ is: 1 . the corolla with $n$ petals, 2. $\mathbb{R}^{n}$ minus a point, 3. the circle $S^{1}, 4$. the torus $T^{2}, 5$. a punctured torus, 6. a Riemann surface of genus $g$.

## 2. Relative homology

2.1. The relative homology complex. Given a topological space $X$, let $A$ be any subspace (that we consider with the relative topology). We fix a coefficient ring $R$ which for the sake of conciseness shall be dropped from the notation. For every $k \geq 0$ there is a natural inclusion (injective morphism of $R$-modules) $S_{k}(A) \subset S_{k}(X)$; the homology operators of the complexes $S_{\bullet}(A), S_{\bullet}(X)$ define a morphism $\delta: S_{k}(X) / S_{k}(A) \rightarrow$ $S_{k-1}(X) / S_{k-1}(A)$ which squares to zero. If we define

$$
Z_{k}^{\prime}(X, A)=\operatorname{ker} \partial: \frac{S_{k}(X)}{S_{k}(A)} \rightarrow \frac{S_{k-1}(X)}{S_{k-1}(A)}
$$

$$
B_{k}^{\prime}(X, A)=\operatorname{Im} \partial: \frac{S_{k+1}(X)}{S_{k+1}(A)} \rightarrow \frac{S_{k}(X)}{S_{k}(A)}
$$

we have $B_{k}^{\prime}(X, A) \subset Z_{k}^{\prime}(X, A)$.
Definition 2.1. The homology groups of $X$ relative to $A$ are the $R$-modules $H_{k}(X, A)=Z_{k}^{\prime}(X, A) / B_{k}^{\prime}(X, A)$. When we want to emphasize the choice of the ring $R$ we write $S_{k}(X, A ; R)$.

The relative homology is more conveniently defined in a slightly different way, which makes clearer its geometrical meaning. It will be useful to consider the following diagram


Let

$$
\begin{gathered}
Z_{k}(X, A)=\left\{c \in S_{k}(X) \mid \partial c \in S_{k-1}(A)\right\} \\
B_{k}(X, A)=\left\{c \in S_{k}(X) \mid c=\partial b+c^{\prime} \text { with } b \in S_{k+1}(X), c^{\prime} \in S_{k}(A)\right\}
\end{gathered}
$$

Thus, $Z_{k}(X, A)$ is formed by the chains whose boundary is in $A$, and $B_{k}(A)$ by the chains that are boundaries up to chains in $A$.

LEmma 2.2. $Z_{k}(X, A)$ is the pre-image of $Z_{k}^{\prime}(X, A)$ under the quotient homomorphism $q_{k}$; that is, an element $c \in S_{k}(X)$ is in $Z_{k}(X, A)$ if and only if $q_{k}(c) \in Z_{k}^{\prime}(X, A)$.

Proof. If $q_{k}(c) \in Z_{k}^{\prime}(X, A)$ then $0=\partial \circ q_{k}(c)=q_{k-1} \circ \partial(c)$ so that $c \in Z_{k}(X, A)$. If $c \in Z_{k}(X, A)$ then $q_{k-1} \circ \partial(c)=0$ so that $q_{k}(c) \in Z_{k}^{\prime}(X, A)$.

Lemma 2.3. $c \in S_{k}(X)$ is in $B_{k}(X, A)$ if and only if $q_{k}(c) \in B_{k}^{\prime}(X, A)$.

Proof. If $c=\partial b+c^{\prime}$ with $b \in S_{k+1}(X)$ and $c^{\prime} \in S_{k}(A)$ then $q_{k}(c)=q_{k} \circ \partial b=$ $\partial \circ q_{k+1}(b) \in B_{k}^{\prime}(X, A)$. Conversely, if $q_{k}(c) \in B_{k}^{\prime}(X, A)$ then $q_{k}(c)=\partial \circ q_{k+1}(b)$ for some $b \in S_{k+1}(X)$, then $c-\partial b \in \operatorname{ker} q_{k-1}$ so that $c=\partial b+c^{\prime}$ with $c^{\prime} \in S_{k}(A)$.

Proposition 2.4. For all $k \geq 0, H_{k}(X, A) \simeq Z_{k}(X, A) / B_{k}(X, A)$.

Proof. What we should do is to prove the commutativity and the exactness of the rows of the diagram


Commutativity is obvious. For the exactness of the first row, it is obvious that $S_{k}(A) \subset$ $B_{k}(X, A)$ and that $q_{k}(c)=0$ if $c \in S_{k}(A)$. On the other hand if $c \in B_{k}(X, A)$ we have $c=\partial b+c^{\prime}$ with $b \in S_{k+1}(X)$ and $c^{\prime} \in S_{k}(A)$, so that $q_{k}(c)=0$ implies $0=q_{k} \circ \partial b=$ $\partial \circ q_{k+1}(b)$, which in turn implies $c \in S_{k}(A)$. To prove the surjectivity of $q_{k}$, just notice that by definition an element in $B_{k}^{\prime}(X, A)$ may be represented as $\partial b$ with $b \in S_{k+1}(X)$.

As for the second row, we have $S_{k}(A) \subset Z_{k}(X, A)$ from the definition of $Z_{k}(X, A)$. If $c \in S_{k}(A)$ then $q_{k}(c)=0$. If $c \in Z_{k}(X, A)$ and $q_{k}(c)=0$ then $c \in S_{k}(A)$ by the definition of $Z_{k}^{\prime}(X, A)$. Moreover $q_{k}$ is surjective by Lemma 2.2.
2.2. Main properties of relative homology. We list here the main properties of the cohomology groups $H_{k}(X, A)$. If a proof is not given the reader should provide one by her/himself.

- If $A$ is empty, $H_{k}(X, A) \simeq H_{k}(X)$.
- The relative cohomology groups are functorial in the following sense. Given topological spaces $X, Y$ with subsets $A \subset X, B \subset Y$, a continous map of pairs is a continuous map $f: X \rightarrow Y$ such that $f(A) \subset B$. Such a map induces in natural way a morphisms of $R$-modules $f_{b}: H_{\bullet}(X, A) \rightarrow H_{\bullet}(Y, B)$. If we consider the inclusion of pairs $(X, \emptyset) \hookrightarrow(X, A)$ we obtain a morphism $H_{\bullet}(X) \rightarrow_{\bullet} H(X, A)$.
- The inclusion map $i: A \hookrightarrow X$ induces a morphism $H_{\bullet}(A) \rightarrow H_{\bullet}(X)$ and the composition $H_{\bullet}(A) \rightarrow H_{\bullet}(X) \rightarrow H_{\bullet}(X, A)$ vanishes (since $\left.Z_{k}(A) \subset B_{k}(X, A)\right)$.
- If $X=\cup_{j} X_{j}$ is a union of pathwise connected components, then $H_{k}(X, A) \simeq$ $\oplus_{j} H_{k}\left(X_{j}, A_{j}\right)$ where $A_{j}=A \cap X_{j}$.

Proposition 2.5. If $X$ is pathwise connected and $A$ is nonempty, then $H_{0}(X, A)$ $=0$.

Proof. If $c=\sum_{j} a_{j} x_{j} \in S_{0}(X)$ and $\gamma_{j}$ is a path from $x_{0} \in A$ to $x_{j}$, then $\partial\left(\sum_{j} a_{j} x_{j}\right)=c-\left(\sum_{j} a_{j}\right) x_{0}$ so that $c \in B_{0}(X, A)$.

Corollary 2.6. $H_{0}(X, A)$ is a free $R$-module generated by the components of $X$ that do not meet $A$.

Indeed $H_{j}\left(X_{j}, A_{j}\right)=0$ if $A_{j}$ is empty.
Proposition 2.7. If $A=\left\{x_{0}\right\}$ is a point, $H_{k}(X, A) \simeq H_{k}(X)$ for $k>0$.

Proof.

$$
\begin{aligned}
Z_{k}(X, A) & =\left\{c \in S_{k}(X) \mid \partial c \in S_{k-1}(A)\right\}=Z_{k}(X) \text { when } k>0 \\
B_{k}(X, A) & =\left\{c \in S_{k}(X) \mid c=\partial b+c^{\prime} \text { with } b \in S_{k+1}(X), c^{\prime} \in S_{k}(A)\right\} \\
& =B_{k}(X) \text { when } k>0 .
\end{aligned}
$$

2.3. The long exact sequence of relative homology. By definition the relative homology of $X$ with respect to $A$ is the homology of the quotient complex $S_{\bullet}(X) / S_{\bullet}(A)$. By Proposition 1.6, adapted to homology by reversing the arrows, one obtains a long exact cohomology sequence

$$
\begin{aligned}
& \cdots \rightarrow H_{2}(A) \rightarrow H_{2}(X) \rightarrow H_{2}(X, A) \\
& \rightarrow H_{1}(A) \rightarrow H_{1}(X)
\end{aligned} \rightarrow H_{1}(X, A) \quad \begin{aligned}
& \rightarrow H_{0}(A) \rightarrow H_{0}(X) \rightarrow H_{0}(X, A) \rightarrow 0
\end{aligned}
$$

Exercise 2.8. Assume to know that $H_{1}\left(S^{1}, R\right) \simeq R$ and $H_{k}\left(S^{1}, R\right)=0$ for $k>$ 1. Use the long relative homology sequence to compute the relative homology groups $H \bullet\left(\mathbb{R}^{2}, S_{1} ; R\right)$.

## 3. The Mayer-Vietoris sequence

The Mayer-Vietoris sequence (in its simplest form, that we are going to consider here) allows one to compute the homology of a union $X=U \cup V$ from the knowledge of the homology of $U, V$ and $U \cap V$. This is quite similar to what happens in de Rham cohomology, but in the case of homology there is a subtlety. Let us denote $A=U \cap V$. One would think that there is an exact sequence

$$
0 \rightarrow S_{k}(A) \xrightarrow{i} S_{k}(U) \oplus S_{k}(V) \xrightarrow{p} S_{k}(X) \rightarrow 0
$$

where $i$ is the morphism induced by the inclusions $A \hookrightarrow U, A \hookrightarrow V$, and $p$ is given by $p\left(\sigma_{1}, \sigma_{2}\right)=\sigma_{1}-\sigma_{2}$ (again using the inclusions $U \hookrightarrow X, V \hookrightarrow X$ ). However, it is not possible to prove that $p$ is surjective (if $\sigma$ is a singular $k$-simplex whose image is not contained in $U$ or $V$, it is not in general possible to write it as a difference of standard $k$-simplexes in $U, V)$. The trick to circumvent this difficulty consists in replacing $S_{\bullet}(X)$ with a different complex that however has the same homology.

Let $\mathfrak{U}=\left\{U_{\alpha}\right\}$ be an open cover of $X$.
Definition 2.1. A singular $k$-chain $\sigma=\sum_{j} a_{j} \sigma_{j}$ is $\mathfrak{U}$-small if every singular $k$ simplex $\sigma_{j}$ maps into an open set $U_{\alpha} \in \mathfrak{U}$ for some $\alpha$. Moreover we define $S_{\bullet}^{\mathfrak{U}}(X)$ as the subcomplex of $S_{\bullet}(X)$ formed by $\mathfrak{U}$-small chains. ${ }^{1}$

The homology differential $\partial$ restricts to $S_{\bullet}^{\mathfrak{U}}(X)$, so that one has a homology $H_{\bullet}^{\mathfrak{U}}(X)$.

[^5]

Figure 5. The join $B\left(<E_{0}, E_{1}>\right)$
Proposition 2.2. $H_{\bullet}^{\mathfrak{U}}(X) \simeq H_{\bullet}(X)$.
To prove this isomorphism we shall build a homotopy between the complexes $S_{\bullet}^{\mathfrak{U}}(X)$ and $S_{\bullet}(X)$. This will be done in several steps.

Given a singular $k$-simplex $<Q_{0}, \ldots, Q_{k}>$ in $\mathbb{R}^{n}$ and a point $B \in \mathbb{R}^{n}$ we consider the singular simplex $<B, Q_{0}, \ldots, Q_{k}>$, called the join of $B$ with $<Q_{0}, \ldots, Q_{k}>$. This operator $B$ is then extended to singular chains in $\mathbb{R}^{n}$ by linearity. The following Lemma is easily proved.

Lemma 2.3. $\partial \circ B+B \circ \partial=\mathrm{Id}$ on $S_{k}\left(\mathbb{R}^{n}\right)$ if $k>0$, while $\partial \circ B(\sigma)=\sigma-\left(\sum_{j} a_{j}\right) B$ if $\sigma=\sum_{j} a_{j} x_{j} \in S_{0}\left(\mathbb{R}^{n}\right)$.

Next we define operators $\Sigma: S_{k}(X) \rightarrow S_{k}(X)$ and $T: S_{k}(X) \rightarrow S_{k+1}(X)$. The operator $\Sigma$ is called the subdivision operator and its effect is that of subdividing a singular simplex into a linear combination of "smaller" simplexes. The operators $\Sigma$ and $T$, analogously to what we did for the prism operator, will be defined for $X=\Delta_{k}$ (the space consisting of the standard $k$-simplex) and for the "identity" singular simplex $\delta_{k}: \Delta_{k} \rightarrow \Delta_{k}$, and then extended by functoriality. This should be done for all $k$. One defines

$$
\Sigma\left(\delta_{0}\right)=\delta_{0}, \quad T\left(\delta_{0}\right)=0
$$

and then extends recursively to positive $k$ :

$$
\Sigma\left(\delta_{k}\right)=B_{k}\left(\Sigma\left(\partial \delta_{k}\right)\right), \quad T\left(\delta_{k}\right)=B_{k}\left(\delta_{k}-\Sigma\left(\delta_{k}\right)-T\left(\partial \delta_{k}\right)\right)
$$

where the point $B_{k}$ is the barycenter of the standard $k$-simplex $\Delta_{k}$,

$$
B_{k}=\frac{1}{k+1} \sum_{j=0}^{k} P_{j}
$$

Example 2.4. For $k=1$ one gets $\Sigma\left(\delta_{1}\right)=<B_{1} P_{1}>-<B_{1} P_{0}>$; for $k=2$, the action of $\Sigma$ splits $\Delta_{2}$ into smaller simplexes as shown in Figure 6.


Figure 6. The subdivision operator $\Sigma$ splits $\Delta_{2}$ into the chain $<B_{2}, M_{0}, P_{2}>-<B_{2}, M_{0}, P_{1}>-<B_{2}, M_{1}, P_{2}>+<B_{2}, M_{1}, P_{0}>$ $+\left\langle B_{2}, M_{2}, P_{1}>-<B_{2}, M_{2}, P_{0}\right\rangle$

The definition of $\Sigma$ and $T$ for every topological space and every singular $k$-simplex $\sigma$ in $X$ is

$$
\Sigma(\sigma)=S_{k}(\sigma)\left(\Sigma\left(\delta_{k}\right)\right), \quad T(\sigma)=S_{k+1}(\sigma)\left(T\left(\delta_{k}\right)\right) .
$$

Lemma 2.5. One has the identities

$$
\partial \circ \Sigma=\Sigma \circ \partial, \quad \partial \circ T+T \circ \partial=\operatorname{Id}-\Sigma .
$$

Proof. These identities are proved by direct computation (it is enough to consider the case $X=\Delta_{k}$ ).

The first identity tells us that $\Sigma$ is a morphism of differential complexes, and the second that $T$ is a homotopy between $\Sigma$ and Id, so that the morphism $\Sigma_{\mathrm{b}}$ induced in homology by $\Sigma$ is an isomorphism.

The diameter of a singular $k$-simplex $\sigma$ in $\mathbb{R}^{n}$ is the maximum of the lengths of the segments contained in $\sigma$. The proof of the following Lemma is an elementary computation.

Lemma 2.6. Let $\sigma=<E_{0}, \ldots, E_{k}>$, with $E_{0}, \ldots, E_{k} \in \mathbb{R}^{n}$. The diameter of every simplex in the singular chain $\Sigma(\sigma) \in S_{k}\left(\mathbb{R}^{n}\right)$ is at most $k / k+1$ times the diameter of $\sigma$.

Proposition 2.7. Let $X$ be a topological space, $\mathfrak{U}=\left\{U_{\alpha}\right\}$ an open cover, and $\sigma$ a singular $k$-simplex in $X$. There is a natural number $r>0$ such that every singular simplex in $\Sigma^{r}(\sigma)$ is contained in a open set $U_{\alpha}$.

Proof. As $\Delta_{k}$ is compact there is a real positive number $\epsilon$ such that $\sigma$ maps a neighbourhood of radius $\epsilon$ of every point of $\Delta_{k}$ into some $U_{\alpha}$. Since

$$
\lim _{r \rightarrow+\infty} \frac{k^{r}}{(k+1)^{r}}=0
$$

there is an $r>0$ such that $\Sigma^{r}\left(\delta_{k}\right)$ is a linear combination of simplexes whose diameter is less than $\epsilon$. But as $\Sigma^{r}(\sigma)=S_{k}(\sigma)\left(\Sigma^{r}\left(\delta_{k}\right)\right)$ we are done.

This completes the proof of Proposition 2.2. We may now prove the exactness of the Mayer-Vietoris sequence in the following sense. If $X=U \cup V$ (union of two open subsets), let $\mathfrak{U}=\{U, V\}$ and $A=U \cap V$.

Proposition 2.8. For every $k$ there is an exact sequence of $R$-modules

$$
0 \rightarrow S_{k}(A) \xrightarrow{i} S_{k}(U) \oplus S_{k}(V) \xrightarrow{p} S_{k}^{\mathfrak{U}}(X) \rightarrow 0
$$

Proof. One has a diagram of inclusions


Defining $i(\sigma)=\left(\ell_{U} \circ \sigma,-\ell_{V} \circ \sigma\right)$ and $p\left(\sigma_{1}, \sigma_{2}\right)=j_{U} \circ \sigma_{1}+j_{V} \circ \sigma_{2}$, the exactness of the Mayer-Vietoris sequence is easily proved.

The morphisms $i$ and $p$ commute with the homology operator $\partial$, so that one obtains a long homology exact sequence involving the homologies $H_{\bullet}(A), H_{\bullet}(V) \oplus H_{\bullet}(V)$ and $H_{\bullet}^{\mathfrak{U}}(X)$. But in view of Proposition 2.2 we may replace $H_{\bullet}^{\mathfrak{U}}(X)$ with the homology $H_{\bullet}(X)$, so that we obtain the exact sequence

$$
\begin{aligned}
& \cdots \rightarrow H_{2}(A) \rightarrow H_{2}(U) \oplus H_{2}(V) \rightarrow H_{2}(X) \\
& \rightarrow H_{1}(A) \rightarrow H_{1}(U) \oplus H_{1}(V) \rightarrow H_{1}(X) \\
& \rightarrow H_{0}(A) \rightarrow H_{0}(U) \oplus H_{0}(V) \rightarrow H_{0}(X) \rightarrow 0
\end{aligned}
$$

Exercise 2.9. Prove that for any ring $R$ the homology of the sphere $S^{n}$ with coefficients in $R, n \geq 2$, is

$$
H_{k}\left(S^{n}, R\right)= \begin{cases}R & \text { for } \quad k=0 \text { and } k=n \\ 0 & \text { for } \quad 0<k<n \text { and } k>n\end{cases}
$$

ExERCISE 2.10. Show that the relative homology of $S^{2} \bmod S^{1}$ with coefficients in $\mathbb{Z}$ is concentrated in degree 2 , and $H_{2}\left(S^{2}, S^{1}\right) \simeq \mathbb{Z} \oplus \mathbb{Z}$.

ExERCISE 2.11. Use the Mayer-Vietoris sequence to compute the homology of a cylinder $S^{1} \times \mathbb{R}$ minus a point with coefficients in $\mathbb{Z}$. (Hint: since the cylinder is homotopic to $S^{1}$, it has the same homology). The result is (calling $X$ the space)

$$
H_{0}(X, Z) \simeq \mathbb{Z}, \quad H_{1}(X, \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}, \quad H_{2}(X, \mathbb{Z})=0
$$

Compare this with the homology of $S^{2}$ minus three points.

## 4. Excision

If a space $X$ is the union of subspaces, the Mayer-Vietoris suquence allows one to compute the homology of $X$ from the homology of the subspaces and of their intersections. The operation of excision in some sense gives us information about the reverse operation, i.e., it tells us what happen to the homology of a space if we "excise" a subpace out of it. Let us recall that given a map $f:(X, A) \rightarrow(Y, B)$ (i.e., a map $f: X \rightarrow Y$ such that $f(A) \subset B)$ there is natural morphism $f_{b}: H_{\bullet}(X, A) \rightarrow H_{\bullet}(Y, B)$.

Definition 2.1. Given nested subspaces $U \subset A \subset X$, the inclusion map $(X-U, A-$ $U) \rightarrow(X, A)$ is said to be an excision if the induced morphism $H_{k}(X-U, A-U) \rightarrow$ $H_{k}(X, A)$ is an isomorphism for all $k$.

If $(X-U, A-U) \rightarrow(X, A)$ is an excision, we say that $U$ "can be excised."
To state the main theorem about excision we need some definitions from topology.
Definition 2.2. 1. Let $i: A \rightarrow X$ be an inclusion of topological spaces. A map $r: X \rightarrow A$ is a retraction of $i$ if $r \circ i=\operatorname{Id}_{A}$.
2. A subspace $A \subset X$ is a deformation retract of $X$ if $\operatorname{Id}_{X}$ is homotopically equivalent to $i \circ r$, where $r: X \rightarrow A$ is a retraction.

If $r: X \rightarrow A$ is a retraction of $i: A \rightarrow X$, then $r_{b} \circ i_{b}=\operatorname{Id}_{H_{\bullet}(A)}$, so that $i_{b}: H_{\bullet}(A) \rightarrow$ $H_{\bullet}(X)$ is injective. Moreover, if $A$ is a deformation retract of $X$, then $H_{\bullet}(A) \simeq H_{\bullet}(X)$. The same notion can be given for inclusions of pairs, $(A, B) \hookrightarrow(X, Y)$; if such a map is a deformation retract, then $H_{\bullet}(A, B) \simeq H_{\bullet}(X, Y)$.

Exercise 2.3. Show that no retraction $S^{n} \rightarrow S^{n-1}$ can exist.
Theorem 2.4. If the closure $\bar{U}$ of $U$ lies in the interior $\operatorname{int}(A)$ of $A$, then $U$ can be excised.

Proof. We consider the cover $\mathfrak{U}=\{X-\bar{U}, \operatorname{int}(A)\}$ of $X$. Let $c=\sum_{j} a_{j} \sigma_{j} \in$ $Z_{k}(X, A)$, so that $\partial c \in S_{k-1}(A)$. In view of Proposition 2.2 we may assume that $c$ is $\mathfrak{U}$ small. If we cancel from $\sigma$ those singular simplexes $\sigma_{j}$ taking values in $\operatorname{int}(A)$, the class $[c] \in H_{k}(X, A)$ is unchanged. Therefore, after the removal, we can regard $c$ as a relative cycle in $X-U \bmod A-U$; this implies that the morphism $H_{k}(X-U, A-U) \rightarrow H_{k}(X, A)$ is surjective.

To prove that it is injective, let $[c] \in H_{k}(X-U, A-U)$ be such that, regarding $c$ as a cycle in $X \bmod A$, it is a boundary, i.e., $c \in B_{k}(X, A)$. This means that

$$
c=\partial b+c^{\prime} \quad \text { with } \quad b \in S_{k+1}(X), c^{\prime} \in S_{k}(A) .
$$

We apply the operator $\Sigma^{r}$ to both sides of this inequality, and split $\Sigma^{r}(b)$ into $b_{1}+b_{2}$, where $b_{1}$ maps into $X-\bar{U}$ and $b_{2}$ into $\operatorname{int}(A)$. We have

$$
\Sigma^{r}(c)-\partial b_{1}=\Sigma^{r}\left(c^{\prime}\right)+\partial b_{2} .
$$

The chain in the left side is in $X-U$ while the chain in the right side is in $A$; therefore, both chains are in $(X-U) \cap A=A-U$. Now we have

$$
\Sigma^{r}(c)=\Sigma^{r}\left(c^{\prime}\right)+\partial b_{2}+\partial b_{1}
$$

with $\Sigma^{r}\left(c^{\prime}\right)+\partial b_{2} \in S_{k}(A-U)$ and $\partial b_{1} \in S_{k+1}(X-U)$ so that $\Sigma^{r}(c) \in B_{k}(X-U, A-U)$, which implies $[c]=0\left(\right.$ in $\left.H_{k}(X-U, A-U)\right)$.

Exercise 2.5. Let $B$ an open band around the equator of $S^{2}$, and $x_{0} \in B$. Compute the relative homology $H_{\bullet}\left(S^{2}-x_{0}, B-x_{0} ; \mathbb{Z}\right)$.

To describe some more applications of excision we need the notion of augmented homology modules. Given a topological space $X$ and a ring $R$, let us define

$$
\begin{aligned}
\partial^{\sharp}: S_{0}(X, R) & \rightarrow R \\
\sum_{j} a_{j} \sigma_{j} & \mapsto \sum_{j} a_{j} .
\end{aligned}
$$

We define the augmented homology modules

$$
H_{0}^{\sharp}(X, R)=\operatorname{ker} \partial^{\sharp} / B_{0}(X, R), \quad H_{k}^{\sharp}(X, R)=H_{k}(X, R) \text { for } k>0
$$

If $A \subset X$, one defines the augmented relative homology modules $H_{k}^{\sharp}(X, A ; R)$ in a similar way, i.e.,

$$
H_{k}^{\sharp}(X, A ; R)=H_{k}(X, A ; R) \text { if } A \neq \emptyset, \quad H_{k}^{\sharp}(X, A ; R)=H_{k}(X, R) \text { if } A=\emptyset
$$

Exercise 2.6. Prove that there is a long exact sequence for the augmented relative homology modules.

Exercise 2.7. Let $B^{n}$ be the closed unit ball in $\mathbb{R}^{n+1}, S^{n}$ its boundary, and let $E_{n}^{ \pm}$ be the two closed (northern, southern) emispheres in $S^{n}$.

1. Use the long exact sequence for the augmented relative homology modules to prove that $H_{k}^{\sharp}\left(S^{n}\right) \simeq H_{k}^{\sharp}\left(S^{n}, E_{n}^{-}\right)$and $H_{k-1}^{\sharp}\left(S^{n-1}\right) \simeq H_{k}^{\sharp}\left(B^{n}, S^{n-1}\right)$. So we have $H_{k}^{\sharp}\left(B^{n}, S^{n-1}\right)=0$ for $k<n, H_{n}^{\sharp}\left(B^{n}, S^{n-1}\right) \simeq R$
2. Use excision to show that $H_{k}^{\sharp}\left(S^{n}, E_{n}^{-}\right) \simeq H_{k}^{\sharp}\left(B^{n}, S^{n-1}\right)$.
3. Deduce that $H_{k}^{\sharp}\left(S^{n}\right) \simeq H_{k-1}^{\sharp}\left(S^{n-1}\right)$.

ExERCISE 2.8. Let $S^{n}$ be the sphere realized as the unit sphere in $\mathbb{R}^{n+1}$, and let $r: S^{n} \rightarrow S^{n} \rightarrow S^{n}$ be the reflection

$$
r\left(x^{0}, x^{1}, \ldots, x^{n}\right)=\left(-x^{0}, x^{1}, \ldots, x^{n}\right)
$$

Prove that $r_{b}: H_{n}\left(S^{n}\right) \rightarrow H_{n}\left(S^{n}\right)$ is the multiplication by -1 . (Hint: this is trivial for $n=0$, and can be extended by induction using the commutativity of the diagram

which follows from Exercise 2.7.
ExERCISE 2.9. 1. The rotation group $O(n+1)$ acts on $S^{n}$. Show that for any $M \in O(n+1)$ the induced morphism $M_{b}: H_{n}\left(S^{n}\right) \rightarrow H_{n}\left(S^{n}\right)$ is the multiplication by $\operatorname{det} M= \pm 1$.
2. Let $a: S^{n} \rightarrow S^{n}$ be the antipodal map, $a(x)=-x$. Show that $a_{b}: H_{n}\left(S^{n}\right) \rightarrow$ $H_{n}\left(S^{n}\right)$ is the multiplication by $(-1)^{n+1}$.

EXAMPLE 2.10. We show that the inclusion map $\left(E_{n}^{+}, S^{n-1}\right) \rightarrow\left(S^{n}, E_{n}^{-}\right)$is an excision. (Here we are excising the open southern emisphere, i.e., with reference to the general theory, $X=S^{n}, U=$ the open southern emisphere, $A=E_{n}^{-}$.)

The hypotheses of Theorem 2.4 are not satisfied. However it is enough to consider the subspace

$$
V=\left\{x \in S^{n} \left\lvert\, x^{0}>-\frac{1}{2}\right.\right\}
$$

$V$ can be excised from $\left(S^{n}, E_{n}^{-}\right)$. But $\left(E_{n}^{+}, S^{n-1}\right)$ is a deformation retract of $\left(S^{n}-\right.$ $\left.V, E_{n}^{-}-V\right)$ so that we are done.

We end with a standard application of algebraic topology. Let us define a vector field on $S^{n}$ as a continous map $v: S^{n} \rightarrow \mathbb{R}^{n+1}$ such that $v(x) \cdot x=0$ for all $x \in S^{n}$ (the product is the standard scalar product in $\left.\mathbb{R}^{n+1}\right)$.

Proposition 2.11. A nowhere vanishing vector field $v$ on $S^{n}$ exists if and only if $n$ is odd.

Proof. If $n=2 m+1$ a nowhere vanishing vector field is given by

$$
v\left(x_{0}, \ldots, x_{2 m+1}\right)=\left(-x_{1}, x_{0},-x_{3}, x_{2}, \ldots,-x_{2 m+1}, x_{2 m}\right)
$$

Conversely, assume that such a vector field exists. Define

$$
w(x)=\frac{v(x)}{\|v(x)\|}
$$

this is a map $S^{n} \rightarrow S^{n}$, with $w(x) \cdot x=0$ for all $x \in S^{n}$. Define

$$
\begin{aligned}
F: S^{n} \times I & \rightarrow S^{n} \\
F(x, t) & =x \cos t \pi+w(x) \sin t \pi
\end{aligned}
$$

Since

$$
F(x, 0)=x, \quad F\left(x, \frac{1}{2}\right)=w(x), \quad F(x, 1)=-x
$$

the three maps Id, $w, a$ are homotopic. But as a consequence of Exercise $2.9, n$ must be odd.

## CHAPTER 3

## Introduction to sheaves and their cohomology

## 1. Presheaves and sheaves

Let $X$ be a topological space.

Definition 3.1. A presheaf of Abelian groups on $X$ is a rule ${ }^{1} \mathcal{P}$ which assigns an Abelian group $\mathcal{P}(U)$ to each open subset $U$ of $X$ and a morphism (called restriction map) $\varphi_{U, V}: \mathcal{P}(U) \rightarrow \mathcal{P}(V)$ to each pair $V \subset U$ of open subsets, so as to verify the following requirements:
(1) $\mathcal{P}(\emptyset)=\{0\}$;
(2) $\varphi_{U, U}$ is the identity map;
(3) if $W \subset V \subset U$ are open sets, then $\varphi_{U, W}=\varphi_{V, W} \circ \varphi_{U, V}$.

The elements $s \in \mathcal{P}(U)$ are called sections of the presheaf $\mathcal{P}$ on $U$. If $s \in \mathcal{P}(U)$ is a section of $\mathcal{P}$ on $U$ and $V \subset U$, we shall write $s_{\mid V}$ instead of $\varphi_{U, V}(s)$. The restriction $\mathcal{P}_{\mid U}$ of $\mathcal{P}$ to an open subset $U$ is defined in the obvious way.

Presheaves of rings are defined in the same way, by requiring that the restriction maps are ring morphisms. If $\mathcal{R}$ is a presheaf of rings on $X$, a presheaf $\mathcal{M}$ of Abelian groups on $X$ is called a presheaf of modules over $\mathcal{R}$ (or an $\mathcal{R}$-module) if, for each open subset $U, \mathcal{M}(U)$ is an $\mathcal{R}(U)$-module and for each pair $V \subset U$ the restriction map $\varphi_{U, V}: \mathcal{M}(U) \rightarrow \mathcal{M}(V)$ is a morphism of $\mathcal{R}(U)$-modules (where $\mathcal{M}(V)$ is regarded as an $\mathcal{R}(U)$-module via the restriction morphism $\mathcal{R}(U) \rightarrow \mathcal{R}(V))$. The definitions in this Section are stated for the case of presheaves of Abelian groups, but analogous definitions and properties hold for presheaves of rings and modules.

Definition 3.2. A morphism $f: \mathcal{P} \rightarrow \mathcal{Q}$ of presheaves over $X$ is a family of morphisms of Abelian groups $f_{U}: \mathcal{P}(U) \rightarrow \mathcal{Q}(U)$ for each open $U \subset X$, commuting with the

[^6]restriction morphisms; i.e., the following diagram commutes:


Definition 3.3. The stalk of a presheaf $\mathcal{P}$ at a point $x \in X$ is the Abelian group

$$
\mathcal{P}_{x}=\underset{U}{\lim } \mathcal{P}(U)
$$

where $U$ ranges over all open neighbourhoods of $x$, directed by inclusion.
REMARK 3.4. We recall here the notion of direct limit. A directed set $I$ is a partially ordered set such that for each pair of elements $i, j \in I$ there is a third element $k$ such that $i<k$ and $j<k$. If $I$ is a directed set, a directed system of Abelian groups is a family $\left\{G_{i}\right\}_{i \in I}$ of Abelian groups, such that for all $i<j$ there is a group morphism $f_{i j}: G_{i} \rightarrow G_{j}$, with $f_{i i}=i d$ and $f_{i j} \circ f_{j k}=f_{i k}$. On the set $\mathfrak{G}=\coprod_{i \in I} G_{i}$, where $\coprod$ denotes disjoint union, we put the following equivalence relation: $g \sim h$, with $g \in G_{i}$ and $h \in G_{j}$, if there exists a $k \in I$ such that $f_{i k}(g)=f_{j k}(h)$. The direct limit $\mathfrak{l}$ of the system $\left\{G_{i}\right\}_{i \in I}$, denoted $\mathfrak{l}=\lim _{i \in I} G_{i}$, is the quotient $\mathfrak{G} / \sim$. Heuristically, two elements in $\mathfrak{G}$ represent the same element in the direct limit if they are 'eventually equal.' From this definition one naturally obtains the existence of canonical morphisms $G_{i} \rightarrow \mathfrak{l}$. The following discussion should make this notion clearer; for more detail, the reader may consult [12].

If $x \in U$ and $s \in \mathcal{P}(U)$, the image $s_{x}$ of $s$ in $\mathcal{P}_{x}$ via the canonical projection $\mathcal{P}(U) \rightarrow \mathcal{P}_{x}$ (see footnote) is called the germ of $s$ at $x$. From the very definition of direct limit we see that two elements $s \in \mathcal{P}(U), s^{\prime} \in \mathcal{P}(V), U, V$ being open neighbourhoods of $x$, define the same germ at $x$, i.e. $s_{x}=s_{x}^{\prime}$, if and only if there exists an open neighbourhood $W \subset U \cap V$ of $x$ such that $s$ and $s^{\prime}$ coincide on $W, s_{\mid W}=s^{\prime}{ }_{\mid W}$.

Definition 3.5. A sheaf on a topological space $X$ is a presheaf $\mathcal{F}$ on $X$ which fulfills the following axioms for any open subset $U$ of $X$ and any cover $\left\{U_{i}\right\}$ of $U$.

S1) If two sections $s \in \mathcal{F}(U), \bar{s} \in \mathcal{F}(U)$ coincide when restricted to any $U_{i}, s_{\mid U_{i}}=$ $\bar{s}_{\mid U_{i}}$, they are equal, $s=\bar{s}$.
S2) Given sections $s_{i} \in \mathcal{F}\left(U_{i}\right)$ which coincide on the intersections, $s_{i \mid U_{i} \cap U_{j}}=$ $s_{j_{\mid U_{i} \cap U_{j}}}$ for every $i, j$, there exists a section $s \in \mathcal{F}(U)$ whose restriction to each $U_{i}$ equals $s_{i}$, i.e. $s_{\mid U_{i}}=s_{i}$.

Thus, roughly speaking, sheaves are presheaves defined by local conditions. The stalk of a sheaf is defined as in the case of a presheaf.

Example 3.6. If $\mathcal{F}$ is a sheaf, and $\mathcal{F}_{x}=\{0\}$ for all $x \in X$, then $\mathcal{F}$ is the zero sheaf, $\mathcal{F}(U)=\{0\}$ for all open sets $U \subset X$. Indeed, if $s \in \mathcal{F}(U)$, since $s_{x}=0$ for all $x \in U$, there is for each $x \in U$ an open neighbourhood $U_{x}$ such that $s_{\mid U_{x}}=0$. The first sheaf axiom then implies $s=0$. This is not true for a presheaf, cf. Example 3.14 below.

A morphism of sheaves is just a morphism of presheaves. If $f: \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of sheaves on $X$, for every $x \in X$ the morphism $f$ induces a morphism between the stalks, $f_{x}: \mathcal{F}_{x} \rightarrow \mathcal{G}_{x}$, in the following way: since the stalk $\mathcal{F}_{x}$ is the direct limit of the groups $\mathcal{F}(U)$ over all open $U$ containing $x$, any $g \in \mathcal{F}_{x}$ is of the form $g=s_{x}$ for some open $U \ni x$ and some $s \in \mathcal{F}(U)$; then set $f_{x}(g)=\left(f_{U}(s)\right)_{x}$.

A sequence of morphisms of sheaves $0 \rightarrow \mathcal{F}^{\prime} \rightarrow \mathcal{F} \rightarrow \mathcal{F}^{\prime \prime} \rightarrow 0$ is exact if for every point $x \in X$, the sequence of morphisms between the stalks $0 \rightarrow \mathcal{F}_{x}^{\prime} \rightarrow \mathcal{F}_{x} \rightarrow \mathcal{F}_{x}^{\prime \prime} \rightarrow 0$ is exact. If $0 \rightarrow \mathcal{F}^{\prime} \rightarrow \mathcal{F} \rightarrow \mathcal{F}^{\prime \prime} \rightarrow 0$ is an exact sequence of sheaves, for every open subset $U \subset X$ the sequence of groups $0 \rightarrow \mathcal{F}^{\prime}(U) \rightarrow \mathcal{F}(U) \rightarrow \mathcal{F}^{\prime \prime}(U)$ is exact, but the last arrow may fail to be surjective. An instance of this situation is contained in Example 3.11 below.

ExERCISE 3.7. Let $0 \rightarrow \mathcal{F}^{\prime} \rightarrow \mathcal{F} \rightarrow \mathcal{F}^{\prime \prime} \rightarrow 0$ be an exact sequence of sheaves. Show that $0 \rightarrow \mathcal{F}^{\prime} \rightarrow \mathcal{F} \rightarrow \mathcal{F}^{\prime \prime}$ is an exact sequence of presheaves.

Example 3.8. Let $G$ be an Abelian group. Defining $\mathcal{P}(U) \equiv G$ for every open subset $U$ and taking the identity maps as restriction morphisms, we obtain a presheaf, called the constant presheaf $\tilde{G}_{X}$. All stalks $\left(\tilde{G}_{X}\right)_{x}$ of $\tilde{G}_{X}$ are isomorphic to the group $G$. The presheaf $\tilde{G}_{X}$ is not a sheaf: if $V_{1}$ and $V_{2}$ are disjoint open subsets of $X$, and $U=V_{1} \cup V_{2}$, the sections $g_{1} \in \tilde{G}_{X}\left(V_{1}\right)=G, g_{2} \in \tilde{G}_{X}\left(V_{2}\right)=G$, with $g_{1} \neq g_{2}$, satisfy the hypothesis of the second sheaf axiom S2) (since $V_{1} \cap V_{2}=\emptyset$ there is nothing to satisfy), but there is no section $g \in \tilde{G}_{X}(U)=G$ which restricts to $g_{1}$ on $V_{1}$ and to $g_{2}$ on $V_{2}$.

Example 3.9. Let $\mathcal{C}_{X}(U)$ be the ring of real-valued continuous functions on an open set $U$ of $X$. Then $\mathcal{C}_{X}$ is a sheaf (with the obvious restriction morphisms), the sheaf of continuous functions on $X$. The stalk $\mathcal{C}_{x} \equiv\left(\mathcal{C}_{X}\right)_{x}$ at $x$ is the ring of germs of continuous functions at $x$.

Example 3.10. In the same way one can define the following sheaves:
The sheaf $\mathcal{C}_{X}^{\infty}$ of differentiable functions on a differentiable manifold $X$.
The sheaves $\Omega_{X}^{p}$ of differential $p$-forms, and all the sheaves of tensor fields on a differentiable manifold $X$.

The sheaf of holomorphic functions on a complex manifold and the sheaves of holomorphic $p$-forms on it.

The sheaves of forms of type $(p, q)$ on a complex manifold $X$.
Example 3.11. Let $X$ be a differentiable manifold, and let $d: \Omega_{X}^{\circ} \rightarrow \Omega_{X}^{\circ}$ be the exterior differential. We can define the presheaves $\mathcal{Z}_{X}^{p}$ of closed differential $p$-forms, and
$\mathcal{B}_{X}^{p}$ of exact $p$-differential forms,

$$
\begin{gathered}
\mathcal{Z}_{X}^{p}(U)=\left\{\omega \in \Omega_{X}^{p}(U) \mid d \omega=0\right\} \\
\mathcal{B}_{X}^{p}(U)=\left\{\omega \in \Omega_{X}^{p}(U) \mid \omega=d \tau \quad \text { for some } \quad \tau \in \Omega_{X}^{p-1}(U)\right\} .
\end{gathered}
$$

$\mathcal{Z}_{X}^{p}$ is a sheaf, since the condition of being closed is local: a differential form is closed if and only if it is closed in a neighbourhood of each point of $X$. On the contrary, $\mathcal{B}_{X}^{p}$ is not a sheaf. In fact, if $X=\mathbb{R}^{2}$, the presheaf $\mathcal{B}_{X}^{1}$ of exact differential 1-forms does not fulfill the second sheaf axiom: consider the form

$$
\omega=\frac{x d y-y d x}{x^{2}+y^{2}}
$$

defined on the open subset $U=X-\{(0,0)\}$. Since $\omega$ is closed on $U$, there is an open cover $\left\{U_{i}\right\}$ of $U$ by open subsets where $\omega$ is an exact form, $\omega_{\mid U_{i}} \in \mathcal{B}_{X}^{1}\left(U_{i}\right)$ (this is Poincaré's lemma). But $\omega$ is not an exact form on $U$ because its integral along the unit circle is different from 0 .

This means that, while the sequence of sheaf morphisms $0 \rightarrow \mathbb{R} \rightarrow \mathcal{C}_{X}^{\infty} \xrightarrow{d} \mathcal{Z}_{X}^{1} \rightarrow 0$ is exact (Poincaré lemma), the morphism $\mathcal{C}_{X}^{\infty}(U) \xrightarrow{d} \mathcal{Z}_{X}^{1}(U)$ may fail to be surjective.
1.1. Étalé space. We wish now to describe how, given a presheaf, one can naturally associate with it a sheaf having the same stalks. As a first step we consider the case of a constant presheaf $G_{X}$ on a topological space $X$, where $G$ is an Abelian group. We can define another presheaf $G_{X}$ on $X$ by putting $G_{X}(U)=$ \{locally constant functions $f: U \rightarrow G\},{ }^{2}$ where $\tilde{G}_{X}(U)=G$ is included as the constant functions. It is clear that $\left(G_{X}\right)_{x}=G_{x}=G$ at each point $x \in X$ and that $G_{X}$ is a sheaf, called the constant sheaf with stalk $G$. Notice that the functions $f: U \rightarrow G$ are the sections of the projection $\pi: \coprod_{x \in X} G_{x} \rightarrow X$ and the locally constant functions correspond to those sections which locally coincide with the sections produced by the elements of $G$.

Now, let $\mathcal{P}$ be an arbitrary presheaf on $X$. Consider the disjoint union of the stalks $\underline{\mathcal{P}}=\coprod_{x \in X} \mathcal{P}_{x}$ and the natural projection $\pi: \underline{\mathcal{P}} \rightarrow X$. The sections $s \in \mathcal{P}(U)$ of the presheaf $\mathcal{P}$ on an open subset $U$ produce sections $\underline{s}: U \hookrightarrow \underline{\mathcal{P}}$ of $\pi$, defined by $\underline{s}(x)=s_{x}$, and we can define a new presheaf $\mathcal{P}^{\natural}$ by taking $\mathcal{P}^{\natural}(U)$ as the group of those sections $\sigma: U \hookrightarrow \underline{\mathcal{P}}$ of $\pi$ such that for every point $x \in U$ there is an open neighbourhood $V \subset U$ of $x$ which satisfies $\sigma_{\mid V}=\underline{s}$ for some $s \in \mathcal{P}(V)$.

That is, $\mathcal{P}^{\natural}$ is the presheaf of all sections that locally coincide with sections of $\mathcal{P}$. It can be described in another way by the following construction.

Definition 3.12. The set $\underline{\mathcal{P}}$, endowed with the topology whose base of open subsets consists of the sets $s(U)$ for $U$ open in $X$ and $s \in \mathcal{P}(U)$, is called the étalé space of the presheaf $\mathcal{P}$.

[^7]ExErcise 3.13. (1) Show that $\pi: \underline{\mathcal{P}} \rightarrow X$ is a local homeomorphism, i.e., every point $u \in \underline{\mathcal{P}}$ has an open neighbourhood $U$ such that $\pi: U \rightarrow \pi(U)$ is a homeomorphism.
(2) Show that for every open set $U \subset X$ and every $s \in \mathcal{P}(U)$, the section $\underline{s}: U \rightarrow \underline{\mathcal{P}}$ is continuous.
(3) Prove that $\mathcal{P}^{\natural}$ is the sheaf of continuous sections of $\pi: \underline{\mathcal{P}} \rightarrow X$.
(4) Prove that for all $x \in X$ the stalks of $\mathcal{P}$ and $\mathcal{P}^{\natural}$ at $x$ are isomorphic.
(5) Show that there is a presheaf morphism $\phi: \mathcal{P} \rightarrow \mathcal{P}^{\natural}$.
(6) Show that $\phi$ is an isomorphism if and only if $\mathcal{P}$ is a sheaf.
$\mathcal{P}^{\natural}$ is called the sheaf associated with the presheaf $\mathcal{P}$. In general, the morphism $\phi: \mathcal{P} \rightarrow \mathcal{P}^{\natural}$ is neither injective nor surjective: for instance, the morphism between the constant presheaf $\tilde{G}_{X}$ and its associated sheaf $G_{X}$ is injective but nor surjective.

Example 3.14. As a second example we study the sheaf associated with the presheaf $\mathcal{B}_{X}^{k}$ of exact $k$-forms on a differentiable manifold $X$. For any open set $U$ we have an exact sequence of Abelian groups (actually of $\mathbb{R}$-vector spaces)

$$
0 \rightarrow \mathcal{B}_{X}^{k}(U) \rightarrow \mathcal{Z}_{X}^{k}(U) \rightarrow \mathcal{H}_{X}^{k}(U) \rightarrow 0
$$

where $\mathcal{H}_{X}^{k}$ is the presheaf that with any open set $U$ associates its $k$-th de Rham cohomology group, $\mathcal{H}_{X}^{k}(U)=H_{D R}^{k}(U)$. Now, the open neighbourhoods of any point $x \in X$ which are diffeomorphic to $\mathbb{R}^{n}$ (where $n=\operatorname{dim} X$ ) are cofinal ${ }^{3}$ in the family of all open neighbourhoods of $x$, so that $\left(\mathcal{H}_{X}^{k}\right)_{x}=0$ by the Poincaré lemma. In accordance with Example 3.6 this means that $\left(\mathcal{H}_{X}^{k}\right)^{\natural}=0$, which is tantamount to $\left(\mathcal{B}_{X}^{k}\right)^{\natural} \simeq \mathcal{Z}_{X}^{k}$.

In this case the natural morhism $\mathcal{H}_{X}^{k} \rightarrow\left(\mathcal{H}_{X}^{k}\right)^{\natural}$ is of course surjective but not injective. On the other hand, $\mathcal{B}_{X}^{k} \rightarrow\left(\mathcal{B}_{X}^{k}\right)^{\natural}=\mathcal{Z}_{X}^{k}$ is injective but not surjective.

Definition 3.15. Given a sheaf $\mathcal{F}$ on a topological space $X$ and a subset (not necessarily open) $S \subset X$, the sections of the sheaf $\mathcal{F}$ on $S$ are the continuous sections $\sigma: S \hookrightarrow \underline{\mathcal{F}}$ of $\pi: \underline{\mathcal{F}} \rightarrow X$. The group of such sections is denoted by $\Gamma(S, \mathcal{F})$.

Definition 3.16. Let $\mathcal{P}, \mathcal{Q}$ be presheaves on a topological space $X .{ }^{4}$
(1) The direct sum of $\mathcal{P}$ and $\mathcal{Q}$ is the presheaf $\mathcal{P} \oplus \mathcal{Q}$ given, for every open subset $U \subset X$, by $(\mathcal{P} \oplus \mathcal{Q})(U)=\mathcal{P}(U) \oplus \mathcal{Q}(U)$ with the obvious restriction morphisms.

[^8](2) For any open set $U \subset X$, let us denote by $\operatorname{Hom}\left(\mathcal{P}_{\mid U}, \mathcal{Q}_{\mid U}\right)$ the space of morphisms between the restricted presheaves $\mathcal{P}_{\mid U}$ and $\mathcal{Q}_{\mid U}$; this is an Abelian group in a natural manner. The presheaf of homomorphisms is the presheaf $\mathcal{H o m}(\mathcal{P}, \mathcal{Q})$ given by $\mathcal{H o m}(\mathcal{P}, \mathcal{Q})(U)=\operatorname{Hom}\left(\mathcal{P}_{\mid U}, \mathcal{Q}_{\mid U}\right)$ with the natural restriction morphisms.
(3) The tensor product of $\mathcal{P}$ and $\mathcal{Q}$ is the presheaf $(\mathcal{P} \otimes \mathcal{Q})(U)=\mathcal{P}(U) \otimes \mathcal{Q}(U)$.

If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then the presheaves $\mathcal{F} \oplus \mathcal{G}$ and $\mathcal{H o m}(\mathcal{F}, \mathcal{G})$ are sheaves. On the contrary, the tensor product of $\mathcal{F}$ and $\mathcal{G}$ previously defined may not be a sheaf. Indeed one defines the tensor product of the sheaves $\mathcal{F}$ and $\mathcal{G}$ as the sheaf associated with the presheaf $U \rightarrow \mathcal{F}(U) \otimes \mathcal{G}(U)$.

It should be noticed that in general $\operatorname{Hom}(\mathcal{F}, \mathcal{G})(U) \nsim \operatorname{Hom}(\mathcal{F}(U), \mathcal{G}(U))$ and $\mathcal{H o m}(\mathcal{F}, \mathcal{G})_{x} \not 千 \operatorname{Hom}\left(\mathcal{F}_{x}, \mathcal{G}_{x}\right)$.
1.2. Direct and inverse images of presheaves and sheaves. Here we study the behaviour of presheaves and sheaves under change of base space. Let $f: X \rightarrow Y$ be a continuous map.

Definition 3.17. The direct image by $f$ of a presheaf $\mathcal{P}$ on $X$ is the presheaf $f_{*} \mathcal{P}$ on $Y$ defined by $\left(f_{*} \mathcal{P}\right)(V)=\mathcal{P}\left(f^{-1}(V)\right)$ for every open subset $V \subset Y$. If $\mathcal{F}$ is a sheaf on $X$, then $f_{*} \mathcal{F}$ turns out to be a sheaf.

Let $\mathcal{P}$ be a presheaf on $Y$.
Definition 3.18. The inverse image of $\mathcal{P}$ by $f$ is the presheaf on $X$ defined by

$$
U \rightarrow \underset{U \subset \overrightarrow{f^{-1}}(V)}{\lim } \mathcal{P}(V)
$$

The inverse image sheaf of a sheaf $\mathcal{F}$ on $Y$ is the sheaf $f^{-1} \mathcal{F}$ associated with the inverse image presheaf of $\mathcal{F}$.

The stalk of the inverse image presheaf at a point $x \in X$ is isomorphic to $\mathcal{P}_{f(x)}$. It follows that if $0 \rightarrow \mathcal{F}^{\prime} \rightarrow \mathcal{F} \rightarrow \mathcal{F}^{\prime \prime} \rightarrow 0$ is an exact sequence of sheaves on $Y$, the induced sequence

$$
0 \rightarrow f^{-1} \mathcal{F}^{\prime} \rightarrow f^{-1} \mathcal{F} \rightarrow f^{-1} \mathcal{F}^{\prime \prime} \rightarrow 0
$$

of sheaves on $X$, is also exact (that is, the inverse image functor for sheaves of Abelian groups is exact).

The étalé space $\underline{f^{-1} \mathcal{F}}$ of the inverse image sheaf is the fibred product ${ }^{5} Y \times_{X} \underline{\mathcal{F}}$. It follows easily that the inverse image of the constant sheaf $G_{X}$ on $X$ with stalk $G$ is the constant sheaf $G_{Y}$ with stalk $G, f^{-1} G_{X}=G_{Y}$.

[^9]
## 2. Cohomology of sheaves

We wish now to describe a cohomology theory which associates cohomology groups to a sheaf on a topological space $X$.
2.1. Čech cohomology. We start by considering a presheaf $\mathcal{P}$ on $X$ and an open cover $\mathfrak{U}$ of $X$. We assume that $\mathfrak{U}$ is labelled by a totally ordered set $I$, and define

$$
U_{i_{0} \ldots i_{p}}=U_{i_{0}} \cap \cdots \cap U_{i_{p}}
$$

We define the Čech complex of $\mathfrak{U}$ with coefficients in $\mathcal{P}$ as the complex whose $p$-th term is the Abelian group

$$
\check{C}^{p}(\mathfrak{U}, \mathcal{P})=\prod_{i_{0}<\cdots<i_{p}} \mathcal{P}\left(U_{i_{0} \ldots i_{p}}\right)
$$

Thus a $p$-cochain $\alpha$ is a collection $\left\{\alpha_{i_{0} \ldots i_{p}}\right\}$ of sections of $\mathcal{P}$, each one belonging to the space of sections over the intersection of $p+1$ open sets in $\mathfrak{U}$. Since the indexes of the open sets are taken in strictly increasing order, each intersection is counted only once.

The Čech differential $\delta: \check{C}^{p}(\mathfrak{U}, \mathcal{P}) \rightarrow \check{C}^{p+1}(\mathfrak{U}, \mathcal{P})$ is defined as follows: if $\alpha=\left\{\alpha_{i_{0} \ldots i_{p}}\right\}$ $\in \check{C}^{p}(\mathfrak{U}, \mathcal{P})$, then

$$
\left\{(\delta \alpha)_{i_{0} \ldots i_{p+1}}\right\}=\sum_{k=0}^{p+1}(-1)^{k} \alpha_{i_{0} \ldots \widehat{\imath}_{k} \ldots i_{p+1} \mid U_{i_{0} \ldots i_{p+1}}}
$$

Here a caret denotes omission of the index. For instance, if $p=0$ we have $\alpha=\left\{\alpha_{i}\right\}$ and

$$
\begin{equation*}
(\delta \alpha)_{i k}=\alpha_{k \mid U_{i} \cap U_{k}}-\alpha_{i \mid U_{i} \cap U_{k}} \tag{3.1}
\end{equation*}
$$

It is an easy exercise to check that $\delta^{2}=0$. Thus we obtain a cohomology theory. We denote the corresponding cohomology groups by $H^{k}(\mathfrak{U}, \mathcal{P})$.

Lemma 3.1. If $\mathcal{F}$ is a sheaf, one has an isomorphism $H^{0}(\mathfrak{U}, \mathcal{F}) \simeq \mathcal{F}(X)$
Proof. We have $H^{0}(\mathfrak{U}, \mathcal{F})=\operatorname{ker} \delta: \check{C}^{0}(\mathfrak{U}, \mathcal{P}) \rightarrow \check{C}^{1}(\mathfrak{U}, \mathcal{P})$. So if $\alpha \in H^{0}(\mathfrak{U}, \mathcal{F})$ by (3.1) we see that

$$
\alpha_{k \mid U_{i} \cap U_{k}}=\alpha_{i \mid U_{i} \cap U_{k}}
$$

By the second sheaf axiom this implies that there is a global section $\tilde{\alpha} \in \mathcal{F}(X)$ such that $\tilde{\alpha}_{\mid U_{i}}=\alpha_{i}$. This yields a morphism $H^{0}(\mathfrak{U}, \mathcal{F}) \rightarrow \mathcal{F}(X)$, which is evidently surjective and is injective because of the first sheaf axiom.

Example 3.2. We consider an open cover $\mathfrak{U}$ of the circle $S^{1}$ formed by three sets which intersect only pairwise. We compute the Čech cohomology of $\mathfrak{U}$ with coefficients in the constant sheaf $\mathbb{R}$. We have $C^{0}(\mathfrak{U}, \mathbb{R})=C^{1}(\mathfrak{U}, \mathbb{R})=\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}, C^{k}(\mathfrak{U}, \mathbb{R})=$ 0 for $k>1$ because there are no triple intersections. The only nonzero differential $d_{0}: C^{0}(\mathfrak{U}, \mathbb{R}) \rightarrow C^{1}(\mathfrak{U}, \mathbb{R})$ is given by

$$
d_{0}\left(x_{0}, x_{1}, x_{2}\right)=\left(x_{1}-x_{2}, x_{2}-x_{0}, x_{0}-x_{1}\right)
$$

Hence

$$
\begin{gathered}
H^{0}(\mathfrak{U}, \mathbb{R})=\operatorname{ker} d_{0} \simeq \mathbb{R} \\
H^{1}(\mathfrak{U}, \mathbb{R})=C^{1}(\mathfrak{U}, \mathbb{R}) / \operatorname{Im} d_{0} \simeq \mathbb{R} .
\end{gathered}
$$

It is possible to define Čech cohomology groups depending only on the pair $(X, \mathcal{F})$, and not on a cover, by letting

$$
H^{k}(X, \mathcal{F})=\underset{\overrightarrow{\mathfrak{U}}}{\lim } \check{H}^{k}(\mathfrak{U}, \mathcal{F})
$$

The direct limit is taken over a cofinal subset of the directed set of all covers of $X$ (the order is of course the refinement of covers: a cover $\mathfrak{V}=\left\{V_{j}\right\}_{j \in J}$ is a refinement of $\mathfrak{U}$ if there is a map $f: I \rightarrow J$ such that $V_{f(i)} \subset U_{i}$ for every $\left.i \in I\right)$. The order must be fixed at the outset, since a cover may be regarded as a refinement of another in many ways. As different cofinal families give rise to the same inductive limit, the groups $H^{k}(X, \mathcal{F})$ are well defined.
2.2. Fine sheaves. Čech cohomology is well-behaved when the base space $X$ is paracompact. (It is indeed the bad behaviour of Čech cohomology on non-paracompact spaces which motivated the introduction of another cohomology theory for sheaves, usually called sheaf cohomology; cf. [5].) In this and in the following sections we consider some properties of Čech cohomology that hold in that case.

Definition 3.3. A sheaf of rings $\mathcal{R}$ on a topological space $X$ is fine if, for any locally finite oper cover $\mathfrak{U}=\left\{U_{i}\right\}_{i \in I}$ of $X$, there is a family $\left\{s_{i}\right\}_{i \in I}$ of global sections of $\mathcal{R}$ such that:
(1) $\sum_{i \in I} s_{i}=1$;
(2) for every $i \in I$ there is a closed subset $S_{i} \subset U_{i}$ such that $\left(s_{i}\right)_{x}=0$ whenever $x \notin S_{i}$.

The family $\left\{s_{i}\right\}$ is called a partition of unity subordinated to the cover $\mathfrak{U}$. For instance, the sheaf of continuous functions on a paracompact topological space as well as the sheaf of smooth functions on a differentiable manifold are fine, while sheaves of complex or real analytic functions are not.

Definition 3.4. A sheaf $\mathcal{F}$ of Abelian groups on a topological space $X$ is said to be acyclic if $H^{k}(X, \mathcal{F})=0$ for $k>0$.

Proposition 3.5. Let $\mathcal{R}$ be a fine sheaf of rings on a paracompact space $X$. Every sheaf $\mathcal{M}$ of $\mathcal{R}$-modules is acyclic.

Proof. Let $\mathfrak{U}=\left\{U_{i}\right\}_{i \in I}$ be a locally finite open cover of $X$, and let $\left\{\rho_{i}\right\}$ be a partition of unity of $\mathcal{R}$ subordinated to $\mathfrak{U}$. For any $\alpha \in \check{C}^{q}(\mathfrak{U}, \mathcal{M})$ with $q>0$ we set

$$
\begin{aligned}
(K \alpha)_{i_{0} \ldots i_{q-1}} & =\sum_{\substack{j \in I \\
j<i_{0}}} \rho_{j} a_{j i_{0} \ldots i_{q-1}}-\sum_{\substack{j \in I \\
i_{0}<j<i_{1}}} \rho_{j} a_{i_{0} j i_{1} \ldots i_{q-1}}+\ldots \\
& =\sum_{k=0}^{q}(-1)^{k} \sum_{\substack{j \in I \\
i_{k-1}<j<i_{k}}} \rho_{j} a_{i_{0} \ldots i_{k-1} j i_{k} \ldots i_{q-1}} .
\end{aligned}
$$

This defines a morphism $K: \check{C}^{k}(\mathfrak{U}, \mathcal{M}) \rightarrow \check{C}^{k-1}(\mathfrak{U}, \mathcal{M})$ such that $\delta K+K \delta=$ id (i.e., $K$ is a homotopy operator); then $\alpha=\delta K \alpha$ if $\delta \alpha=0$, so that $H^{k}(\mathfrak{U}, \mathcal{M})=0$ for $k>0$. Since on a paracompact space the locally finite open covers are cofinal in the family of all covers, we can take direct limit on such covers, thus getting $H^{k}(X, \mathcal{M})=0$ for $k>0$.

Example 3.6. Using this result we may recast the proof of the exactness of the Mayer-Vietoris sequence for de Rham cohomology in a slightly different form. Given a differentiable manifold $X$, let $\mathfrak{U}$ be the open cover formed by two sets $U$ and $V$. Since $\check{C}^{2}\left(\mathfrak{U}, \Omega^{k}\right)=0$ (there are no triple intersections) we have an exact sequence

$$
0 \rightarrow H^{0}\left(\mathfrak{U}, \Omega^{k}\right) \rightarrow \check{C}^{0}\left(\mathfrak{U}, \Omega^{k}\right) \xrightarrow{\delta} \check{C}^{1}\left(\mathfrak{U}, \Omega^{k}\right) \rightarrow 0
$$

which in principle is exact everywhere but at $C^{1}\left(\mathfrak{U}, \Omega^{k}\right)$. However since the sheaves $\Omega^{k}$ are acyclic by Proposition 3.5, one has $H^{1}\left(\mathfrak{U}, \Omega^{k}\right)=0$, which means that $\delta$ is surjective, and the sequence is exact at that place as well. We have the identifications

$$
H^{0}\left(\mathfrak{U}, \Omega^{k}\right)=\Omega^{k}(X), \quad C^{0}\left(\mathfrak{U}, \Omega^{k}\right)=\Omega^{k}(U) \oplus \Omega^{k}(V), \quad C^{1}\left(\mathfrak{U}, \Omega^{k}\right)=\Omega^{k}(U \cap V)
$$

so that we obtain the exactness of the Mayer-Vietoris sequence.
2.3. Long exact sequences in Čech cohomology. We wish to show that when $X$ is paracompact, any exact sequence of sheaves induces a corresponding long exact sequence in Čech cohomology.

Lemma 3.7. Let $X$ be any topological space, and let

$$
\begin{equation*}
0 \rightarrow \mathcal{P}^{\prime} \rightarrow \mathcal{P} \rightarrow \mathcal{P}^{\prime \prime} \rightarrow 0 \tag{3.2}
\end{equation*}
$$

be an exact sequence of presheaves on $X$. Then one has a long exact sequence

$$
\begin{aligned}
& 0 \rightarrow H^{0}\left(X, \mathcal{P}^{\prime}\right) \rightarrow H^{0}(X, \mathcal{P}) \rightarrow H^{0}\left(X, \mathcal{P}^{\prime \prime}\right) \rightarrow H^{1}\left(X, \mathcal{P}^{\prime}\right) \rightarrow \ldots \\
& \rightarrow H^{k}\left(X, \mathcal{P}^{\prime}\right) \rightarrow H^{k}(X, \mathcal{P}) \rightarrow H^{k}\left(X, \mathcal{P}^{\prime \prime}\right) \rightarrow H^{k+1}\left(X, \mathcal{P}^{\prime}\right) \rightarrow \ldots
\end{aligned}
$$

Proof. For any open cover $\mathfrak{U}$ the exact sequence (3.2) induces an exact sequence of differential complexes

$$
0 \rightarrow \check{C}^{\bullet}\left(\mathfrak{U}, \mathcal{P}^{\prime}\right) \rightarrow \check{C}^{\bullet}(\mathfrak{U}, \mathcal{P}) \rightarrow \check{C}^{\bullet}\left(\mathfrak{U}, \mathcal{P}^{\prime \prime}\right) \rightarrow 0
$$

which induces the long cohomology sequence

$$
\begin{aligned}
& 0 \rightarrow H^{0}\left(\mathfrak{U}, \mathcal{P}^{\prime}\right) \rightarrow H^{0}(\mathfrak{U}, \mathcal{P}) \rightarrow H^{0}\left(\mathfrak{U}, \mathcal{P}^{\prime \prime}\right) \rightarrow H^{1}\left(\mathfrak{U}, \mathcal{P}^{\prime}\right) \rightarrow \ldots \\
& \rightarrow H^{k}\left(\mathfrak{U}, \mathcal{P}^{\prime}\right) \rightarrow H^{k}(\mathfrak{U}, \mathcal{P}) \rightarrow H^{k}\left(\mathfrak{U}, \mathcal{P}^{\prime \prime}\right) \rightarrow H^{k+1}\left(\mathfrak{U}, \mathcal{P}^{\prime}\right) \rightarrow \ldots
\end{aligned}
$$

Since the direct limit of a family of exact sequences yields an exact sequence, by taking the direct limit over the open covers of $X$ one obtains the required exact sequence.

LEMmA 3.8. Let $X$ be a paracompact topological space, $\mathcal{P}$ a presheaf on $X$ whose associated sheaf is the zero sheaf, let $\mathfrak{U}$ be an open cover of $X$, and let $\alpha \in \check{C}^{k}(\mathfrak{U}, \mathcal{P})$. There is a refinement $\mathfrak{W}$ of $\mathfrak{U}$ such that $\tau(\alpha)=0$, where $\tau: \check{C}^{k}(\mathfrak{U}, \mathcal{P}) \rightarrow \check{C}^{k}(\mathfrak{W}, \mathcal{P})$ is the morphism induced by restriction.

Proof. The proof relies on a standard paracompactness argument. See [13] §2.9.

Proposition 3.9. Let $\mathcal{P}$ be a presheaf on a paracompact space $X$, and let $\mathcal{P}^{\natural}$ be the associated sheaf. For all $k \geq 0$, the natural morphism $H^{k}(X, \mathcal{P}) \rightarrow H^{k}\left(X, \mathcal{P}^{\natural}\right)$ is an isomorphism.

Proof. One has an exact sequence of presheaves

$$
0 \rightarrow \mathcal{Q}_{1} \rightarrow \mathcal{P} \rightarrow \mathcal{P}^{\natural} \rightarrow \mathcal{Q}_{2} \rightarrow 0
$$

with

$$
\begin{equation*}
\mathcal{Q}_{1}^{\natural}=\mathcal{Q}_{2}^{\mathfrak{G}}=0 \tag{3.3}
\end{equation*}
$$

This gives rise to

$$
\begin{equation*}
0 \rightarrow \mathcal{Q}_{1} \rightarrow \mathcal{P} \rightarrow \mathcal{T} \rightarrow 0, \quad 0 \rightarrow \mathcal{T} \rightarrow \mathcal{P}^{\natural} \rightarrow \mathcal{Q}_{2} \rightarrow 0 \tag{3.4}
\end{equation*}
$$

where $\mathcal{T}$ is the quotient presheaf $\mathcal{P} / \mathcal{Q}_{1}$, i.e. the presheaf $U \rightarrow \mathcal{P}(U) / \mathcal{Q}_{1}(U)$. By Lemma 3.8 the isomorphisms (3.3) yield $H^{k}\left(X, \mathcal{Q}_{1}\right)=H^{k}\left(X, \mathcal{Q}_{2}\right)=0$. Then by taking the long exact sequences of cohomology from the exact sequences (3.4) we obtain the desired isomorphism.

Using these results we may eventually prove that on paracompact spaces one has long exact sequences in Čech cohomology.

Theorem 3.10. Let

$$
0 \rightarrow \mathcal{F}^{\prime} \rightarrow \mathcal{F} \rightarrow \mathcal{F}^{\prime \prime} \rightarrow 0
$$

be an exact sequence of sheaves on a paracompact space $X$. There is a long exact sequence of Čech cohomology groups

$$
\begin{aligned}
& 0 \rightarrow H^{0}\left(X, \mathcal{F}^{\prime}\right) \rightarrow H^{0}(X, \mathcal{F}) \rightarrow H^{0}\left(X, \mathcal{F}^{\prime \prime}\right) \rightarrow H^{1}\left(X, \mathcal{F}^{\prime}\right) \rightarrow \ldots \\
& \rightarrow H^{k}\left(X, \mathcal{F}^{\prime}\right) \rightarrow H^{k}(X, \mathcal{F}) \rightarrow H^{k}\left(X, \mathcal{F}^{\prime \prime}\right) \rightarrow H^{k+1}\left(X, \mathcal{F}^{\prime}\right) \rightarrow \ldots
\end{aligned}
$$

Proof. Let $\mathcal{P}$ be the quotient presheaf $\mathcal{F} / \mathcal{F}^{\prime} ;$ then $\mathcal{P}^{\natural} \simeq \mathcal{F}^{\prime \prime}$. One has an exact sequence of presheaves

$$
0 \rightarrow \mathcal{F}^{\prime} \rightarrow \mathcal{F} \rightarrow \mathcal{P} \rightarrow 0 .
$$

By taking the associated long exact sequence in cohomology (cf. Lemma 3.7) and using the isomorphism $H^{k}(X, \mathcal{P})=H^{k}\left(X, \mathcal{F}^{\prime \prime}\right)$ one obtains the required exact sequence.
2.4. Abstract de Rham theorem. We describe now a very useful way of computing cohomology groups; this result is sometimes called "abstract de Rham theorem." As a particular case it yields one form of the so-called de Rham theorem, which states that the de Rham cohomology of a differentiable manifold and the Čech cohomology of the constant sheaf $\mathbb{R}$ are isomorphic.

Definition 3.11. Let $\mathcal{F}$ be a sheaf of abelian groups on $X$. A resolution of $\mathcal{F}$ is a collection of sheaves of abelian groups $\left\{\mathcal{L}^{k}\right\}_{k \in \mathbb{N}}$ with morphisms $i: \mathcal{F} \rightarrow \mathcal{L}^{0}$, $d_{k}: \mathcal{L}^{k} \rightarrow$ $\mathcal{L}^{k+1}$ such that the sequence

$$
0 \rightarrow \mathcal{F} \xrightarrow{i} \mathcal{L}^{0} \xrightarrow{d_{0}} \mathcal{L}^{1} \xrightarrow{d_{1}} \ldots
$$

is exact. If the sheaves $\mathcal{L}^{\bullet}$ are acyclic (fine) the resolution is said to be acyclic (fine).
Lemma 3.12. If $0 \rightarrow \mathcal{F} \rightarrow \mathcal{L}^{\bullet}$ is a resolution, the morphism $i_{X}: \mathcal{F}(X) \rightarrow \mathcal{L}^{0}(X)$ is injective.

Proof. Let $\mathcal{Q}$ be the quotient $\mathcal{L}^{0} / \mathcal{F}$. Then the sequence of sheaves

$$
0 \rightarrow \mathcal{F} \rightarrow \mathcal{L}^{0} \rightarrow \mathcal{Q} \rightarrow 0
$$

is exact. By Exercise 3.7, the sequence of abelian groups

$$
0 \rightarrow \mathcal{F}(X) \rightarrow L^{0}(X) \rightarrow \mathcal{Q}(X)
$$

is exact. This implies the claim.
However the sequence of abelian groups

$$
0 \rightarrow \mathcal{L}^{0}(X) \xrightarrow{d_{0}} \mathcal{L}^{1}(X) \xrightarrow{d_{1}} \ldots
$$

is not exact. We shall consider its cohomology $H^{\bullet}\left(\mathcal{L}^{\bullet}(X), d\right)$. By the previous Lemma we have $H^{0}\left(\mathcal{L}^{\bullet}(X), d\right) \simeq H^{0}(X, \mathcal{F})$.

Theorem 3.13. If $0 \rightarrow \mathcal{F} \rightarrow \mathcal{L}^{\bullet}$ is an acyclic resolution there is an isomorphism $H^{k}(X, \mathcal{F}) \simeq H^{k}\left(\mathcal{L}^{\bullet}(X), d\right)$ for all $k \geq 0$.

Proof. Define $\mathcal{Q}^{k}=\operatorname{ker} d_{k}: \mathcal{L}^{k} \rightarrow \mathcal{L}^{k+1}$. The resolution may be split into

$$
0 \rightarrow \mathcal{F} \rightarrow \mathcal{L}_{X}^{0} \rightarrow \mathcal{Q}^{1} \rightarrow 0, \quad 0 \rightarrow \mathcal{Q}^{k} \rightarrow \mathcal{L}^{k} \rightarrow \mathcal{Q}^{k+1} \rightarrow 0 . \quad k \geq 1
$$

Since the sheaves $\mathcal{L}^{k}$ are acyclic by taking the long exact sequences of cohomology we obtain a chain of isomorphisms

$$
H^{k}(X, \mathcal{F}) \simeq H^{k-1}\left(X, \mathcal{Q}^{1}\right) \simeq \cdots \simeq H^{1}\left(X, \mathcal{Q}^{k-1}\right) \simeq \frac{H^{0}\left(X, \mathcal{Q}^{k}\right)}{\operatorname{Im} H^{0}\left(X, \mathcal{L}^{k-1}\right)}
$$

By Exercise $3.7 H^{0}\left(X, \mathcal{Q}^{k}\right)=\mathcal{Q}^{k}(X)$ is the kernel of $d_{k}: \mathcal{L}^{k}(X) \rightarrow \mathcal{L}^{k+1}$ so that the claim is proved.

Corollary 3.14. (de Rham theorem.) Let $X$ be a differentiable manifold. For all $k \geq 0$ the cohomology groups $H_{D R}^{k}(X)$ and $H^{k}(X, \mathbb{R})$ are isomorphic.

Proof. Let $n=\operatorname{dim} X$. The sequence

$$
\begin{equation*}
0 \rightarrow \mathbb{R} \rightarrow \Omega_{X}^{0} \xrightarrow{d} \Omega_{X}^{1} \xrightarrow{d} \cdots \rightarrow \Omega_{X}^{n} \rightarrow 0 \tag{3.5}
\end{equation*}
$$

(where $\Omega_{X}^{0} \equiv \mathcal{C}_{X}^{\infty}$ ) is exact (this is Poincaré's lemma). Moreover the sheaves $\Omega_{X}^{\bullet}$ are modules over the fine sheaf of rings $\mathcal{C}_{X}^{\infty}$, hence are acyclic. The claim then follows for the previous theorem.

Corollary 3.15. Let $U$ be a subset of a differentiable manifold $X$ which is diffeomorphic to $\mathbb{R}^{n}$. Then $H^{k}(U, \mathbb{R})=0$ for $k>0$.
2.5. Soft sheaves. For later use we also introduce and study the notion of soft sheaf. However, we do not give the proofs of most claims, for which the reader is referred to $[\mathbf{2}, \mathbf{5}, \mathbf{2 2}]$. The contents of this subsection will only be used in Section 4.5.

Definition 3.16. Let $\mathcal{F}$ be a sheaf $a$ on a topological space $X$, and let $U \subset X$ be a closed subset of $X$. The space $\mathcal{F}(U)$ (called "the space of sections of $\mathcal{F}$ over $U$ ") is defined as

$$
\mathcal{F}(U)={\underset{V \supset U}{\lim } \mathcal{F}(V), ~(x)}^{V \supset \longrightarrow}
$$

where the direct limit is taken over all open neighbourhoods $V$ of $U$.

A consequence of this definition is the existence of a natural restriction morphism $\mathcal{F}(X) \rightarrow \mathcal{F}(U)$.

Definition 3.17. A sheaf $\mathcal{F}$ is said to be soft if for every closed subset $U \subset X$ the restriction morphism $\mathcal{F}(X) \rightarrow \mathcal{F}(U)$ is surjective.

PROPOSITION 3.18. If $0 \rightarrow \mathcal{F}^{\prime} \rightarrow \mathcal{F} \rightarrow \mathcal{F}^{\prime \prime} \rightarrow 0$ is an exact sequence of soft sheaves on a paracompact space $X$, for every open subset $U \subset X$ the sequence of groups

$$
0 \rightarrow \mathcal{F}^{\prime}(U) \rightarrow \mathcal{F}(U) \rightarrow \mathcal{F}^{\prime \prime}(U) \rightarrow 0
$$

is exact.

Proof. One can e.g. adapt the proof of Proposition II.1.1 in [2].
Corollary 3.19. The quotient of two soft sheaves on a paracompact space is soft.
Proposition 3.20. Any soft sheaf of rings $\mathcal{R}$ on a paracompact space is fine.
Proof. Cf. Lemma II.3.4 in [2].
Proposition 3.21. Every sheaf $\mathcal{F}$ on a paracompact space admits soft resolutions.

Proof. Let $\mathcal{S}^{0}(\mathcal{F})$ be the sheaf of discontinuous sections of $\mathcal{F}$ (i.e., the sheaf of all sections of the sheaf space $(\underline{\mathcal{F}})$. The sheaf $\mathcal{S}^{0}(\mathcal{F})$ is obviously soft. Now we have an exact sequence $0 \rightarrow \mathcal{F} \rightarrow \mathcal{S}^{0}(\mathcal{F}) \rightarrow \mathcal{F}_{1} \rightarrow 0$. The sheaf $\mathcal{F}_{1}$ is not soft in general, but it may embedded into the soft sheaf $\mathcal{S}^{0}\left(\mathcal{F}_{1}\right)$, and we have an exact sequence $0 \rightarrow \mathcal{F}_{1} \rightarrow$ $\mathcal{S}^{0}\left(\mathcal{F}_{1}\right) \rightarrow \mathcal{F}_{2} \rightarrow 0$. Upon iteration we have exact sequences

$$
0 \rightarrow \mathcal{F}_{k} \xrightarrow{i_{k}} \mathcal{S}^{k}(\mathcal{F}) \xrightarrow{p_{k}} \mathcal{F}_{k+1} \rightarrow 0
$$

where $\mathcal{S}^{k}(\mathcal{F})=\mathcal{S}^{0}\left(\mathcal{F}_{k}\right)$. One can check that the sequence of sheaves

$$
0 \rightarrow \mathcal{F} \rightarrow \mathcal{S}^{0}(\mathcal{F}) \xrightarrow{f_{0}} \mathcal{S}^{1}(\mathcal{F}) \xrightarrow{f_{1}} \ldots
$$

(where $f_{k}=i_{k+1} \circ p_{k}$ ) is exact.
Proposition 3.22. If $\mathcal{F}$ is a sheaf on a paracompact space, the sheaf $\mathcal{S}^{0}(\mathcal{F})$ is acyclic.

Proof. The endomorphism sheaf $\mathcal{E} n d\left(\mathcal{S}^{0}(\mathcal{F})\right)$ is soft, hence fine by Proposition 3.20. Since $\mathcal{S}^{0}(\mathcal{F})$ is an $\mathcal{E} \operatorname{nd}\left(\mathcal{S}^{0}(\mathcal{F})\right)$-module, it is acyclic. ${ }^{6}$

Proposition 3.23. On a paracompact space soft sheaves are acyclic.
Proof. If $\mathcal{F}$ is a soft sheaf, the sequence $0 \rightarrow \mathcal{F}(X) \rightarrow \mathcal{S}^{0} \mathcal{F}(X) \rightarrow \mathcal{F}_{1}(X) \rightarrow 0$ obtained from $0 \rightarrow \mathcal{F} \rightarrow \mathcal{S}^{0} \mathcal{F} \rightarrow \mathcal{F}_{1} \rightarrow 0$ is exact (Proposition 3.18). Since $\mathcal{F}$ and $\mathcal{S}^{0} \mathcal{F}$ are soft, so is $\mathcal{F}_{1}$ by Corollary 3.19, and the sequence $0 \rightarrow \mathcal{F}_{1}(X) \rightarrow \mathcal{S}^{1} \mathcal{F}(X) \rightarrow$ $\mathcal{F}_{2}(X) \rightarrow 0$ is also exact. With this procedure we can show that the complex $\mathcal{S}^{\bullet}(\mathcal{F})(X)$ is exact. But since all sheaves $\mathcal{S}^{\bullet}(\mathcal{F})$ are acyclic by the previous Proposition, by the abstract de Rham theorem the claim is proved.

Note that in this way we have shown that for any sheaf $\mathcal{F}$ on a paracompact space there is a canonical soft resolution.

[^10]2.6. Leray's theorem for Čech cohomology. If an open cover $\mathfrak{U}$ of a topological space $X$ is suitably chosen, the Čech cohomologies $H^{\bullet}(\mathfrak{U}, \mathcal{F})$ and $H^{\bullet}(X, \mathcal{F})$ are isomorphic. Leray's theorem establishes a sufficient condition for such an isomorphism to hold. Since the cohomology $H^{\bullet}(\mathfrak{U}, \mathcal{F})$ is in generally much easier to compute, this turns out to be a very useful tool in the computation of Čech cohomology groups.

We say that an open cover $\mathfrak{U}=\left\{U_{i}\right\}_{i \in I}$ of a topological space $X$ is acyclic for a sheaf $\mathcal{F}$ if $H^{k}\left(U_{i_{0} \ldots i_{p}}, \mathcal{F}\right)=0$ for all $k>0$ and all nonvoid intersections $U_{i_{0} \ldots i_{p}}=U_{i_{0}} \cap \cdots \cap U_{i_{p}}$, $i_{0} \ldots i_{p} \in I$.

Theorem 3.24. (Leray's theorem) Let $\mathcal{F}$ be a sheaf on a paracompact space $X$, and let $\mathfrak{U}$ be an open cover of $X$ which is acyclic for $\mathcal{F}$ and is indexed by an ordered set. Then, for all $k \geq 0$, the cohomology groups $H^{k}(\mathfrak{U}, \mathcal{F})$ and $H^{k}(X, \mathcal{F})$ are isomorphic.

To prove this theorem we need to construct the so-called Čech sheaf complex. For every nonvoid intersection $U_{i_{0} \ldots i_{p}}$ let $j_{i_{0} \ldots i_{p}}: U_{i_{0} \ldots i_{p}} \rightarrow X$ be the inclusion. For every $p$ define the sheaf

$$
\check{\mathcal{C}}^{p}(\mathfrak{U}, \mathcal{F})=\prod_{i_{0}<\cdots<i_{p}}\left(j_{i_{0} \ldots i_{p}}\right)_{*} \mathcal{F}_{\mid U_{i_{0} \ldots i_{p}}}
$$

(every factor $\left(j_{i_{0} \ldots i_{p}}\right)_{*} \mathcal{F}_{\mid U_{i_{0} \ldots i_{p}}}$ is the sheaf $\mathcal{F}$ first restricted to $U_{i_{0} \ldots i_{p}}$ and the exteded by zero to the whole of $X$ ). The Čech differential induces sheaf morphisms $\delta: \check{\mathcal{C}}^{p}(\mathfrak{U}, \mathcal{F}) \rightarrow$ $\breve{\mathcal{C}}^{p+1}(\mathfrak{U}, \mathcal{F})$. From the definition, we get isomorphisms

$$
\begin{equation*}
\check{\mathcal{C}}^{p}(\mathfrak{U}, \mathcal{F})(X) \simeq \check{C}^{p}(\mathfrak{U}, \mathcal{F}) \tag{3.6}
\end{equation*}
$$

i.e., by taking global sections of the Čech sheaf complex we get the Čech cochain group complex. Moreover we have:

Lemma 3.25. For all $p$ and $k$,

$$
H^{k}\left(X, \breve{\mathcal{C}}^{p}(\mathfrak{U}, \mathcal{F})\right) \simeq \prod_{i_{0}<\cdots<i_{p}} H^{k}\left(U_{i_{0} \ldots i_{p}}, \mathcal{F}\right)
$$

We may now prove Leray's theorem. It is not difficult to prove that the complex $\breve{\mathcal{C}}^{\bullet}(\mathfrak{U}, \mathcal{F})$ is a resolution of $\mathcal{F}$ (cf. [2], Prop. II.3.3). Under the hypothesis of Leray's theorem, by Lemma 3.25 this resolution is acyclic. By the abstract de Rham theorem, the cohomology of the global sections of the resolution is isomorphic to the cohomology of $\mathcal{F}$. But, due to the isomorphisms (3.6), the cohomology of the global sections of the resolution is the cohomology $H^{\bullet}(\mathfrak{U}, \mathcal{F})$.
2.7. Good covers. By means of Leray's theorem we may reduce the problem of computing the Čech cohomology of a differentiable manifold with coefficients in the constant sheaf $\mathbb{R}$ (which, via de Rham theorem, amounts to computing its de Rham cohomology) to the computation of the cohomology of a cover with coefficients in $\mathbb{R}$; thus a problem which in principle would need the solution of differential equations on
topologically nontrivial manifolds is reduced to a simpler problem which only involves the intersection pattern of the open sets of a cover.

Definition 3.26. A locally finite open cover $\mathfrak{U}$ of a differentiable manifold is good if all nonempty intersections of its members are diffeomorphic to $\mathbb{R}^{n}$.

Good covers exist on any differentiable manifold (cf. [17]). Due to Corollary 3.15, good covers are acyclic for the constant sheaf $\mathbb{R}$. We have therefore

Proposition 3.27. For any good cover $\mathfrak{U}$ of a differentiable manifold $X$ one has isomorphisms

$$
H^{k}(\mathfrak{U}, \mathbb{R}) \simeq H^{k}(X, \mathbb{R}), \quad k \geq 0
$$

The cover of Example 3.2 was good, so we computed there the de Rham cohomology of the circle $S^{1}$.
2.8. Flabby sheaves. Another kind of sheaves which can be introduced is that of flabby sheaves (also called "flasque"). A sheaf $\mathcal{F}$ on a topological space $X$ is said to be flabby if for every open subset $U \subset X$ the restriction morphism $\mathcal{F}(X) \rightarrow \mathcal{F}(U)$ is surjective. It is easy to prove that flabby sheaves are soft: if $U \subset X$ is a closed subset, by definition of direct limit, for every $s \in \mathcal{F}(U)$ there is an open neighbourhood $V$ of $U$ and a section $s^{\prime} \in \mathcal{F}(V)$ which restricts to $s$. Since $\mathcal{F}$ is flabby, $s^{\prime}$ can be extended to the whole of $X$. So on a paracompact space, flabby sheaves are acyclic, and by the abstract de Rham theorem flabby resolution can be used to compute cohomology. We should also notice that the canonical soft resolution $\mathcal{S}^{\bullet}(\mathcal{F})$ we constructed in Section 2.5 is flabby, as one can easily check by the definition itself.

One can further pursue this line and use flabby resolutions (for instance, the canonical flabby resolution of Section 2.5) to define cohomology. That is, for every sheaf $\mathcal{F}$, its cohomology is by definition the cohomology of the global sections of its canonical flabby resolution (it then turns out that cohomology can be computed with any acyclic resolution). This has the advantage of producing a cohomology theory (called sheaf cohomology) which is bell-behaved (e.g., it has long exact sequences in cohomology) on every topological space, not just on paracompact ones. In this connection the reader may refer to $[\mathbf{5}, \mathbf{4}, \mathbf{2}]$, or to $[\mathbf{2 0}]$ where a different and more general approach to sheaf cohomology (using injective resolutions) is pursued; also the original paper by Grothendieck [8] can be fruitfully read. It follows from our treatment that on a paracompact topological space the sheaf and Čech cohomology coincide, but in general they do not (cf. [11], especially the exercise section, for a discussion of the comparison between the two cohomologies).
2.9. Comparison with other cohomologies. In algebraic topology one attaches to a topological space $X$ several cohomologies with coefficients in an abelian group $G$. Loosely speaking, whenever $X$ is paracompact and locally Euclidean, all these cohomologies coincide with the Coch cohomology of $X$ with coefficients in the constant sheaf $G$. In particular, we have the following result:

Proposition 3.28. Let $X$ be a paracompact locally Euclidean topological space, and let $G$ be an abelian group. The singular cohomology of $X$ with coefficients in $G$ is isomorphic to the Čech cohomology of $X$ with coefficients in the constant sheaf $G$.

## CHAPTER 4

## Spectral sequences

Spectral sequences are a powerful tool for computing homology, cohomology and homotopy groups. Often they allow one to trade a difficult computation for an easier one. Examples that we shall consider are another proof of the Čech-de Rham theorem, the Leray spectral sequence, and the Künneth theorem.

Spectral sequences are a difficult topic, basically because the theory is quite intrincate and the notation is correspondingly cumbersome. Therefore we have chosen what seems to us to be the simplest approach, due to Massey [18]. Our treatment basically follows [3].

## 1. Filtered complexes

Let $(K, d)$ be a graded differential module, i.e.,

$$
K=\bigoplus_{n \in \mathbb{Z}} K^{n}, \quad d: K^{n} \rightarrow K^{n+1}, \quad d^{2}=0
$$

A graded submodule of $(K, d)$ is a graded subgroup $K^{\prime} \subset K$ such that $d K^{\prime} \subset K^{\prime}$, i.e.,

$$
K^{\prime}=\bigoplus_{n \in \mathbb{Z}} K^{\prime n}, \quad K^{\prime n} \subset K^{n}, \quad d: K^{\prime n} \rightarrow K^{\prime n+1}
$$

A sequence of nested graded submodules

$$
K=K_{0} \supset K_{1} \supset K_{2} \supset \ldots
$$

is a filtration of $(K, d)$. We then say that $(K, d)$ is filtered, and associate with it the graded complex ${ }^{1}$

$$
\operatorname{Gr}(K)=\bigoplus_{p \in \mathbb{Z}} K_{p} / K_{p+1}, \quad K_{p}=K \text { if } p \leq 0
$$

Note that by assumption (since every $K_{p+1}$ is a graded subgroup of $K_{p}$ ) the filtration is compatible with the grading, i.e., if we define $K_{p}^{i}=K^{i} \cap K_{p}$, then

$$
\begin{equation*}
K^{n}=K_{0}^{n} \supset K_{1}^{n} \supset K_{2}^{n} \supset \ldots \tag{4.1}
\end{equation*}
$$

is a filtration of $K^{i}$, and moreover $d K_{p}^{n} \subset K_{p}^{n+1}$.

[^11]Example 4.1. A double complex is a collection of abelian groups $K^{p, q}$, with $p, q \geq$ $0,{ }^{2}$ and morphisms $\delta_{1}: K^{p, q} \rightarrow K^{p+1, q}, \delta_{2}: K^{p, q} \rightarrow K^{p, q+1}$ such that

$$
\delta_{1}^{2}=\delta_{2}^{2}=0, \quad \delta_{1} \delta_{2}+\delta_{2} \delta_{1}=0 .
$$

Let $(T, d)$ be the associated total complex:

$$
T^{i}=\bigoplus_{p+q=i} K^{p, q}, \quad d: T^{i} \rightarrow T^{i+1} \text { defined by } d=\delta_{1}+\delta_{2}
$$

(note that the definition of $d$ implies $d^{2}=0$ ). Then letting

$$
T_{p}=\bigoplus_{i \geq p, q \geq 0} K^{i, q}
$$

we obtain a filtration of $(T, d)$. This satifies $T_{p} \simeq T$ for $p \leq 0$. The successive quotients of the filtration are

$$
T_{p} / T_{p+1}=\bigoplus_{q \in \mathbb{N}} K^{p, q} .
$$

Definition 4.2. A filtration $K_{\bullet}$ of $(K, d)$ is said to be regular if for every $i \geq 0$ the filtration (4.1) is finite; in other words, for every $i$ there is a number $\ell(i)$ such that $K_{p}^{i}=0$ for $p>\ell(i)$.

For instance, the filtration in Example 4.1 is regular since $T_{p}^{i}=0$ for $p>i$, and indeed

$$
T_{p}^{i}=T^{i} \cap T_{p}=\bigoplus_{j=0}^{i-p} K^{i-j, j}
$$

## 2. The spectral sequence of a filtered complex

At first we shall not consider the grading. Let $K$ • be a filtration of a differential module ( $K, d$ ), and let

$$
G=\bigoplus_{p \in \mathbb{Z}} K_{p}
$$

The inclusions $K_{p+1} \rightarrow K_{p}$ induce a morphism $i: G \rightarrow G$ ("the shift by the filtering degree"), and one has an exact sequence

$$
\begin{equation*}
0 \rightarrow G \xrightarrow{i} G \xrightarrow{j} E \rightarrow 0 \tag{4.2}
\end{equation*}
$$

[^12]with $E \simeq \operatorname{Gr}(K)$. The differential $d$ induces differentials in $G$ and $E$, so that from (4.2) one gets an exact triangle in cohomology (cf. Section 1.1)

where $k$ is the connecting morphism.
Let us now assume that the filtration $K_{\bullet}$ has finite length, i.e., $K_{p}=0$ for $p$ greater than some $\ell$ (called the length of the filtration).

Since $d K_{p} \subset K_{p}$ for every $p$, we may consider the cohomology groups $H\left(K_{p}\right)$. The morphism $i$ induces morphisms $i: H\left(K_{p+1}\right) \rightarrow H\left(K_{p}\right)$. Define $G_{1}$ to be the direct sum of the terms on the sequence (which is not exact)

$$
\begin{aligned}
0 \rightarrow H\left(K_{\ell}\right) \stackrel{i}{\longrightarrow} & H\left(K_{\ell-1}\right) \xrightarrow{i} \\
& \xrightarrow{i} H\left(K_{1}\right) \xrightarrow{i} H(K) \xrightarrow{\sim} H\left(K_{-1}\right) \xrightarrow{\sim} \ldots,
\end{aligned}
$$

i.e., $G_{1}=\bigoplus_{p \in \mathbb{Z}} H\left(K_{p}\right) \simeq H(G)$. Next we define $G_{2}$ as the sum of the terms of the sequence

$$
\begin{aligned}
\left.0 \rightarrow i\left(H\left(K_{\ell}\right)\right)\right) \rightarrow i\left(H\left(K_{\ell-1}\right)\right) \rightarrow \ldots & \\
& \rightarrow i\left(H\left(K_{1}\right)\right) \rightarrow H(K) \xrightarrow{\sim} H\left(K_{-1}\right) \xrightarrow{\sim} \ldots
\end{aligned}
$$

Note that the morphism $i\left(H\left(K_{1}\right)\right) \rightarrow H(K)$ is injective, since it is the inclusion of the image of $i: H\left(K_{1}\right) \rightarrow H(K)$ into $H(K)$. This procedure is then iterated: $G_{3}$ is the sum of the terms in the sequence

$$
\begin{aligned}
\left.0 \rightarrow i\left(i\left(H\left(K_{\ell}\right)\right)\right)\right) \rightarrow i\left(i\left(H\left(K_{\ell-1}\right)\right)\right) & \rightarrow i\left(i\left(H\left(K_{2}\right)\right)\right. \\
& \rightarrow i\left(H\left(K_{1}\right)\right) \rightarrow H(K) \xrightarrow{\sim} H\left(K_{-1}\right) \xrightarrow{\sim} \ldots
\end{aligned}
$$

and now the morphisms $i\left(i\left(H\left(K_{2}\right)\right) \rightarrow i\left(H\left(K_{1}\right)\right)\right.$ and $i\left(H\left(K_{1}\right)\right) \rightarrow H(K)$ are injective. When we reach the step $\ell$, all the morphisms in the sequence

$$
\begin{aligned}
\left.0 \rightarrow i^{\ell}\left(H\left(K_{\ell}\right)\right)\right) \rightarrow i^{\ell-1}\left(H\left(K_{\ell-1}\right)\right) \rightarrow & \ldots \\
& \rightarrow i\left(H\left(K_{1}\right)\right) \rightarrow H(K) \xrightarrow{\sim} H\left(K_{-1}\right) \xrightarrow{\sim} \ldots
\end{aligned}
$$

are injective, so that $G_{\ell+2} \simeq G_{\ell+1}$, and the procedure stabilizes: $G_{r} \simeq G_{r+1}$ for $r \geq \ell+1$. We define $G_{\infty}=G_{\ell+1}$; we have

$$
G_{\infty} \simeq \bigoplus_{p \in \mathbb{Z}} F_{p}
$$

where $F_{p}=i^{p}\left(H\left(K_{p}\right)\right)$, i.e., $F_{p}$ is the image of $H\left(K_{p}\right)$ into $H(K)$. The groups $F_{p}$ provide a filtration of $H(K)$,

$$
\begin{equation*}
H(K)=F_{0} \supset F_{1} \supset \cdots \supset F_{\ell} \supset F_{\ell+1}=0 . \tag{4.4}
\end{equation*}
$$

We come now to the construction of the spectral sequence. Recall that since $d K_{p} \subset$ $K_{p}$, and $E=\bigoplus_{p} K_{p} / K_{p+1}$, the differential $d$ acts on $E$, and one has a cohomology group $H(E)$ wich splits into a direct sum

$$
H(E) \simeq \bigoplus_{p \in \mathbb{Z}} H\left(K_{p} / K_{p+1}, d\right)
$$

The cohomology group $H(E)$ fits into the exact triangle (4.3), that we rewrite as

where $E_{1}=H(E)$. We define $d_{1}: E_{1} \rightarrow E_{1}$ by letting $d_{1}=j_{1} \circ k_{1}$; then $d_{1}^{2}=0$ since the triangle is exact. Let $E_{2}=H\left(E_{1}, d_{1}\right)$ and recall that $G_{2}$ is the image of $G_{1}$ under $i: G_{1} \rightarrow G_{1}$. We have morphisms

$$
i_{2}: G_{2} \rightarrow G_{2},, \quad j_{2}: G_{2} \rightarrow E_{2}, \quad k_{2}: E_{2} \rightarrow G_{2}
$$

where
(i) $i_{2}$ is induced by $i_{1}$ by letting $i_{2}\left(i_{1}(x)\right)=i_{1}\left(i_{1}(x)\right)$ for $x \in G_{1}$;
(ii) $j_{2}$ is induced by $j_{1}$ by letting $j_{2}\left(i_{1}(x)\right)=\left[j_{1}(x)\right]$ for $x \in G_{1}$, where [] denotes taking the homology class in $E_{2}=H\left(E_{1}, d_{1}\right)$.
(iii) $k_{2}$ is induced by $k_{1}$ by letting $k_{2}([e])=i_{1}\left(k_{1}(e)\right)$.

Exercise 4.1. Show that the morphisms $j_{2}$ and $k_{2}$ are well defined, and that the triangle

is exact.
We call (4.6) the derived triangle of (4.5). The procedure leading from (4.5) to the triangle (4.6) can be iterated, and we get a sequence of exact triangles

where each group $E_{r}$ is the cohomology group of the differential module $\left(E_{r-1}, d_{r-1}\right)$, with $d_{r-1}=j_{r-1} \circ k_{r-1}$.

As we have already noticed, due to the assumption that the filtration $K_{\bullet}$ has finite length $\ell$, the groups $G_{r}$ stabilize when $r \geq \ell+1$, and the morphisms $i_{r}: G_{r} \rightarrow G_{r}$
become injective. Thus all morphisms $k_{r}: E_{r} \rightarrow G_{r}$ vanish in that range, which implies $d_{r}=0$, so that the groups $E_{r}$ stabilize as well: $E_{r+1} \simeq E_{r}$ for $r \geq \ell+1$. We denote by $E_{\infty}=E_{\ell+1}$ the stable value.

Thus, the sequence

$$
0 \rightarrow G_{\infty} \xrightarrow{i_{\infty}} G_{\infty} \rightarrow E_{\infty} \rightarrow 0
$$

is exact, which implies that $E_{\infty}$ is the associated graded module of the filtration (4.4) of $H(K)$ :

$$
E_{\infty} \simeq \bigoplus_{p \leq \ell} F_{p} / F_{p+1}
$$

Definition 4.2. A sequence of differential modules $\left\{\left(E_{r}, d_{r}\right)\right\}$ such that $H\left(E_{r}, d_{r}\right)$ $\simeq E_{r+1}$ is said to be a spectral sequence. If the groups $E_{r}$ eventually become stationary, we denote the stationary value by $E_{\infty}$. If $E_{\infty}$ is isomorphic to the associated graded module of some filtered group $H$, we say that the spectral sequence converges to $H$.

So what we have seen so far in this section is that if $(K, d)$ is a differential module with a filtration of finite length, one can build a spectral sequence which converges to $H(K)$.

Remark 4.3. It may happen in special cases that the groups $E_{r}$ stabilize before getting the value $r=\ell+1$. That happens if and only if $d_{r}=0$ for some value $r=r_{0}$. This implies that $d_{r}=0$ also for $r>r_{0}$, and $E_{r+1} \simeq E_{r}$ for all $r \geq r_{0}$. When this happens we say that the spectral sequence degenerates at step $r_{0}$.

Now we consider the presence of a grading.
Theorem 4.4. Let $(K, d)$ be a graded differential module, and $K$ • a regular filtration. There is a spectral sequence $\left\{\left(E_{r}, d_{r}\right)\right\}$, where each $E_{r}$ is graded, which converges to the graded group $H^{\bullet}(K, d)$.

Note that the filtration need not be of finite length: the length $\ell(i)$ of the filtration of $K^{i}$ is finite for every $i$, but may increase with $i$.

Proof. For every $n$ and $p$ we have $d\left(K_{p}^{n}\right) \subset K_{p}^{n+1}$, therefore we have cohomology groups $H^{n}\left(K_{p}\right)$. As a consequence, the groups $G_{r}$ are graded:

$$
G_{r} \simeq \bigoplus_{n \in \mathbb{Z}} F_{r}^{n}=\bigoplus_{n, p \in \mathbb{Z}} i^{r-1}\left(H^{n}\left(K_{p}\right)\right)
$$

and the groups $E_{r}$ are accordingly graded. We may construct the derived triangles as before, but now we should pay attention to the grading: the morphisms $i$ and $j$ have degree zero, but $k$ has degree one (just check the definition: $k$ is basically a connecting morphism).

Fix a natural number $n$, and let $r \geq \ell(n+1)+1$; for every $p$ the morphisms

$$
i_{r}: F_{r}^{n+1} \rightarrow F_{r}^{n+1}
$$

are injective, and the morphisms

$$
k_{r}: E_{r}^{n} \rightarrow F_{r}^{n+1}
$$

are zero. These are the same statements as in the ungraded case. Therefore, as it happened in the ungraded case, the groups $E_{r}^{n}$ become stationary for $r$ big enough. Note that $G_{\infty}^{n}=\oplus_{p \in \mathbb{Z}} F_{p}^{n}$, where $F_{p+1}^{n+1}=i^{\ell(n+1)}\left(H^{n+1}\left(K_{p+1}\right)\right)$, and that the morphism $i_{\infty}$ sends $F_{p+1}^{n}$ injectively into $F_{p}^{n}$ for every $n$, and there is an exact sequence

$$
0 \rightarrow G_{\infty}^{n} \xrightarrow{i_{\infty}} G_{\infty}^{n} \rightarrow E_{\infty}^{n} \rightarrow 0
$$

This implies that $E_{r}$ is the graded module associated with the graded complex $H^{\bullet}(K, d)$.

The last statement in the proof means that for each $n, F_{\bullet}^{n}$ is a filtration of $H^{n}(K, d)$, and $E_{\infty}^{n} \simeq \bigoplus_{p \in \mathbb{Z}} F_{p}^{n} / F_{p+1}^{n}$.

## 3. The bidegree and the five-term sequence

The terms $E_{r}$ of the spectral sequence are actually bigraded; for instance, since the filtration and the degree of $K$ are compatible, we have

$$
K_{p} / K_{p+1} \simeq \bigoplus_{q \in \mathbb{Z}} K_{p}^{q} / K_{p+1}^{q} \simeq \bigoplus_{q \in \mathbb{Z}} K_{p}^{p+q} / K_{p+1}^{p+q}
$$

and $E_{0}=E$ is bigraded by

$$
E_{0}=\bigoplus_{p, q \in \mathbb{Z}} E_{0}^{p, q} \quad \text { with } \quad E_{0}^{p, q}=K_{p}^{p+q} / K_{p+1}^{p+q}
$$

Note that the total complex associated with this bidegree yields the gradation of $E$.
Let us go to next step. Since $d: K_{p}^{p+q} \rightarrow K_{p}^{p+q+1}$, i.e., $d: E_{0}^{p, q} \rightarrow E_{0}^{p, q+1}$, and $E_{1}=H(E, d)$, if we set

$$
E_{1}^{p, q}=H^{q}\left(E_{0}^{p, \bullet}, d\right) \simeq H^{p+q}\left(K_{p} / K_{p+1}\right)
$$

we have $E_{1} \simeq \bigoplus_{p, q \in \mathbb{Z}} E_{1}^{p, q}$.
If we go one step further we can show that

$$
d_{1}: E_{1}^{p, q} \rightarrow E_{1}^{p+1, q} .
$$

Indeed if $x \in E_{1}^{p, q} \simeq H^{p+q}\left(K_{p} / K_{p+1}\right)$ we write $x$ as $x=[e]$ where $e \in K_{p}^{p+q} / K_{p+1}^{p+q}$ so that $k_{1}(x)=i_{1}(k(e)) \in H^{p+q+1}\left(K_{p+1}\right)$ and

$$
d_{1}(x)=j_{1}\left(k_{1}(x)\right)=j_{1}(k(e)) \in H^{p+q+1}\left(K_{p+1} / K_{p+2}\right) \simeq E_{1}^{p+1, q} .
$$

As a result we have $E_{2} \simeq \bigoplus_{p, q \in \mathbb{Z}} E_{2}^{p, q}$ with

$$
E_{2}^{p, q} \simeq H^{p}\left(E_{1}^{\bullet, q}, d_{1}\right) .
$$

The same analysis shows that in general $E_{r} \simeq \bigoplus_{p, q \in \mathbb{Z}} E_{r}^{p, q}$ with

$$
d_{r}: E_{r}^{p, q} \rightarrow E_{r}^{p+r, q-r+1}
$$

and moreover we have

$$
E_{\infty}^{p, q} \simeq F_{p}^{p+q} / F_{p+1}^{p+q}
$$

The next two Lemmas establish the existence of the morphisms that we shall use to introduce the so-called five-term sequence, and will anyway be useful in the following.

Lemma 4.1. There are canonical morphisms $H^{q}(K) \rightarrow E_{r}^{0, q}$.
Proof. Since $K_{p} \simeq K$ for $p \leq 0$ we have $F_{p}^{n} \simeq H^{n}\left(K_{p}\right)=H^{n}(K)$ for $p \leq 0$, hence $E_{\infty}^{p, q}=0$ for $p<0$ and $E_{\infty}^{0, q} \simeq F_{0}^{q} / F_{1}^{q} \simeq H^{q}(K) / F_{1}^{q}$, so that there is a surjective morphism $H^{q}(K) \rightarrow E_{\infty}^{0, q}$.

Note now that a nonzero class in $E_{r}^{0, q}$ cannot be a boundary, since then it should come from $E_{r}^{-r, q+r-1}=0$. So cohomology classes are cycles. Since cohomology classes are elements in $E_{r+1}^{0, q}$, we have inclusions $E_{r+1}^{0, q} \subset E_{r}^{0, q}\left(E_{r+1}^{0, q}\right.$ is the subgroup of cycles in $\left.E_{r}^{0, q}\right)$. This yields an inclusion $E_{\infty}^{0, q} \subset E_{r}^{0, q}$ for all $r$.

Combining the two arguments we obtain morphisms $H^{q}(K) \rightarrow E_{r}^{0, q}$.
LEMMA 4.2. Assume that $K_{p}^{n}=0$ if $p>n$ (so, in particular, the filtration is regular). Then for every $r \geq 2$ there is a morphism $E_{r}^{p, 0} \rightarrow H^{p}(K)$.

Proof. The hypothesis of the Lemma implies that $E_{r}^{p, q}=0$ for $q<0$ (indeed, $F_{p}^{p+q}=i^{r}\left(H^{p+q}\left(K_{p}\right)\right)$ for $r$ big enough, so that $F_{q}^{p+q}=0$ if $q<0$ since then $\left.K_{p}^{p+1}=0\right)$. As a consequence, for $r \geq 2$ the differential $d_{r}: E_{r}^{p, 0} \rightarrow E_{r}^{p+r, 1-r}$ maps to zero, i.e., all elements in $E_{r}^{p, 0}$ are cycles, and determine cohomology classes in $E_{r+1}^{p, 0}$. This means we have a morphism $E_{r}^{p, 0} \rightarrow E_{r+1}^{p, 0}$, and composing, morphisms $E_{r}^{p, 0} \rightarrow E_{\infty}^{p, 0}$.

Since $F_{p}^{n}=0$ for $p>n$ we have $E_{\infty}^{p, 0} \simeq F_{p}^{p} / F_{p+1}^{p} \simeq F_{p}^{p}$ so that one has an injective morphism $E_{\infty}^{p, 0} \rightarrow H^{p}(K)$. Composing we have a morphism $E_{r}^{p, 0} \rightarrow H^{p}(K)$.

Proposition 4.3. (The five-term sequence). Assume that $K_{p}^{n}=0$ if $p>n$. There is an exact sequence

$$
0 \rightarrow E_{2}^{1,0} \rightarrow H^{1}(K) \rightarrow E_{2}^{0,1} \xrightarrow{d_{2}} E_{2}^{2,0} \rightarrow H^{2}(K)
$$

Proof. The morphisms involved in the sequence in addition to $d_{2}$ have been defined in the previous two Lemmas. We shall not prove the exactness of the sequence here, for a proof cf. e.g. [5].

## 4. The spectral sequences associated with a double complex

In this Section we consider a double complex as we have defined in Example 4.1. Due to the presence of the bidegree, the result in Theorem 4.4 may be somehow refined.

We shall use the notation in Example 4.1. The group

$$
G=\bigoplus_{p \in \mathbb{Z}} T_{p}=\bigoplus_{p \in \mathbb{Z}} \bigoplus_{n \geq p, q \in \mathbb{N}} K^{i, q}
$$

has natural gradation $G=\oplus_{n \in \mathbb{Z}} G^{n}$ given by

$$
\begin{equation*}
G^{n}=\bigoplus_{p \in \mathbb{Z}} T_{p}^{n} \simeq \bigoplus_{p \in \mathbb{Z}} \bigoplus_{j=0}^{n-p} K^{n-j, j} \tag{4.7}
\end{equation*}
$$

but it also bigraded, with bidegree

$$
G^{p, q}=T_{q}^{p+q}
$$

Notice that if we form the total complex $\bigoplus_{p+q=n} G^{p, q}$ we obtain the complex (4.7) back:

$$
\bigoplus_{p+q=n} G^{p, q} \simeq \bigoplus_{p+q=n} \bigoplus_{j=0}^{q} K^{p+q-j, j}=\bigoplus_{j=0}^{n-p} K^{n-j, j}=G^{n}
$$

The operators $\delta_{1}, \delta_{2}$ and $d=\delta_{1}+\delta_{2}$ act on $G$ :

$$
\delta_{1}: G^{n, q} \rightarrow G^{n+1, q}, \quad \delta_{2}=G^{n, q} \rightarrow G^{n, q+1}, \quad d: G^{k} \rightarrow G^{k+1}
$$

We analyze the spectral sequence associated with these data. The first three terms are easily described. One has

$$
E_{0}^{p, q} \simeq T_{p}^{p+q} / T_{p+1}^{p+q} \simeq K^{p, q}
$$

so that the differential $d_{0}: E_{0}^{p, q} \rightarrow E_{0}^{p, q+1}$ coincides with $\delta_{2}: K^{p, q} \rightarrow K^{p, q+1}$, and one has

$$
\begin{equation*}
E_{1}^{p, q} \simeq H^{q}\left(K^{p, \bullet}, \delta_{2}\right) \tag{4.8}
\end{equation*}
$$

At next step we have $d_{1}: E_{1}^{p, q} \rightarrow E_{1}^{p+1, q}$ with $E_{1}^{p, q} \simeq H^{p+q}\left(T_{p} / T_{p+1}\right)$ and $T_{p} / T_{p+1} \simeq$ $\bigoplus_{q \in Z} K^{p, q}$. Hence the differential

$$
d_{1}: H^{p+q}\left(\bigoplus_{n \in \mathbb{Z}} K^{p, n}\right) \rightarrow H^{p+q+1}\left(\bigoplus_{n \in Z} K^{p+1, n}\right)
$$

is identified with $\delta_{1}$, and

$$
\begin{equation*}
E_{2}^{p, q} \simeq H^{p}\left(E_{1}^{\bullet, q}, \delta_{1}\right) \tag{4.9}
\end{equation*}
$$

One should notice that by exchanging the two degrees in $K$ (i.e., considering another double complex ' $K$ such that ${ }^{\prime} K^{p, q}=K^{q, p}$ ), we obtain another spectral sequence, that we denote by $\left\{{ }^{\prime} E_{r},{ }^{\prime} d_{r}\right\}$. Both sequences converge to the same graded group, i.e., the cohomology of the total complex (but the corresponding filtrations are in general different), and this often provides interesting information. For the second spectral sequence we get

$$
\begin{align*}
{ }^{\prime} E_{1}^{q, p} & \simeq H^{p}\left(K^{\bullet, q}, \delta_{1}\right)  \tag{4.10}\\
{ }^{\prime} E_{2}^{q, p} & \simeq H^{q}\left({ }^{\prime} E_{1}^{p, \bullet}, \delta_{2}\right) \tag{4.11}
\end{align*}
$$

Example 4.1. A simple application of the two spectral sequences associated with a double complex provides another proof of the Čech-de Rham theorem, i.e., the isomorphism $H^{\bullet}(X, \mathbb{R}) \simeq H_{D R}^{\bullet}(X)$ for a differentiable manifold $X$. Let $\mathfrak{U}=\left\{U_{i}\right\}$ be a good cover of $X$, and define the double complex

$$
K^{p, q}=\check{C}^{p}\left(\mathfrak{U}, \Omega^{q}\right),
$$

i.e., $K^{\bullet, q}$ is the complex of Čech cochains of $\mathfrak{U}$ with coefficients in the sheaf of differential $q$-forms. The first differential $\delta_{1}$ is basically the Čech differential $\delta$, while $\delta_{2}$ is the exterior differential $d .{ }^{3}$ Actually $\delta$ and $d$ commute rather than anticommute, but this is easily settled by defining the action of $\delta_{1}$ on $K^{p, q}$ as $\delta_{1}=(-1)^{q} \delta$ (this of course leaves the spaces of boundaries and cycles unchanged). We start analyzing the spectral sequences from the terms $E_{1}$. For the first, we have

$$
E_{1}^{p, q} \simeq H^{q}\left(K^{p, \bullet}, d\right) \simeq \prod_{i_{0}<\cdots<i_{p}} H_{D R}^{q}\left(U_{i_{0} \ldots i_{p}}\right)
$$

Since all $U_{i_{0} \ldots i_{p}}$ are contractible we have

$$
\begin{aligned}
& E_{1}^{p, 0} \simeq \check{C}^{p}(\mathfrak{U}, \mathbb{R}) \\
& E_{1}^{p, q}=0 \text { for } q \neq 0 .
\end{aligned}
$$

As a consequence we have $E_{2}^{p, q}=0$ for $q \neq 0$, while

$$
E_{2}^{p, 0} \simeq H^{p}(\check{C} \bullet(\mathfrak{U}, \mathbb{R}), \delta)=H^{p}(\mathfrak{U}, \mathbb{R})
$$

This implies that $d_{2}=0$, so that the spectral sequence degenerates at the second step, and $E_{\infty}^{p, q}=0$ for $q \neq 0$ and $E_{\infty}^{p, 0} \simeq H^{p}(\mathfrak{U}, \mathbb{R})$. The resulting filtration of $H^{p}(T, D)$ has only one nonzero quotient, so that $H^{p}(T, D) \simeq H^{p}(\mathfrak{U}, \mathbb{R})$.

Let us now consider the second spectral sequence. We have

$$
{ }^{\prime} E_{1}^{p, q} \simeq H^{q}\left(K^{\bullet, p}, \delta\right)=H^{q}\left(C^{\bullet}\left(\mathfrak{U}, \Omega^{p}\right), \delta\right)=H^{q}\left(\mathfrak{U}, \Omega^{p}\right) .
$$

Since the sheaves $\Omega^{p}$ are acyclic, we have

$$
\begin{aligned}
& E_{1}^{p, 0} \simeq H^{0}\left(\mathfrak{U}, \Omega^{p}\right) \simeq \Omega^{p}(X) \\
& E_{1}^{p, q}=0 \text { for } q \neq 0
\end{aligned}
$$

At next step we have therefore ' $E_{2}^{p, q}=0$ for $q \neq 0$, and

$$
\left.{ }^{\prime} E_{2}^{p, 0} \simeq H^{p}\left(\Omega^{\bullet}(X), d\right)\right) \simeq H_{D R}^{p}(X)
$$

Again the spectral sequence degenerates at the second step, and we have $H^{p}(T, D) \simeq$ $H_{D R}^{p}(X)$. Comparing with what we got from the first sequence, we obtain $H_{D R}^{p}(X) \simeq$ $H^{p}(\mathfrak{U}, \mathbb{R})$. Taking a direct limit on good covers, we obtain $H^{p}(X, \mathbb{R}) \simeq H_{D R}^{p}(X)$.

[^13]Remark 4.2. From this example we may get the general result that if at step $r$, with $r \geq 1$, we have $E_{r}^{p, q}=0$ for $q \neq 0$ (or for $p \neq 0$ ) then the sequence degenerates at step $r$, and $E_{r}^{p, 0} \simeq H^{p}(T, d)\left(\right.$ or $\left.E_{r}^{0, q} \simeq H^{q}(T, d)\right)$.

## 5. Some applications

5.1. The spectral sequence of a resolution. In this section we extend Example 4.1 to a much general situation. Let $\left(\mathcal{L}^{\bullet}, f\right)$ be a complex of sheaves on a paracompact topological space $X$, and let $\mathfrak{U}$ be an open cover of $X$. We introduce the double complex $K^{p, q}=\check{C}^{p}\left(\mathfrak{U}, \mathcal{L}^{q}\right)$. We shall denote by $\mathcal{H}^{q}\left(\mathcal{L}^{\bullet}\right)$ the cohomology sheaves of the complex $\mathcal{L}^{\bullet}$. These are the sheaves associated with the quotient presheaves

$$
\tilde{\mathcal{H}}^{q}(U)=\frac{\operatorname{ker} f: \mathcal{L}^{q}(U) \rightarrow \mathcal{L}^{q+1}(U)}{\operatorname{Im} f: \mathcal{L}^{q-1}(U) \rightarrow \mathcal{L}^{q}(U)}
$$

The $E_{1}$ term of the first spectral sequence is

$$
\left.E_{1}^{p, q} \simeq H^{q}\left(K^{p, \bullet}, \delta_{2}\right)=H^{q}\left(\check{C}^{p}\left(\mathfrak{U}, \mathcal{L}^{\bullet}\right), f\right)\right) \simeq \check{C}^{p}\left(\mathscr{U}, \tilde{\mathcal{H}}^{q}\left(\mathcal{L}^{\bullet}\right)\right) .
$$

The second term of the sequence is

$$
E_{2}^{p, q} \simeq H^{p}\left(E_{1}^{\bullet, q}, \delta_{1}\right) \simeq H^{p}\left(\check{C}^{\bullet}\left(\mathfrak{U}, \tilde{\mathcal{H}}^{q}\left(\mathcal{L}^{\bullet}\right)\right), \delta\right) \simeq H^{p}\left(\mathfrak{U}, \mathcal{H}^{q}\left(\mathcal{L}^{\bullet}\right)\right)
$$

where, since $X$ is paracompact, we have replaced the presheaves $\tilde{\mathcal{H}}^{\bullet}$ with the corresponding sheaves $\mathcal{H}^{\bullet}$ (possibly replacing the cover $\mathfrak{U}$ by a suitable refinement).

For the second spectral sequence we have

$$
\begin{gathered}
{ }^{\prime} E_{1}^{p, q} \simeq H^{q}\left(K^{\bullet, p}, \delta_{1}\right) \simeq H^{q}\left(\check{C}^{\bullet}\left(\mathfrak{U}, \mathcal{L}^{p}\right), \delta_{1}\right) \simeq H^{p}\left(\mathfrak{U}, \mathcal{L}^{q}\right) \\
\quad{ }^{\prime} E_{2}^{q, p} \simeq H^{p}\left({ }^{\prime} E_{1}^{q, \bullet}, \delta_{2}\right) \simeq H^{p}\left(H^{q}\left(\mathfrak{U}, \mathcal{L}^{\bullet}\right), f\right) .
\end{gathered}
$$

Let assume now that $\mathcal{L}^{\bullet}$ is a resolution of a sheaf $\mathcal{F}$; then $\mathcal{H}^{q}\left(\mathcal{L}^{\bullet}\right)=0$ for $q \neq 0$, and $\mathcal{H}^{0}\left(\mathcal{L}^{\bullet}\right) \simeq \mathcal{F}$. The first spectral sequence degenerates at the second step, and we have $E_{2}^{p, q}=0$ for $q \neq 0$ and $E_{2}^{p, 0} \simeq H^{p}(\mathfrak{U}, \mathcal{F})$. The second spectral sequence does not degenerate, but we may say that it converges to the graded group $H^{\bullet}(\mathfrak{U}, \mathcal{F})$ (since the same does the first sequence). By taking direct limit over the cover $\mathfrak{U}$, we have:

Proposition 4.1. Given a resolution $\mathcal{L}^{\bullet}$ of a sheaf $\mathcal{F}$ on a paracompact space $X$, there is a spectral sequence $\mathfrak{E}$ • whose second term is $\mathfrak{E}_{2}^{p, q}=H^{q}\left(H^{p}\left(X, \mathcal{L}^{\bullet}\right), f\right)$, which converges to the graded group $H^{\bullet}(X, \mathcal{F})$.

The canonical filtrations of a double complex always satisfy the hypothesis of Lemma 4.2. So, considering the first spectral sequence, we obtain morphisms (again taking a direct limit)

$$
H^{q}\left(\mathcal{L}^{\bullet}(X), f\right) \rightarrow H^{q}(X, \mathcal{F})
$$

In general these are not isomorphisms. The same morphisms could be obtained by breaking the exact sequence $0 \rightarrow \mathcal{F} \rightarrow \mathcal{L}^{\bullet}$ into short exact sequences, taking the associated long exact cohomology sequences and suitably composing the morphisms, as in the proof of the abstract de Rham theorem 3.13.

A further specialization is obtained if the resolution $\mathcal{L}^{\bullet}$ is acyclic; then the second spectral sequence degenerates at the second step as well, and we get isomorphisms $H^{p}(X, \mathcal{F}) \simeq H^{p}\left(\mathcal{L}^{\bullet}(X), f\right)$, i.e., we have another proof of the abstract de Rham theorem 3.13.
5.2. The spectral sequence of a fibred space. Let $\mathcal{F}$ be a sheaf on a paracompact space $X$ and $\pi: X \rightarrow Y$ a continuous map, where $Y$ is a second paracompact space. We shall use the fact that every sheaf of abelian groups on space admits flabby resolutions (cf. Sections 3.2.5 and 3.2.8). We shall associate a spectral sequence to these data. We consider the complex

$$
\begin{equation*}
0 \rightarrow \pi_{*} \mathcal{F} \rightarrow \pi_{*} \mathcal{L}_{0} \xrightarrow{f} \pi_{*} \mathcal{L}_{1} \xrightarrow{f} \ldots \tag{4.12}
\end{equation*}
$$

where $\left(\mathcal{L}^{\bullet}, f\right)$ is a flabby resolution of $\mathcal{F}$. The morphism $\pi_{*} \mathcal{F} \rightarrow \pi_{*} \mathcal{L}_{0}$ is injective, but otherwise the complex (4.12) is no longer exact. However, the sheaves $\pi_{*} \mathcal{L}^{\bullet}$ are flabby. We denote by $R^{k} \pi_{*} \mathcal{F}$ the cohomology sheaves $\mathcal{H}^{k}\left(\pi_{*} \mathcal{L}^{\bullet}\right)$. These sheaves are called the higher direct images of $\mathcal{F}$. Note that $R^{0} \pi_{*} \mathcal{F} \simeq \pi_{*} \mathcal{F}$.

Proposition 4.2. The sheaf $R^{k} \pi_{*} \mathcal{F}$ is isomorphic to the sheaf associated with the presheaf $\mathcal{P}^{k}$ on $Y$ defined by $\mathcal{P}^{k}(U)=H^{k}\left(\pi^{-1}(U), \mathcal{F}\right)$.

This implies that the sheaves $R^{k} \pi_{*} \mathcal{F}$ do not depend, up to isomorphism, on the choice of the resolution.

Proof. $R^{k} \pi_{*} \mathcal{F}$ is by definition the sheaf associated with the presheaf

$$
U \rightsquigarrow \frac{\operatorname{ker} f: \mathcal{L}^{k}\left(\pi^{-1}(U)\right) \rightarrow \mathcal{L}^{k+1}\left(\pi^{-1}(U)\right)}{\operatorname{Im} f: \mathcal{L}^{k-1}\left(\pi^{-1}(U)\right) \rightarrow \mathcal{L}^{k}\left(\pi^{-1}(U)\right)}=H^{k}\left(\mathcal{L}^{\bullet}\left(\pi^{-1}(U), f\right) .\right.
$$

Since the restriction of a flabby sheaf to an open subset is flabby, by the abstract de Rham theorem we have isomorphisms

$$
H^{k}\left(\mathcal{L}^{\bullet}\left(\pi^{-1}(U), f\right) \simeq H^{k}\left(\pi^{-1}(U), \mathcal{F}\right)\right.
$$

whence the claim follows.
Let us consider the double complex $\check{C}^{p}\left(\mathfrak{U}, \pi_{*} \mathcal{L}^{q}\right)$, where $\mathfrak{U}$ is a locally finite open cover of $Y$. The two spectral sequences we have previously studied yield at the second term

$$
\begin{gathered}
E_{2}^{p, q} \simeq H^{p}\left(\mathfrak{U}, R^{q} \pi_{*} \mathcal{F}\right) \\
{ }^{\prime} E_{2}^{p, q} \simeq H^{q}\left(H^{p}\left(\mathfrak{U}, \pi_{*} \mathcal{L}^{\bullet}\right), f\right)
\end{gathered}
$$

Since the sheaves $\pi_{*} \mathcal{L}^{\bullet}$ are soft (hence acyclic) the second spectral sequence degenerates, and one has ${ }^{\prime} E_{\infty}^{p, q}=0$ for $p \neq 0$, and

$$
\begin{aligned}
{ }^{\prime} E_{\infty}^{0, q} & \simeq{ }^{\prime} E_{2}^{0, q} \simeq H^{q}\left(H^{0}\left(Y, \pi_{*} \mathcal{L}^{\bullet}\right), f\right) \\
& \simeq H^{q}\left(\mathcal{L}^{\bullet}(X), f\right) \simeq H^{q}(X, \mathcal{F}) .
\end{aligned}
$$

Again after taking a direct limit, we have:
Proposition 4.3. Given a continuous map of paracompact spaces $\pi: X \rightarrow Y$ and a sheaf $\mathcal{F}$ on $X$, there is a spectral sequence $\mathfrak{E}$ • whose second term is $\mathfrak{E}_{2}^{p, q}=H^{p}\left(Y, R^{q} \pi_{*} \mathcal{F}\right)$, which converges to the graded group $H^{\bullet}(X, \mathcal{F})$.

We describe without proof the relation between the stalks of the sheaf $R^{k} \pi_{*} \mathcal{F}$ at points $y \in Y$ and the cohomology groups $H^{k}\left(\pi^{-1}(y), \mathcal{F}\right)$; here $\mathcal{F}$ is to be considered as restricted to $\pi^{-1}$, i.e., more precisely we should write $H^{k}\left(\pi^{-1}(y), i_{y}^{-1} \mathcal{F}\right)$ where $i_{y}: \pi^{-1}(y) \rightarrow X$ is the inclusion. Since

$$
\left(R^{k} \pi_{*} F\right)_{y}=\underset{y \in U}{\lim }\left(R^{k} \pi_{*} \mathcal{F}\right)(U) \simeq \underset{y \in U}{\lim _{\vec{U}}} H^{k}\left(\pi^{-1}(U), \mathcal{F}\right),
$$

while $H^{k}\left(\pi^{-1}(y), \mathcal{F}\right)$ is the direct limit of the groups $H^{k}(V, \mathcal{F})$ where $V$ ranges over all open neighbourhoods of $\pi^{-1}(y)$, there is a natural map

$$
\begin{equation*}
\left(R^{k} \pi_{*} \mathcal{F}\right)_{y} \rightarrow H^{k}\left(\pi^{-1}(y), \mathcal{F}\right) . \tag{4.13}
\end{equation*}
$$

This is an isomorphism under some conditions, e.g., if $Y$ is locally compact and $\pi$ is proper (cf. [5]). This happens for instance when both $X$ and $Y$ are compact.

As a simple Corollary to Proposition 4.3 one obtains Leray's theorem:
Corollary 4.4. If every point $y \in Y$ has a system of neighbourhoods whose preimages are acyclic for $\mathcal{F}$, then $H^{k}(X, \mathcal{F}) \simeq H^{k}\left(Y, \pi_{*} \mathcal{F}\right)$ for all $k \geq 0$.

Proof. The hypothesis of the Corollary means that every $y \in Y$ has a system of neighbourhoods $\{U\}$ such that $H^{k}\left(\pi^{-1}(U), \mathcal{F}\right)=0$ for all $k>0$. This implies that $R^{k} \pi_{*} \mathcal{F}=0$ for $k>0$, so that the only nonzero terms in the spectral sequence $\mathfrak{E}_{2}$ are $\mathfrak{E}_{2}^{p, 0} \simeq H^{p}\left(Y, \pi_{*} \mathcal{F}\right)$. The sequence degenerates and the claim follows.
5.3. The Künneth theorem. Let $X, Y$ be topological spaces, and $G$ an abelian group. We shall denote by the same symbol $G$ the corresponding constant sheaves on the spaces $X, Y$ and $X \times Y$. The Künneth theorem computes the cohomology groups $H^{\bullet}(X \times Y, G)$ in terms of the groups $H^{\bullet}(X, \mathbb{Z})$ and $H^{\bullet}(Y, G)$.

We shall need the following version of the universal coefficient theorem.
Proposition 4.5. If $X$ is a paracompact topological space and $G$ a torsion-free group, then $H^{k}(X, G) \simeq H^{k}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} G$ for all $k \geq 0$.

Proof. Cf. [19].

Proposition 4.6. Assume that the groups $H^{\bullet}(Y, G)$ have no torsion over $\mathbb{Z}$, and that $X$ and $Y$ are compact Hausdorff and locally Euclidean. Then,

$$
H^{k}(X \times Y, G) \simeq \bigoplus_{p+q=k} H^{p}(X, \mathbb{Z}) \otimes H^{q}(Y, G)
$$

Proof. Let $\pi: X \times Y \rightarrow X$ be the projection onto the first factor. If $U$ is a contractible open set in $U$, then by the homotopic invariance of the cohomology with coefficients in a constant sheaf (which follows e.g. from its isomorphism with singular cohomology) we have $H^{\bullet}(U \times Y, G) \simeq H^{\bullet}(Y, G)$. If $V \subset U$, the morphism $H^{\bullet}(U \times$ $Y, G) \rightarrow H^{\bullet}(V \times Y, G)$ corresponds to the identity of $H^{\bullet}(Y, G)$. Under the present hypotheses the morphism (4.13) is an isomorphism. These facts imply that $R^{p} \pi_{*} G$ is the constant sheaf on $X$ with stalk $H^{p}(Y, G)$. The second term of the spectral sequence of Proposition 4.3 becomes $\mathfrak{E}_{2}^{p, q} \simeq H^{p}\left(X, H^{q}(Y, G)\right)$. By the universal coefficient theorem, since the groups $H^{q}(Y, G)$ have no torsion over $\mathbb{Z}$, we have $\mathfrak{E}_{2}^{p, q} \simeq H^{p}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} H^{q}(Y, G)$.

## Part 2

## Introduction to algebraic geometry

## CHAPTER 5

## Complex manifolds and vector bundles

In this chapter we give a sketchy introduction to complex manifolds. The reader is assumed to be acquainted with the rudiments of the theory of differentiable manifolds.

## 1. Basic definitions and examples

1.1. Holomorphic functions. Let $U \subset \mathbb{C}$ be an open subset. We say that a function $f: U \rightarrow \mathbb{C}$ is holomorphic if it is $C^{1}$ and for all $x \in U$ its differential $D f_{x}: \mathbb{C} \rightarrow$ $C$ is not only $\mathbb{R}$-linear but also $\mathbb{C}$-linear. If elements in $\mathbb{C}$ are written $z=x+i y$, and we set $f(x, y)=\alpha(x, y)+i \beta(x, y)$, then this condition can be written as

$$
\begin{equation*}
\alpha_{x}=\beta_{y}, \quad \alpha_{y}=-\beta_{x} \tag{5.1}
\end{equation*}
$$

(these are the Cauchy-Riemann conditions). If we use $z, \bar{z}$ as variables, the CauchyRiemann conditions read $f_{\bar{z}}=0$, i.e. the holomorphic functions are the $C^{1}$ function of the variable $z$. Moreover, one can show that holomorphic functions are analytic.

The same definition can be given for holomorphic functions of several variables.
DEFINITION 5.1. Two open subsets $U, V$ of $\mathbb{C}^{n}$ are said to biholomorphic if there exists a bijective holomorphic map $f: U \rightarrow V$ whose inverse is holomorphic. The map $f$ itself is then said to be biholomorphic.
1.2. Complex manifolds. Complex manifolds are defined as differentiable manifolds, but requiring that the local model is $\mathbb{C}^{n}$, and that the transition functions are biholomorphic.

Definition 5.2. An n-dimensional complex manifold is a second countable Hausdorff topological space $X$ together with an open cover $\left\{U_{i}\right\}$ and maps $\psi_{i}: U_{i} \rightarrow \mathbb{C}^{n}$ which are homeomorphisms onto their images, and are such that all transition functions

$$
\psi_{i} \circ \psi_{j}^{-1}: \psi_{j}\left(U_{i} \cap U_{j}\right) \rightarrow \psi_{i}\left(U_{i} \cap U_{j}\right)
$$

are biholomorphisms.
Example 5.3. (The Riemann sphere) Consider the sphere in $\mathbb{R}^{3}$ centered at the origin and having radius $\frac{1}{2}$, and identify the tangent planes at $\left(0,0, \frac{1}{2}\right)$ and $\left(0,0,-\frac{1}{2}\right)$ with $\mathbb{C}$. The stereographic projections give local complex coordinates $z_{1}, z_{2}$; the transition function $z_{2}=1 / z_{1}$ is defined in $\mathbb{C}^{\star}=\mathbb{C}-\{0\}$ and is biholomorphic.

1-dimensional complex manifolds are called Riemann surfaces. Compact Riemann surfaces play a distinguished role in algebraic geometry; they are all algebraic (i.e. they are sets of zeroes of systems of homogeneous polynomials), as we shall see in Chapter 7.

Example 5.4. (Projective spaces) We define the $n$-dimensional complex projective space as the space of complex lines through the origin of $\mathbb{C}^{n+1}$, i.e.

$$
\mathbb{P}_{n}=\frac{\mathbb{C}^{n+1}-\{0\}}{\mathbb{C}^{*}}
$$

By standard topological arguments $\mathbb{P}_{n}$ with the quotient topology is a Hausdorff secondcountable space.

Let $\pi: \mathbb{C}^{n+1}-\{0\} \rightarrow \mathbb{P}_{n}$ be the projection, If $w=\left(w^{0}, \ldots, w^{n}\right) \in \mathbb{C}^{n+1}$ we shall denote $\pi(w)=\left[w^{0}, \ldots, w^{n}\right]$. The numbers $\left(w^{0}, \ldots, w^{n}\right)$ are said to be the homogeneous coordinates of the point $\pi(w)$. If $\left(u^{0}, \ldots, u^{n}\right)$ is another set of homogeneous coordinates for $\pi(w)$, then $u^{i}=\lambda w^{i}$, with $\lambda \in \mathbb{C}^{*}(i=0, \ldots, n)$.

Denote by $\tilde{U}_{i} \subset \mathbb{C}^{n+1}$ the open set where $w^{i} \neq 0$, let $U_{i}=\pi\left(\tilde{U}_{i}\right)$, and define a map

$$
\psi_{i}: U_{i} \rightarrow \mathbb{C}^{n}, \quad \psi\left(\left[w^{0}, \ldots, w^{n}\right]\right)=\left(\frac{w^{0}}{w^{i}}, \ldots, \frac{w^{i-1}}{w^{i}}, \frac{w^{i+1}}{w^{i}}, \ldots, \frac{w^{n}}{w^{i}}\right)
$$

The sets $U_{i}$ cover $\mathbb{P}_{n}$, the maps $\psi_{i}$ are homeomorphisms, and their transition functions

$$
\begin{array}{r}
\psi_{i} \circ \psi_{j}^{-1}: \quad \psi_{j}\left(U_{j}\right) \rightarrow \psi_{i}\left(U_{i}\right) \\
\psi_{i} \circ \psi_{j}^{-1}\left(z^{1}, \ldots, z^{n}\right)=\left(\frac{z^{1}}{z^{i}}, \ldots, \frac{z^{i-1}}{z^{i}}, \frac{z^{i+1}}{z^{i}}, \ldots, \frac{1}{z^{i}}, \ldots \frac{z^{n}}{z^{i}}\right), \\
j \text {-th argument }
\end{array}
$$

are biholomorphic, so that $\mathbb{P}_{n}$ is a complex manifold (we have assumed that $i<j$ ). The map $\pi$ restricted to the unit sphere in $\mathbb{C}^{n+1}$ is surjective, so that $\mathbb{P}_{n}$ is compact. The previous formula for $n=1$ shows that $\mathbb{P}_{1}$ is biholomorphic to the Riemann sphere.

The coordinates defined by the maps $\psi_{i}$, usually denoted $\left(z^{1}, \ldots, z^{n}\right)$, are called affine or Euclidean coordinates.

Example 5.5. (The general linear complex group). Let

$$
\begin{gathered}
M_{k, n}=\{k \times n \text { matrices with complex entries, } k \leq n\} \\
\hat{M}_{k, n}=\left\{\text { matrices in } M_{k, n} \text { of rank } k\right\}, \quad \text { i.e. } \\
\hat{M}_{k, n}=\bigcup_{i=1}^{\ell}\left\{A \in M_{k, n} \quad \text { such that } \quad \operatorname{det} A_{i} \neq 0\right\}
\end{gathered}
$$

where $A_{i}, \ldots, A_{\ell}$ are the $k \times k$ minors of $A . M_{k, n}$ is a complex manifold of dimension $k n ; \hat{M}_{k, n}$ is an open subset in $M_{k, n}$, as its second description shows, so it is a complex manifold of dimension $k n$ as well. In particular, the general linear group $G l(n, \mathbb{C})=$ $\hat{M}_{n, n}$ is a complex manifold of dimension $n^{2}$. Here are some of its relevant subgroups:
(i) $U(n)=\left\{A \in G l(n, \mathbb{C})\right.$ such that $\left.A A^{\dagger}=I\right\}$;
(ii) $S U(n)=\{A \in U(n)$ such that $\operatorname{det} A=1\}$;
these two groups are real (not complex!) manifolds, and $\operatorname{dim}_{\mathbb{R}} U(n)=n^{2}, \operatorname{dim}_{\mathbb{R}} S U(n)=$ $n^{2}-1$.
(iii) the group $G l(k, n ; \mathbb{C})$ formed by invertible complex matrices having a block form

$$
M=\left(\begin{array}{ll}
A & 0  \tag{5.2}\\
B & C
\end{array}\right)
$$

where the matrices $A, B, C$ are $k \times k,(n-k) \times k$, and $(n-k) \times(n-k)$, respectively. $G l(k, n ; \mathbb{C})$ is a complex manifold of dimension $k^{2}+n^{2}-n k$. Since a matrix of the form (5.2) is invertible if and only if $A$ and $C$ are, while $B$ can be any matrix, $G l(k, n ; \mathbb{C})$ is biholomorphic to the product manifold $G l(k, \mathbb{C}) \times G l(n-k, \mathbb{C}) \times M_{k, n}$.
1.3. Submanifolds. Given a complex manifold $X$, a submanifold of $X$ is a pair $(Y, \iota)$, where $Y$ is a complex manifold, and $\iota: Y \rightarrow X$ is an injective holomorphic map whose jacobian matrix has rank equal to the dimension of $Y$ at any point of $Y$ (of course $Y$ can be thought of as a subset of $X$ ).

Example 5.6. $G l(k, n ; \mathbb{C})$ is a submanifold of $G l(n, \mathbb{C})$.
Example 5.7. For any $k<n$ the inclusion of $\mathbb{C}^{k+1}$ into $\mathbb{C}^{n+1}$ obtained by setting to zero the last $n-k$ coordinates in $\mathbb{C}^{n+1}$ yields a map $\mathbb{P}_{k} \rightarrow \mathbb{P}_{n}$; the reader may check that this realizes $\mathbb{P}_{k}$ as a submanifold of $\mathbb{P}_{n}$.

Example 5.8. (Grassmann varieties) Let

$$
G_{k, n}=\left\{\text { space of } k \text {-dimensional planes in } \mathbb{C}^{n}\right\}
$$

(so $G_{1, n} \equiv \mathbb{P}_{n}-1$ ). This is the Grassmann variety of $k$-planes in $\mathbb{C}^{n}$. Given a $k$-plane, the action of $G l(n, \mathbb{C})$ on it yields another plane (possibly coinciding with the previous one). The subgroup of $\operatorname{Gl}(n, \mathbb{C})$ which leaves the given $k$-plane fixed is isomorphic to $G l(k, n ; \mathbb{C})$, so that

$$
G_{k, n} \simeq \frac{G l(n, \mathbb{C})}{G l(k, n ; \mathbb{C})} .
$$

As the reader may check, this representation gives $G_{k, n}$ the structure of a complex manifold of dimension $k(n-k)$. Since in the previous reasoning $G l(n, \mathbb{C})$ can be replaced by $U(n)$, and since $G l(k, n ; \mathbb{C}) \cap U(n)=U(k) \times U(n-k)$, we also have the representation

$$
G_{k, n} \simeq \frac{U(n)}{U(k) \times U(n-k)}
$$

showing that $G_{k, n}$ is compact.
An element in $G_{k, n}$ singles out (up to a complex factor) a decomposable element in $\Lambda^{k} \mathbb{C}^{n}$,

$$
\lambda=v_{1} \wedge \cdots \wedge v_{k}
$$

where the $v_{i}$ are a basis of tangent vectors to the given $k$-plane. So $G_{k, n}$ imbeds into $\mathbb{P}\left(\Lambda^{k} \mathbb{C}^{n}\right)=\mathbb{P}_{N}$, where $N=\left(\binom{n}{k}\right)-1$ (this is called the Plücker embedding. If a basis $\left\{v_{1}, \ldots, v_{n}\right\}$ is fixed in $\mathbb{C}^{n}$, one has a representation

$$
\lambda=\sum_{i_{1}, \ldots, i_{k}=1}^{n} P_{i_{1} \ldots i_{k}} v_{i_{1}} \wedge \cdots \wedge v_{i_{k}}
$$

the numbers $P_{i_{1} \ldots i_{k}}$ are the Plücker coordinates on the Grassmann variety.

## 2. Some properties of complex manifolds

2.1. Orientation. All complex manifolds are oriented. Consider for simplicity the 1-dimensional case; the jacobian matrix of a transition function $z^{\prime}=f(z)=\alpha(x, y)+$ $i \beta(x, y)$ is (by the Cauchy-Riemann conditions)

$$
J=\left(\begin{array}{cc}
\alpha_{x} & \alpha_{y} \\
\beta_{x} & \beta_{y}
\end{array}\right)=\left(\begin{array}{cc}
\alpha_{x} & \alpha_{y} \\
-\alpha_{y} & \alpha_{x}
\end{array}\right)
$$

so that $\operatorname{det} J=\alpha_{x}^{2}+\alpha_{y}^{2}>0$, and the manifold is oriented.
Notice that we may always conjugate the complex structure, considering (e.g. in the 1-dimensional case) the coordinate change $z \mapsto \bar{z}$; in this case we have $J=\left(\begin{array}{cc}1 & 0 \\ 0 & -1\end{array}\right)$, so that the orientation gets reversed.
2.2. Forms of type $(p, q)$. Let $X$ be an $n$-dimensional complex manifold; by the identification $\mathbb{C}^{n} \simeq \mathbb{R}^{2 n}$, and since a biholomorphic map is a $C^{\infty}$ diffeomorphism, $X$ has an underlying structure of $2 n$-dimensional real manifold. Let $T X$ be the smooth tangent bundle (i.e. the collection of all ordinary tangent spaces to $X$ ). If $\left(z^{1}, \ldots, z^{n}\right)$ is a set of local complex coordinates around a point $x \in X$, then the complexified tangent space $T_{x} X \otimes_{\mathbb{R}} \mathbb{C}$ admits the basis

$$
\left(\left(\frac{\partial}{\partial z^{1}}\right)_{x}, \ldots,\left(\frac{\partial}{\partial z^{n}}\right)_{x},\left(\frac{\partial}{\partial \bar{z}^{1}}\right)_{x}, \ldots,\left(\frac{\partial}{\partial \bar{z}^{n}}\right)_{x}\right) .
$$

This yields a decomposition

$$
T X \otimes \mathbb{C}=T^{\prime} X \oplus T^{\prime \prime} X
$$

which is intrinsic because $X$ has a complex structure, so that the transition functions are holomorphic and do not mix the vectors $\frac{\partial}{\partial z^{i}}$ with the $\frac{\partial}{\partial z^{i}}$. As a consequence one has a decomposition

$$
\Lambda^{i} T^{*} X \otimes \mathbb{C}=\bigoplus_{p+q=i} \Omega^{p, q} X \quad \text { where } \quad \Omega^{p, q} X=\Lambda^{p}\left(T^{\prime} X\right)^{*} \otimes \Lambda^{q}\left(T^{\prime \prime} X\right)^{*}
$$

The elements in $\Omega^{p, q} X$ are called differential forms of type ( $p, q$ ), and can locally be written as

$$
\eta=\eta_{i_{1} \ldots i_{p}, j_{1} \ldots j_{q}}(z, \bar{z}) d z^{i_{1}} \wedge \cdots \wedge d z^{i_{p}} \wedge d \bar{z}^{j_{1}} \wedge \cdots \wedge d \bar{z}^{j_{q}} .
$$

The compositions

define differential operators $\partial, \bar{\partial}$ such that

$$
\partial^{2}=\bar{\partial}^{2}=\partial \bar{\partial}+\bar{\partial} \partial=0
$$

(notice that the Cauchy-Riemann condition can be written as $\bar{\partial} f=0$ ).

## 3. Dolbeault cohomology

Another interesting cohomology theory one can consider is the Dolbeault cohomology associated with a complex manifold $X$. Let $\Omega^{p, q}$ denote the sheaf of forms of type $(p, q)$ on $X$. The Dolbeault (or Cauchy-Riemann) operator $\bar{\partial}: \Omega^{p, q} \rightarrow \Omega^{p, q+1}$ squares to zero. Therefore, the pair $\left(\Omega^{p, \bullet}(X), \bar{\partial}\right)$ is for any $p \geq 0$ a cohomology complex. Its cohomology groups are denoted by $H_{\bar{\partial}}^{p, q}(X)$, and are called the Dolbeault cohomology groups of $X$.

We have for this theory an analogue of the Poincaré Lemma, which is sometimes called the $\bar{\partial}$-Poincaré Lemma (or Dolbeault or Grothendieck Lemma).

Proposition 5.1. Let $\Delta$ be a polycylinder in $\mathbb{C}^{n}$ (that is, the cartesian product of disks in $\mathbb{C})$. Then $H_{\bar{\partial}}^{p, q}(\Delta)=0$ for $q \geq 1$.

Proof. Cf. [9].
Moreover, the kernel of the morphism $\bar{\partial}: \Omega^{p, 0} \rightarrow \Omega^{p, 1}$ is the sheaf of holomorphic $p$-forms $\Omega^{p}$. Therefore, the Dolbeault complex of sheaves $\Omega^{p, \bullet}$ is a resolution of $\Omega^{p}$, i.e. for all $p=0, \ldots, n$ (where $n=\operatorname{dim}_{\mathbb{C}} X$ ) the sheaf sequence

$$
0 \rightarrow \Omega^{p} \rightarrow \Omega^{p, 0} \xrightarrow{\bar{\partial}} \Omega^{p, 1} \xrightarrow{\bar{\partial}} \ldots \xrightarrow{\bar{\partial}} \Omega^{p, 1} \rightarrow 0
$$

is exact. Moreover, the sheaves $\Omega^{p, q}$ are fine (they are $\mathcal{C}_{X}^{\infty}$-modules). Then, exactly as one proves the de Rham theorem (Theorem 3.3.14), one obtains the Dolbeault theorem:

Proposition 5.2. Let $X$ be a complex manifold. For all $p, q \geq 0$, the cohomology groups $H_{\bar{\partial}}^{p, q}(X)$ and $H^{q}\left(X, \Omega^{p}\right)$ are isomorphic.

## 4. Holomorphic vector bundles

4.1. Basic definitions. Holomorphic vector bundles on a complex manifold $X$ are defined in the same way than smooth complex vector bundles, but requiring that all the maps involved are holomorphic.

Definition 5.1. A complex manifold $E$ is a rank $n$ holomorphic vector bundle on $X$ if there are
(i) an open cover $\left\{U_{\alpha}\right\}$ of $X$
(ii) a holomorphic map $\pi: E \rightarrow X$
(iii) holomorphic maps $\psi_{\alpha}: \pi^{-1}\left(U_{\alpha}\right) \rightarrow U_{\alpha} \times \mathbb{C}^{n}$
such that
(i) $\pi=\operatorname{pr}_{1} \circ \psi_{\alpha}$, where $=\operatorname{pr}_{1}$ is the projection onto the first factor of $U_{\alpha} \times \mathbb{C}^{n}$;
(ii) for all $p \in U_{\alpha} \cap U_{\beta}$, the map

$$
\operatorname{pr}_{2} \circ \psi_{\beta} \circ \psi_{\alpha}^{-1}(p, \bullet): \mathbb{C}^{n} \rightarrow \mathbb{C}^{n}
$$

is a linear isomorphism.
Vector bundles of rank 1 are called line bundles.
With the data that define a holomorphic vector bundle we may construct holomorphic maps

$$
g_{\alpha \beta}: U_{\alpha} \cap U_{\beta} \rightarrow G l(n, \mathbb{C})
$$

given by

$$
g_{\alpha \beta}(p) \cdot x=\operatorname{pr}_{2} \circ \psi_{\alpha} \circ \psi_{\beta}^{-1}(\psi, x) .
$$

These maps satisfy the cocycle condition

$$
g_{\alpha \beta} g_{\beta \gamma} g_{\gamma \alpha}=\mathrm{Id} \quad \text { on } \quad U_{\alpha} \cap U_{\beta} \cap U_{\gamma} .
$$

The collection $\left\{U_{\alpha}, \psi_{\alpha}\right\}$ is a trivialization of $E$.
For every $x \in X$, the subset $E_{x}=\pi^{-1}(x) \subset E$ is called the fibre of $E$ over $x$. By means of a trivialization around $x, E_{x}$ is given the structure of a vector space, which is actually independent of the trivialization.

A morphism between two vector bundles $E, F$ over $X$ is a holomorphic map $f: E \rightarrow$ $F$ such that for every $x \in X$ one has $f\left(E_{x}\right) \subset F_{x}$, and such that the resulting map $f_{x}: E_{x} \rightarrow F_{x}$ is linear. If $f$ is a biholomorphism, it is said to be an isomorphism of vector bundles, and $E$ and $F$ are said to be isomorphic.

A holomorphic section of $E$ over an open subset $U \subset X$ is a holomorphic map $s: U \rightarrow E$ such that $\pi \circ s=$ Id. With reference to the notation previously introduced, the maps

$$
s_{(\alpha) i}: U_{\alpha} \rightarrow E, \quad s_{(\alpha) i}(x)=\psi_{\alpha}^{-1}\left(x, e_{i}\right), \quad i=1, \ldots, n
$$

where $\left\{e_{i}\right\}$ is the canonical basis of $\mathbb{C}^{n}$, are sections of $E$ over $U_{\alpha}$. Let $E\left(U_{\alpha}\right)$ denote the set of sections of $E$ over $U_{\alpha}$; it is a free module over the ring $\mathcal{O}\left(U_{\alpha}\right)$ of holomorphic functions on $U_{\alpha}$, and its subset $\left\{s_{(\alpha) i}\right\}_{i=1, \ldots, n}$ is a basis. On an intersection $U_{\alpha} \cap U_{\beta}$ one has the relation

$$
s_{(\alpha) i}=\sum_{k=1}^{n}\left(g_{\alpha \beta}\right)_{i k} s_{(\beta) k} .
$$

Exercise 5.2. Show that two trivializations are equivalent (i.e. describe isomorphic bundles) if there exist holomorphic maps $\lambda_{\alpha}: U_{\alpha} \rightarrow G l(n, \mathbb{C})$ such that

$$
\begin{equation*}
g_{\alpha \beta}^{\prime}=\lambda_{\alpha} g_{\alpha \beta} \lambda_{\beta}^{-1} \tag{5.3}
\end{equation*}
$$

Exercise 5.3. Show that the rule that to any open subset $U \subset X$ assigns the $\mathcal{O}_{X}^{\infty}(U)$-module of sections of a holomorphic vector bundle $E$ defines a sheaf $\mathcal{E}$ (which actually is a sheaf of $\mathcal{O}_{X}$-modules).

If $E$ is a holomorphic (or smooth complex) vector bundle, with transition functions $g_{\alpha \beta}$, then the maps

$$
\begin{equation*}
g_{\alpha \beta}^{\prime}=\left(g_{\alpha \beta}^{T}\right)^{-1} \tag{5.4}
\end{equation*}
$$

(where $T$ denotes transposition) define another vector bundle, called the dual vector bundle to $E$, and denoted by $E^{*}$. Sections of $E^{*}$ can be paired with (or act on) sections of $E$, yielding holomorphic (smooth complex-valued) functions on (open sets of) $X$.

Example 5.4. The space $E=X \times \mathbb{C}^{n}$, with the projection onto the first factor, is obviously a holomorphic vector bundle, called the trivial vector bundle of rank $n$. We shall denote such a bundle by $\mathbb{C}^{n}$ (in particular, $\mathbb{C}$ denotes the trivial line bundle). A holomorphic vector bundle is said to be trivial when it is isomorphic to $\mathbb{\mathbb { C }}^{n}$.

Every holomorphic vector bundle has an obvious structure of smooth complex vector bundle. A holomorphic vector bundle may be trivial as a smooth bundle while not being trivial as a holomorphic bundle. (In the next sections we shall learn some homological techniques that can be used to handle such situations).

Example 5.5. (The tangent and cotangent bundles) If $X$ is a complex manifold, the "holomorphic part" $T^{\prime} X$ of the complexified tangent bundle is a holomorphic vector bundle, whose rank equals the complex dimension of $X$. Given a holomorphic atlas for $X$, the locally defined holomorphic vector fields $\frac{\partial}{\partial z^{1}} \ldots, \frac{\partial}{\partial z^{n}}$ provide a holomorphic trivialization of $X$, such that the transition functions of $T^{\prime} X$ are the jacobian matrices of the transition functions of $X$. The dual of $T^{\prime} X$ is the holomorphic cotangent bundle of $X$.

Example 5.6. (The tautological bundle) Let $\left(w^{1}, \ldots, w^{n+1}\right)$ be homogeneous coordinates in $\mathbb{P}_{n}$. If to any $p \in \mathbb{P}_{n}$ (which is a line in $\mathbb{C}^{n+1}$ ) we associate that line we obtain a line bundle, the tautological line bundle $L$ of $\mathbb{P}_{n}$. To be more concrete, let us exhibit a trivialization for $L$ and the related transition functions. If $\left\{U_{i}\right\}$ is the standard cover of $\mathbb{P}_{n}$, and $p \in U_{i}$, then $w^{i}$ can be used to parametrize the points in the line $p$. So if $p$ has homogeneous coordinates $\left(w^{0}, \ldots, w^{n}\right)$, we may define $\psi_{i}: \pi^{-1}\left(U_{i}\right) \rightarrow U_{i} \times \mathbb{C}$ as $\psi_{i}(u)=\left(p, w^{i}\right)$ if $p=\pi(u)$. The transition function is then $g_{i k}=w^{i} / w^{k}$. The dual bundle $H=L^{*}$ acts on $L$, so that its fibre at $p=\pi(u), u \in \mathbb{C}^{n+1}$ can be regarded as
the space of linear functionals on the line $\mathbb{C} u \equiv L_{p}$, i.e. as hyperplanes in $\mathbb{C}^{n+1}$. Hence $H$ is called the hyperplane bundle. Often $L$ is denoted $\mathcal{O}(-1)$, and $H$ is denoted $\mathcal{O}(1)$ - the reason of this notation will be clear in Chapter 6.

In the same way one defines a tautological bundle on the Grassmann variety $G_{k, n}$; it has rank $k$.

Exercise 5.7. Show that that the elements of a basis of the vector space of global sections of $L$ can be identified homogeneous coordinates, so that $\operatorname{dim} H^{0}\left(\mathbb{P}_{n}, L\right)=n+$ 1. Show that the global sections of $H$ can be identified with the linear polynomials in the homogeneous coordinates. Hence, the global sections of $H^{r}$ are homogeneous polynomials of order $r$ in the homogeneous coordinates.
4.2. More constructions. Additional operations that one can perform on vector bundles are again easily described in terms of transition functions.
(1) Given two vector bundles $E_{1}$ and $E_{2}$, of rank $r_{1}$ and $r_{2}$, their direct sum $E_{1} \oplus E_{2}$ is the vector bundle of rank $r_{1}+r_{2}$ whose transition functions have the block matrix form

$$
\left(\begin{array}{cc}
g_{\alpha \beta}^{(1)} & 0 \\
0 & g_{\alpha \beta}^{(2)}
\end{array}\right)
$$

(2) We may also define the tensor product $E_{1} \otimes E_{2}$, which has rank $r_{1} r_{2}$ and has transition functions $g_{\alpha \beta}^{(1)} g_{\alpha \beta}^{(2)}$. This means the following: assume that $E_{1}$ and $E_{2}$ trivialize over the same cover $\left\{U_{\alpha}\right\}$, a condition we may always meet, and that in the given trivializations, $E_{1}$ and $E_{2}$ have local bases of sections $\left\{s_{(\alpha) i}\right\}$ and $\left\{t_{(\alpha) k}\right\}$. Then $E_{1} \otimes E_{2}$ has local bases of sections $\left\{s_{(\alpha) i} \otimes t_{(\alpha) k}\right\}$ and the corresponding transition functions are given by

$$
s_{(\alpha) i} \otimes t_{(\alpha) k}=\sum_{m=1}^{r_{1}} \sum_{n=1}^{r_{2}}\left(g_{\alpha \beta}^{(1)}\right)_{i m}\left(g_{\alpha \beta}^{(2)}\right)_{k n} s_{(\beta) m} \otimes t_{(\beta) n} .
$$

In particular the tensor product of line bundles is a line bundle. If $L$ is a line bundle, one writes $L^{n}$ for $L \otimes \cdots \otimes L$ ( $n$ factors). If $L$ is the tautological line bundle on a projective space, one often writes $L^{n}=\mathcal{O}(-n)$, and similarly $H^{n}=\mathcal{O}(n)$ (notice that $\left.\mathcal{O}(-n)^{*}=\mathcal{O}(n)\right)$.
(3) If $E$ is a vector bundle with transition functions $g_{\alpha \beta}$, we define its determinant $\operatorname{det} E$ as the line bundle whose transition functions are the functions $\operatorname{det} g_{\alpha \beta}$. The determinant bundle of the holomorphic tangent bundle to a complex manifold is called the canonical bundle $K$.

Exercise 5.8. Show that the canonical bundle of the projective space $\mathbb{P}_{n}$ is isomorphic to $\mathcal{O}(-n-1)$.

Example 5.9. Let $\pi: \mathbb{C}^{n+1}-\{0\} \rightarrow \mathbb{P}_{n}$ be the usual projection, and let ( $w^{1}, \ldots$, $w^{n+1}$ ) be homogeneous coordinates in $\mathbb{P}_{n}$. The tangent spaces to $\mathbb{P}_{n}$ are generated by
the vectors $\pi_{*} \frac{\partial}{\partial w^{i}}$, and these are subject to the relation

$$
\sum_{i=1}^{n+1} w^{i} \pi_{*} \frac{\partial}{\partial w^{i}}=0
$$

If $\ell$ is a linear functional on $\mathbb{C}^{n+1}$ the vector field

$$
v(w)=\ell(w) \frac{\partial}{\partial w^{i}}
$$

( $i$ is fixed) satisfies $v(\lambda w)=\lambda v(w)$ and therefore descends to $\mathbb{P}_{n}$. One can then define a map

$$
\begin{aligned}
E: H^{\oplus(n+1)} & \rightarrow T \mathbb{P}_{n} \\
\left(\sigma_{1}, \ldots, \sigma_{n+1}\right) & \mapsto \sum_{i=1}^{n+1} \sigma_{i}(w) \frac{\partial}{\partial w^{i}}
\end{aligned}
$$

(recall that the sections of $H$ can be regarded as linear functionals on the homogeneous coordinates). The map $E$ is apparently surjective. Its kernel is generated by the section $\sigma_{i}(w)=w^{i}, i=1, \ldots, n+1$; notice that this is the image of the map

$$
\underline{\mathbb{C}} \rightarrow H^{\oplus(n+1)}, \quad 1 \mapsto\left(w^{1}, \ldots, w^{n+1}\right)
$$

The morphism $H^{\oplus(n+1)} \rightarrow T \mathbb{P}_{n}$ may be regarded as a sheaf morphism $\mathcal{O}_{\mathbb{P}_{n}}(1)^{\oplus(n+1)}$ $\rightarrow T \mathbb{P}_{n}$, the second sheaf being the tangent sheaf of $\mathbb{P}_{n}$, i.e., the sheaf of germs of holomorphic vector fields on $\mathbb{P}_{n}$, and one has an exact sequence

$$
0 \rightarrow \mathcal{O}_{\mathbb{P}_{n}} \rightarrow \mathcal{O}_{\mathbb{P}_{n}}(1)^{\oplus(n+1)} \rightarrow T \mathbb{P}_{n} \rightarrow 0
$$

called the Euler sequence.

## 5. Chern class of line bundles

5.1. Chern classes of holomorphic line bundles. Let $X$ a complex manifold. We define $\operatorname{Pic}(X)$ (the Picard group of $X$ ) as the set of holomorphic line bundles on $X$ modulo isomorphism. The group structure of $\operatorname{Pic}(X)$ is induced by the tensor product of line bundles $L \otimes L^{\prime}$; in particular one has $L \otimes L^{*} \simeq \mathbb{C}$ (think of it in terms of transition functions - here $\mathbb{C}$ denotes the trivial line bundle, whose class $[\mathbb{C}]$ is the identity in $\operatorname{Pic}(X))$, so that the class $\left[L^{*}\right]$ is the inverse in $\operatorname{Pic}(X)$ of the class $[L]$.

Let $\mathcal{O}$ denote the sheaf of holomorphic functions on $X$, and $\mathcal{O}^{*}$ the subsheaf of nowhere vanishing holomorphic funtions. If $L \simeq L^{\prime}$ then the transition functions $g_{\alpha \beta}$, $g_{\alpha \beta}^{\prime}$ of the two bundles with respect to a cover $\left\{U_{\alpha}\right\}$ of $X$ are 2-cocycles $\mathcal{O}^{*}$, and satisfy

$$
g_{\alpha \beta}^{\prime}=g_{\alpha \beta} \frac{\lambda_{\alpha}}{\lambda_{\beta}} \quad \text { with } \quad \lambda_{\alpha} \in \mathcal{O}^{*}\left(U_{\alpha}\right)
$$

so that one has an identification $\operatorname{Pic}(X) \simeq H^{1}\left(X, \mathcal{O}^{*}\right)$. The long cohomology sequence associated with the exact sequence

$$
0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O} \xrightarrow{\exp } \mathcal{O}^{*} \rightarrow 0
$$

(where $\exp f=e^{2 \pi i f}$ ) contains the segment

$$
H^{1}(X, Z) \rightarrow H^{1}(X, \mathcal{O}) \rightarrow H^{1}\left(X, \mathcal{O}^{*}\right) \xrightarrow{\delta} H^{2}(X, \mathbb{Z}) \rightarrow H^{2}(X, \mathcal{O})
$$

where $\delta$ is the connecting morphism. Given a line bundle $L$, the element

$$
c_{1}(L)=\delta([L]) \in H^{2}(X, \mathbb{Z})
$$

is the first Chern class ${ }^{1}$ of $L$. The fact that $\delta$ is a group morphism means that

$$
c_{1}\left(L \otimes L^{\prime}\right)=c_{1}(L)+c_{1}\left(L^{\prime}\right) .
$$

In general, the morphism $\delta$ is neither injective nor surjective, so that
(i) the first Chern class does not classify the holomorphic line bundles on $X$; the group

$$
\operatorname{Pic}^{0}(X)=\operatorname{ker} \delta \simeq H^{1}(X, \mathcal{O}) / \operatorname{Im} H^{1}(X, \mathbb{Z})
$$

classifies the line bundles having the same first Chern class.
(ii) not every element in $H^{2}(X, \mathbb{Z})$ is the first Chern class of a holomorphic line bundle.

The image of $c_{1}$ is a subgroup $\operatorname{NS}(X)$ of $H^{2}(X, \mathbb{Z})$, called the Néron-Severi group of $X$.
Exercise 5.1. Show that all line bundles on $\mathbb{C}^{n}$ are trivial.
Exercise 5.2. Show that there exist nontrivial holomorphic line bundles which are trivial as smooth complex line bundles.

Notice that when $X$ is compact the sequence

$$
0 \rightarrow H^{0}(X, \mathbb{Z}) \rightarrow H^{0}(X, \mathcal{O}) \rightarrow H^{0}\left(X, \mathcal{O}^{*}\right) \rightarrow 0
$$

is exact, so that $\operatorname{Pic}^{0}(X)=H^{1}(X, \mathcal{O}) / H^{1}(X, \mathbb{Z})$. If in addition $\operatorname{dim} X=1$ we have $H^{2}(X, \mathcal{O})=0$, so that every element in $H^{2}(X, \mathbb{Z})$ is the first Chern class of a holomorphic line bundle. ${ }^{2}$

From the definition of connecting morphism we can deduce an explicit formula for a Čech cocycle representing $c_{1}(L)$ with respect to the cover $\left\{U_{\alpha}\right\}$ :

$$
\left\{c_{1}(L)\right\}_{\alpha \beta \gamma}=\frac{1}{2 \pi i}\left(\log g_{\alpha \beta}+\log g_{\beta \gamma}+\log g_{\gamma \alpha}\right) .
$$

From this one can easily prove that, if $f: X \rightarrow Y$ is a holomorphic map, and $L$ is a line bundle on $Y$, then

$$
c_{1}\left(f^{*} L\right)=f^{\sharp}\left(c_{1}(L)\right) .
$$

[^14]5.2. Smooth line bundles. The first Chern class can equally well be defined for smooth complex line bundles. In this case we consider the sheaf $\mathcal{C}$ of complexvalued smooth functions on a differentiable manifold $X$, and the subsheaf $\mathcal{C}^{*}$ of nowhere vanishing functions of such type. The set of isomorphism classes of smooth complex line bundles is identified with the cohomology group $H^{1}\left(X, \mathcal{C}^{*}\right)$. However now the sheaf $\mathcal{C}$ is acyclic, so that the obstruction morphism $\delta$ establishes an isomorphism $H^{1}\left(X, \mathcal{C}^{*}\right) \simeq$ $H^{2}(X, \mathbb{Z})$. The first Chern class of a line bundle $L$ is again defined as $c_{1}(L)=\delta([L])$, but now $c_{1}(L)$ classifies the bundle (i.e. $L \simeq L^{\prime}$ if and only if $c_{1}(L)=c_{1}\left(L^{\prime}\right)$ ).

Exercise 5.3. (A rather pedantic one, to be honest...) Show that if $X$ is a complex manifold, and $L$ is a holomorphic line bundle on it, the first Chern classes of $L$ regarded as a holomorphic or smooth complex line bundle coincide. (Hint: start from the inclusion $\mathcal{O} \hookrightarrow \mathcal{C}$, write from it a diagram of exact sequences, and take it to cohomology ...)

## 6. Chern classes of vector bundles

In this section we define higher Chern classes for complex vector bundles of any rank. Since the Chern classes of a vector bundle will depend only on its smooth structure, we may consider a smooth complex vector bundle $E$ on a differentiable manifold $X$. We are already able to define the first Chern class $c_{1}(L)$ of a line bundle $L$, and we know that $c_{1}(L) \in H^{2}(X, \mathbb{Z})$. We proceed in two steps:
(1) we first define Chern classes of vector bundles that are direct sums of line bundles;
(2) and then show that by means of an operation called cohomology base change we can always reduce the computation of Chern classes to the previous situation.

Step 1. Let $\sigma_{i}, i=1 \ldots k$, denote the symmetric function of order $i$ in $k$ arguments. ${ }^{3}$. Since these functions are polynomials with integer coefficients, they can be regarded as functions on the cohomology ring $H^{\bullet}(X, \mathbb{Z})$. In particular, if $\alpha_{1}, \ldots, \alpha_{k}$ are classes in $H^{2}(X, \mathbb{Z})$, we have $\sigma_{i}\left(\alpha_{1}, \ldots, \alpha_{k}\right) \in H^{2 i}(X, \mathbb{Z})$.

If $E=L_{1} \oplus \cdots \oplus L_{k}$, where the $L_{i}$ 's are line bundles, for $i=1 \ldots k$ we define the $i$-th Chern class of $E$ as

$$
c_{i}(E)=\sigma_{i}\left(c_{1}\left(L_{1}\right), \ldots, c_{1}\left(L_{k}\right)\right) \in H^{2 i}(X, \mathbb{Z})
$$

[^15]As a first reference for symmetric functions see e.g. [21].

We also set $c_{0}(E)=1$; identifying $H^{0}(X, \mathbb{Z})$ with $\mathbb{Z}$ (assuming that $X$ is connected) we may think that $c_{0}(E) \in H^{0}(X, \mathbb{Z})$.

Step 2 relies on the following result (sometimes called the splitting principle), which we do not prove here.

Proposition 5.1. Let $E$ be a complex vector bundle on a differentiable manifold $X$. There exists a differentiable map $f: Y \rightarrow X$, where $Y$ is a differentiable manifold, such that
(1) the pullback bundle $f^{*} E$ is a direct sum of line bundles;
(2) the morphism $f^{\sharp}: H^{\bullet}(X, \mathbb{Z}) \rightarrow H^{\bullet}(Y, \mathbb{Z})$ is injective;
(3) the Chern classes $c_{i}\left(f^{*} E\right)$ lie in the image of the morphism $f^{\sharp}$.

Definition 5.2. The $i$-th Chern class $c_{i}(E)$ of $E$ is the unique class in $H^{2 i}(X, \mathbb{Z})$ such that $f^{\sharp}\left(c_{i}(E)\right)=c_{i}\left(f^{*} E\right)$.

We also define the total Chern class of $E$ as

$$
c(E)=\sum_{i=0}^{k} c_{i}(E) \in H^{\bullet}(X, \mathbb{Z})
$$

The main property of the Chern classes are the following.
(1) If two vector bundles on $X$ are isomorphic, their Chern classes coincide.
(2) Functoriality: if $f: Y \rightarrow X$ is a differentiable map of differentiable manifolds, and $E$ is a complex vector bundle on $X$, then

$$
f^{\sharp}\left(c_{i}(E)\right)=c_{i}\left(f^{*} E\right) .
$$

(3) Whitney product formula: if $E, F$ are complex vector bundles on $X$, then

$$
c(E \oplus F)=c(E) \cup c(F)
$$

(4) Normalization: identify the cohomology group $H^{2}\left(\mathbb{P}_{n}, \mathbb{Z}\right)$ with $\mathbb{Z}$ by identifying the class of the hyperplane $H$ with $1 \in \mathbb{Z}$. Then $c_{1}(H)=1$.

These properties characterize uniquely the Chern classes (cf. e.g. [13]). Notice that, in view of the splitting principle, it is enough to prove the properties (1), (2), (3) when $E$ and $F$ are line bundles. Then (1) and (2) are already known, and (3) follows from elementary properties of the symmetric functions.

The reader can easily check that all Chern classes (but for $c_{0}$, obviously) of a trivial vector bundle vanish. Thus, Chern classes in some sense measure the twisting of a bundle. It should be noted that, even in smooth case, Chern classes do not in general classify vector bundles, even as smooth bundles (i.e., generally speaking, $c(E)=c(F)$ does not imply $E \simeq F$ ). However, in some specific instances this may happen.

Exercise 5.3. Prove that for any vector bundle $E$ one has $c_{1}(E)=c_{1}(\operatorname{det} E)$.

## 7. Kodaira-Serre duality

In this section we introduce Kodaira-Serre duality, which will be one of the main tools in our study of algebraic curves. To start with a simple situation, let us study the analogous result in de Rham theory. Let $X$ be a differentiable manifold. Since the exterior product of two closed forms is a closed form, one can define a bilinear map

$$
H_{D R}^{i}(X) \otimes H_{D R}^{j}(X) \rightarrow H_{D R}^{i+j}(X), \quad[\tau] \otimes[\omega] \rightarrow[\tau \wedge \omega]
$$

As we already know, via the Čech-de Rham isomorphism this product can be identified with the cup product. If $X$ is compact and oriented, by composition with the map ${ }^{4}$

$$
\int_{X}: H_{D R}^{n}(X) \rightarrow \mathbb{R}, \quad \int_{X}[\omega]=\int_{X} \omega
$$

where $n=\operatorname{dim} X$, we obtain a pairing

$$
H_{D R}^{i}(X) \otimes H_{D R}^{n-i}(X) \rightarrow \mathbb{R}, \quad[\tau] \otimes[\omega] \rightarrow \int_{X}[\tau \wedge \omega]
$$

which is quite easily seen to be nondegenerate. Thus one has an isomorphism

$$
H_{D R}^{i}(X)^{*} \simeq H_{D R}^{n-i}(X)
$$

(this is a form of Poincaré duality).
If $X$ is an $n$-dimensional compact complex manifold, in the same way we obtain a nondegenerate pairing between Dolbeault cohomology groups

$$
\begin{equation*}
H_{\bar{\partial}}^{p, q}(X) \otimes H_{\bar{\partial}}^{n-p, n-q}(X) \rightarrow \mathbb{C} \tag{5.5}
\end{equation*}
$$

and a duality

$$
H_{\bar{\partial}}^{p, q}(X)^{*} \simeq H_{\bar{\partial}}^{n-p, n-q}(X)
$$

Exercise 5.1. (1) Let $E$ be a holomorphic vector bundle on a complex manifold $X$, denote by $\mathcal{E}$ the sheaf of its holomorphic sections, and by $\mathcal{E}^{\infty}$ the sheaf of its smooth sections. Show (using a local trivialization and proving that the result is independent of the trivialization) that one can define a $\mathbb{C}$-linear sheaf morphism

$$
\begin{equation*}
\bar{\partial}_{E}: \mathcal{E}^{\infty} \rightarrow \Omega^{0,1} \otimes \mathcal{E}^{\infty} \tag{5.6}
\end{equation*}
$$

which obeys a Leibniz rule

$$
\bar{\partial}_{E}(f s)=f \bar{\partial}_{E} s+\bar{\partial} f \otimes s
$$

for $s \in \mathcal{E}^{\infty}(U), f \in C^{\infty}(U)$.
(2) Show that $\bar{\partial}_{E}$ defines an exact sequence of sheaves

$$
\begin{equation*}
0 \rightarrow \Omega^{p} \otimes \mathcal{E} \rightarrow \Omega^{p, 0} \otimes \mathcal{E}^{\infty} \xrightarrow{\bar{\partial}_{E}} \Omega^{p, 1} \otimes \mathcal{E}^{\infty} \xrightarrow{\bar{\partial}_{E}} \ldots \xrightarrow{\bar{\partial}_{E}} \Omega^{p, n} \otimes \mathcal{E}^{\infty} \rightarrow 0 \tag{5.7}
\end{equation*}
$$

[^16]Here $\Omega^{p}$ is the sheaf of holomorphic $p$-forms. In particular, $\mathcal{E}=\operatorname{ker}\left(\bar{\partial}_{E}: \mathcal{E}^{\infty} \rightarrow \Omega^{0,1} \otimes\right.$ $\left.\mathcal{E}^{\infty}\right)$.
(3) By taking global sections in (5.7), and taking coholomology from the resulting (in general) non-exact sequence, one defines Dolbeault cohomology groups with coefficients in $E$, denoted $H_{\bar{\partial}}^{p, q}(X, E)$. Use the same argument as in the proof of de Rham's theorem to prove an isomorphism

$$
\begin{equation*}
H_{\bar{\partial}}^{p, q}(X, E) \simeq H^{q}\left(X, \Omega^{p} \otimes \mathcal{E}\right) \tag{5.8}
\end{equation*}
$$

By combining the pairing (5.5) with the action of the sections of $E^{*}$ on the sections of $E$ we obtain a nondegenerate pairing

$$
H_{\bar{\partial}}^{p, q}(X, E) \otimes H_{\bar{\partial}}^{n-p, n-q}\left(X, E^{*}\right) \rightarrow \mathbb{C}
$$

and therefore a duality

$$
H_{\bar{\partial}}^{p, q}(X, E)^{*} \simeq H_{\bar{\partial}}^{n-p, n-q}\left(X, E^{*}\right)
$$

Using the isomorphism (5.8) we can express this duality in the form

$$
H^{p}\left(X, \Omega^{q} \otimes \mathcal{E}\right)^{*} \simeq H^{n-p}\left(X, \Omega^{n-q} \otimes \mathcal{E}^{*}\right)
$$

This is the Kodaira-Serre duality. In particular for $q=0$ we get (denoting $K=\Omega^{n}=$ $\operatorname{det} T^{*} X$, the canonical bundle of $X$ )

$$
H^{p}(X, \mathcal{E})^{*} \simeq H^{n-p}\left(X, K \otimes \mathcal{E}^{*}\right)
$$

This is usually called Serre duality.

## 8. Connections

In this section we give the basic definitions and sketch the main properties of connections. The concept of connection provides the correct notion of differential operator to differentiate the sections of a vector bundle.
8.1. Basic definitions. Let $E$ a complex, in general smooth, vector bundle on a differentiable manifold $X$. We shall denote by $\mathcal{E}$ the sheaf of sections of $E$, and by $\Omega_{X}^{1}$ the sheaf of differential 1-forms on $X$. A connection is a sheaf morphism

$$
\nabla: \mathcal{E} \rightarrow \Omega_{X}^{1} \otimes \mathcal{E}
$$

satisfying a Leibniz rule

$$
\nabla(f s)=f \nabla(s)+d f \otimes s
$$

for every section $s$ of $E$ and every function $f$ on $X$ (or on an open subset). The Leibniz rule also shows that $\nabla$ is $\mathbb{C}$-linear. The connection $\nabla$ can be made to act on all sheaves $\Omega_{X}^{k} \otimes \mathcal{E}$, thus getting a morphism

$$
\nabla: \Omega_{X}^{k} \otimes \mathcal{E} \rightarrow \Omega_{X}^{k+1} \otimes \mathcal{E}
$$

by letting

$$
\nabla(\omega \otimes s)=d \omega \otimes s+(-1)^{k} \omega \otimes \nabla(s)
$$

If $\left\{U_{\alpha}\right\}$ is a cover of $X$ over which $E$ trivializes, we may choose on any $U_{\alpha}$ a set $\left\{s_{\alpha}\right\}$ of basis sections of $\mathcal{E}\left(U_{\alpha}\right)$ (notice that this is a set of $r$ sections, with $r=\operatorname{rk} E$ ). Over these bases the connection $\nabla$ is locally represented by matrix-valued differential 1 -forms $\omega_{\alpha}$ :

$$
\nabla\left(s_{\alpha}\right)=\omega_{\alpha} \otimes s_{\alpha}
$$

Every $\omega_{\alpha}$ is as an $r \times r$ matrix of 1-forms. The $\omega_{\alpha}$ 's are called connection 1-forms.
Exercise 5.1. Prove that if $g_{\alpha \beta}$ denotes the transition functions of $E$ with respect to the chosen local basis sections (i.e., $s_{\alpha}=g_{\alpha \beta} s_{\beta}$ ), the transformation formula for the connection 1-forms is

$$
\begin{equation*}
\omega_{\alpha}=g_{\alpha \beta} \omega_{\beta} g_{\alpha \beta}^{-1}+d g_{\alpha \beta} g_{\alpha \beta}^{-1} \tag{5.9}
\end{equation*}
$$

The connection is not a tensorial morphism, but rather satifies a Leibniz rule; as a consequence, the transformation properties of the connection 1-forms are inhomogeneous and contain an affine term.

Exercise 5.2. Prove that if $E$ and $F$ are vector bundles, with connections $\nabla_{1}$ and $\nabla_{2}$, then the rule

$$
\nabla(s \otimes t)=\nabla_{1}(s) \otimes t+s \otimes \nabla_{2}(t)
$$

(minimal coupling) defines a connection on the bundle $E \otimes F$ (here $s$ and $t$ are sections of $E$ and $F$, respectively).

Exercise 5.3. Prove that is $E$ is a vector bundle with a connection $\nabla$, the rule

$$
<\nabla^{*}(\tau), s>=d<\tau, s>-<\tau, \nabla(s)>
$$

defines a connection on the dual bundle $E^{*}$ (here $\tau, s$ are sections of $E^{*}$ and $E$, respectively, and $<,>$ denotes the pairing between sections of $E^{*}$ and $\left.E\right)$.

It is an easy exercise, which we leave to the reader, to check that the square of the connection

$$
\nabla^{2}: \Omega_{X}^{k} \otimes \mathcal{E} \rightarrow \Omega_{X}^{k+2} \otimes \mathcal{E}
$$

is $f$-linear, i.e., it satisfies the property

$$
\nabla^{2}(f s)=f \nabla^{2}(s)
$$

for every function $f$ on $X$. In other terms, $\nabla^{2}$ is an endomorphism of the bundle $E$ with coefficients in 2 -forms, namely, a global section of the bundle $\Omega_{X}^{2} \otimes \operatorname{End}(E)$. It is called the curvature of the connection $\nabla$, and we shall denote it by $\Theta$. On local basis sections $s_{\alpha}$ it is represented by the curvature 2 -forms $\Theta_{\alpha}$ defined by

$$
\Theta\left(s_{\alpha}\right)=\Theta_{\alpha} \otimes s_{\alpha}
$$

Exercise 5.4. Prove that the curvature 2-forms may be expressed in terms of the connection 1-forms by the equation (Cartan's structure equation)

$$
\begin{equation*}
\Theta_{\alpha}=d \omega_{\alpha}-\omega_{\alpha} \wedge \omega_{\alpha} \tag{5.10}
\end{equation*}
$$

Exercise 5.5. Prove that the transformation formula for the curvature 2 -forms is

$$
\Theta_{\alpha}=g_{\alpha \beta} \Theta_{\beta} g_{\alpha \beta}^{-1} .
$$

Due to the tensorial nature of the curvature morphism, the curvature 2 -forms obey a homogeneous transformation rule, without affine term.

Since we are able to induce connections on tensor products of vector bundles (and also on direct sums, in the obvious way), and on the dual of a bundle, we can induce connections on a variety of bundles associated to given vector bundles with connections, and thus differentiate their sections. The result of such a differentiation is called the covariant differential of the section. In particular, given a vector bundle $E$ with connection $\nabla$, we may differentiate its curvature as a section of $\Omega_{X}^{2} \otimes \operatorname{End}(E)$.

Proposition 5.6. (Bianchi identity) The covariant differential of the curvature of a connection is zero, $\nabla \Theta=0$.

Proof. A simple computation shows that locally $\nabla \Theta$ is represented by the matrixvalued 3 -forms

$$
d \Theta_{\alpha}+\omega_{\alpha} \wedge \Theta_{\alpha}-\Theta_{\alpha} \wedge \omega_{\alpha}
$$

By plugging in the structure equation (5.10) we obtain $\nabla \Theta=0$.
8.2. Connections and holomorphic structures. If $X$ is a complex manifold, and $E$ a $C^{\infty}$ complex vector bundle on it with a connection $\nabla$, we may split the latter into its $(1,0)$ and $(0,1)$ parts, $\nabla^{\prime}$ and $\nabla^{\prime \prime}$, according to the splitting $\Omega_{X}^{1} \otimes \mathbb{C}=\Omega_{X}^{1,0} \oplus \Omega_{X}^{0,1}$. Analogously, the curvature splits into its $(2,0),(1,1)$ and $(0,2)$ parts,

$$
\Theta=\Theta^{2,0}+\Theta^{1,1}+\Theta^{0,2}
$$

Obviously we have

$$
\Theta^{2,0}=\left(\nabla^{\prime}\right)^{2}, \quad \Theta^{1,1}=\nabla^{\prime} \circ \nabla^{\prime \prime}+\nabla^{\prime \prime} \circ \nabla^{\prime}, \quad \Theta^{0,2}=\left(\nabla^{\prime \prime}\right)^{2}
$$

In particular $\nabla^{\prime \prime}$ is a morphism $\Omega_{X}^{p, q} \otimes \mathcal{E} \rightarrow \Omega_{X}^{p, q+1} \otimes \mathcal{E}$. If $\Theta^{0,2}=0$, then $\nabla^{\prime \prime}$ is a differential for the complex $\Omega_{X}^{p, \bullet} \otimes \mathcal{E}$. The same condition implies that the kernel of the map

$$
\begin{equation*}
\nabla^{\prime \prime}: \mathcal{E} \rightarrow \Omega_{X}^{0,1} \otimes \mathcal{E} \tag{5.11}
\end{equation*}
$$

has enough sections to be the sheaf of sections of a holomorphic vector bundle.

Proposition 5.7. If $\Theta^{0,2}=0$, then the $C^{\infty}$ vector bundle $E$ admits a unique holomorphic structure, such that the corresponding sheaf of holomorphic sections is isomorphic to the kernel of the operator (5.11). Moreover, under this isomorphism the operator (5.11) concides with the operator $\bar{\delta}_{E}$ defined in Exercise 5.1.

Proof. Cf. [16], p. 9.
Conversely, if $E$ is a holomophic vector bundle, a connection $\nabla$ on $E$ is said to be compatible with the holomorphic structure of $E$ if $\nabla^{\prime \prime}=\partial_{E}$.
8.3. Hermitian bundles. A Hermitian metric $h$ of a complex vector bundle $E$ is a global section of $E \otimes \overline{E^{*}}$ which when restricted to the fibres yields a Hermitian form on them (more informally, it is a smoothly varying assignation of Hermitian structures on the fibres of $E)$. On a local basis of sections $\left\{s_{\alpha}\right\}$, of $E, h$ is represented by matrices $h_{\alpha}$ of functions on $U_{\alpha}$ which, when evaluated at any point of $U_{\alpha}$, are Hermitian and positive definite. The local basis is said to be unitary if the corresponding matrix $h$ is the identity matrix.

A pair $(E, h)$ formed by a holomorphic vector bundle with a hermitian metric is called a hermitian bundle. A connection $\nabla$ on $E$ is said to be metric with respect to $h$ if for every pair $s, t$ of sections of $E$ one has

$$
d h(s, t)=h(\nabla s, t)+h(s, \nabla t)
$$

In terms of connection forms and matrices representing $h$ this condition reads

$$
\begin{equation*}
d h_{\alpha}=\tilde{\omega}_{\alpha} h_{\alpha}+h_{\alpha} \bar{\omega}_{\alpha} \tag{5.12}
\end{equation*}
$$

where ${ }^{\sim}$ denotes transposition and ${ }^{-}$denotes complex conjugation (but no transposition, i.e., it is not the hermitian conjugation). This equation implies right away that on a unitary frame, the connection forms are skew-hermitian matrices.

Proposition 5.8. Given a hermitian bundle $(E, h)$, there is a unique connection $\nabla$ on $E$ which is metric with respect to $h$ and is compatible with the holomorphic structure of $E$.

Proof. If we use holomophic local bases of sections, the connection forms are of type $(1,0)$. Then equation (5.12) yields

$$
\begin{equation*}
\tilde{\omega}_{\alpha}=\partial h_{\alpha} h_{\alpha}^{-1} \tag{5.13}
\end{equation*}
$$

and this equations shows the uniqueness. As for the existence, one can easily check that the connection forms as defined by equation (5.13) satisfy the condition (5.9) and therefore define a connection on $E$. This is metric w.r.t. $h$ and compatible with the holomorphic structure of $E$ by construction.

Example 5.9. (Chern classes and Maxwell theory) The Chern classes of a complex vector bundle $E$ can be calculated in terms of a connection on $E$ via the so-called ChernWeil representation theorem. Let us discuss a simple situation. Let $L$ be a complex line bundle on a smooth 2-dimensional manifold $X$, endowed with a connection, and let $F$ be the curvature of the connection. $F$ can be regarded as a 2 -form on $X$. In this case the Chern-Weil theorem states that

$$
\begin{equation*}
c_{1}(L)=\frac{i}{2 \pi} \int_{X} F \tag{5.14}
\end{equation*}
$$

where we regard $c_{1}(L)$ as an integer number via the isomorphism $H^{2}(X, \mathbb{Z}) \simeq \mathbb{Z}$ given by integration over $X$. Notice that the Chern class of $F$ is independent of the connection we have chosen, as it must be. Alternatively, we notice that the complex-valued form $F$ is closed (Bianchi identity) and therefore singles out a class $[F]$ in the complexified de Rham group $H_{D R}^{2}(X) \otimes_{\mathbb{R}} \mathbb{C} \simeq H^{2}(X, \mathbb{C})$; the class $\frac{i}{2 \pi}[F]$ is actually real, and one has the equality

$$
c_{1}(L)=\frac{i}{2 \pi}[F]
$$

in $H_{D R}^{2}(X)$. If we consider a static spherically symmetric magnetic field in $\mathbb{R}^{3}$, by solving the Maxwell equations we find a solution which is singular at the origin. If we do not consider the dependence from the radius the vector potential defines a connection on a bundle $L$ defined on an $S^{2}$ which is spanned by the angular spherical coordinates. The fact that the Chern class of $L$ as given by (5.14) can take only integer values is known in physics as the quantization of the Dirac monopole.

## CHAPTER 6

## Divisors

Divisors are a powerful tool to study complex manifolds. We shall start with the onedimensional case. The notion will be later generalized to higher dimensional manifolds.

## 1. Divisors on Riemann surfaces

Let $S$ be a Riemann surface (a complex manifold of dimension 1). A divisor $D$ on $S$ is a locally finite formal linear combinations of points of $S$ with integer coefficients,

$$
D=\sum a_{i} p_{i}, \quad a_{i} \in \mathbb{Z}, \quad p_{i} \in S,
$$

where "locally finite" means that every point $p$ in $S$ has a neighbourhood which contains only a finite number of $p_{i}$ 's. If $S$ is compact, this means that the number of points is finite. We say that the divisor $D$ is effective if $a_{i} \geq 0$ for all $i$. We shall then write $D \geq 0$.

The set of all divisors of $S$ forms an abelian group, denoted by $\operatorname{Div}(S)$.
Let $f$ a holomorphic function defined in a neighbourhood of $p$, and let $z$ be a local coordinate around $p$. There exists a unique nonnegative integer $a$ and a holomorphic function $h$ such that

$$
f(z)=(z-z(p))^{a} h(z)
$$

and $h(p) \neq 0$. We define

$$
\operatorname{ord}_{p} f=a .
$$

Notice that

$$
\begin{equation*}
\operatorname{ord}_{p} f g=\operatorname{ord}_{p} f+\operatorname{ord}_{p} g . \tag{6.1}
\end{equation*}
$$

If $f$ is a meromorphic function which in a neighbourhood of $p$ can be written as $f=g / h$, with $g$ and $h$ holomorphic, we define

$$
\operatorname{ord}_{p} f=\operatorname{ord}_{p} g-\operatorname{ord}_{p} h .
$$

We say that $f$ has a zero of order $a$ at $p$ if $\operatorname{ord}_{p} f=a>0$ (then $f$ is holomorphic in a neighbourhood of $p$ ), and that it has a pole of order $a$ if $\operatorname{ord}_{p} f=-a<0$.

With each meromorphic function $f$ we may associate the divisor

$$
(f)=\sum_{p \in S} \operatorname{ord}_{p} f \cdot p
$$

if $f=g / h$ with $g$ and $h$ relatively prime, then $(f)=(g)-(h)$.
1.1. Sheaf-theoretic description of divisors. The group of divisors Div can be described in sheaf-theoretic terms as follows. Let $\mathcal{M}^{*}$ be the sheaf of meromorphic functions that are not identically zero. We have an exact sequence

$$
0 \rightarrow \mathcal{O}^{*} \rightarrow \mathcal{M}^{*} \rightarrow \mathcal{M}^{*} / \mathcal{O}^{*} \rightarrow 0
$$

of sheaves of abelian groups (notice that the group structure is multiplicative).
Proposition 6.1. There is a group isomorphism $\operatorname{Div}(S) \simeq H^{0}\left(S, \mathcal{M}^{*} / \mathcal{O}^{*}\right)$.

Proof. Given a cover $\mathfrak{U}=\left\{U_{\alpha}\right\}$ of $X$, one has a commutative diagram of exact sequences

where $H^{1}\left(U_{\alpha}, \mathcal{O}^{*}\right)=0$ because $U_{\alpha} \simeq \mathbb{C}$ holomorphically (here $\delta$ denotes the Čech cohomology operator). This diagram shows that a global section $s \in H^{0}\left(S, \mathcal{M}^{*} / \mathcal{O}^{*}\right)$ can be represented by a 0 -cochain $\left\{f_{\alpha} \in \mathcal{M}^{*}\left(U_{\alpha}\right)\right\} \in \check{C}^{0}\left(\mathfrak{U}, \mathcal{M}^{*}\right)$ subject to the condition $f_{\alpha} / f_{\beta} \in \mathcal{O}^{*}\left(U_{\alpha} \cap U_{\beta}\right)$, so that $\operatorname{ord}_{p} f_{\alpha}$ does not depend on $\alpha$, and the quantity $\operatorname{ord}_{p} s$ is well defined. We set $D=\sum_{p} \operatorname{ord}_{p} s \cdot p$.

Conversely, given $D=\sum a_{i} p_{i}$, we may choose an open cover $\left\{U_{\alpha}\right\}$ such that each $U_{\alpha}$ contains at most one $p_{i}$, and functions $g_{i \alpha} \in \mathcal{O}\left(U_{\alpha}\right)$ such that that $g_{i \alpha}$ has a zero of order one at $p_{i}$ if $p_{i} \in U_{\alpha}$. We set

$$
f_{\alpha}=\prod_{i} g_{i \alpha}^{a_{i}}
$$

Then $f_{\alpha} / f_{\beta} \in \mathcal{O}^{*}\left(U_{\alpha} \cap U_{\beta}\right)$, so that $\left\{f_{\alpha}\right\}$ determines a global section of $\mathcal{M}^{*} / \mathcal{O}^{*}$.
The two constructions are one the inverse of the other, so that they establish an isomorphism of sets. The fact that this is also a group homomorphism follows from the formula (6.1), which holds also for meromorphic functions.
1.2. Correspondence between divisors and line bundles. Let $D \in \operatorname{Div}(S)$, and let $\left\{U_{\alpha}\right\}$ be an open cover of $S$ with meromorphic functions $\left\{f_{\alpha}\right\}$ which define the divisor, according to Proposition 6.1. Then the functions

$$
g_{\alpha \beta}=\frac{f_{\alpha}}{f_{\beta}} \in \mathcal{O}^{*}\left(U_{\alpha} \cap U_{\beta}\right)
$$

obviously satisfy the cocycle condition, and define a line bundle, which we denote by $[D]$. The line bundle $[D]$ in independent, up to isomorphism, of the set of functions defining $D$; if $\left\{f_{\alpha}^{\prime}\right\}$ is another set, then $\operatorname{ord}_{p_{i}} f_{\alpha}=\operatorname{ord}_{p_{i}} f_{\alpha}^{\prime}$, so that the functions $h_{\alpha}=f_{\alpha} / f_{\alpha}^{\prime}$ are holomorphic and nowhere vanishing, and

$$
g_{\alpha \beta}^{\prime}=\frac{f_{\alpha}^{\prime}}{f_{\beta}^{\prime}}=\frac{f_{\alpha}}{f_{\beta}} \frac{h_{\beta}}{h_{\alpha}}=g_{\alpha \beta} \frac{h_{\beta}}{h_{\alpha}},
$$

so that the transition functions $g_{\alpha \beta}^{\prime}$ define an isomorphic line bundle.
If $D=D^{(1)}+D^{(2)}$ then $f_{\alpha}=f_{\alpha}^{(1)} f_{\alpha}^{(2)}$ by eq. (6.1), so that $\left[D^{(1)}+D^{(2)}\right]=\left[D^{(1)}\right] \otimes$ $\left[D^{(2)}\right]$, and one has a homomorphism $\operatorname{Div}(S) \rightarrow \operatorname{Pic}(S)$.

We offer now a sheaf-theoretic description of this homomorphism. Let $f=\left\{f_{\alpha}\right\} \in$ $H^{0}\left(S, \mathcal{M}^{*}\right)$; let us set $f_{\alpha}=g_{\alpha} / h_{\alpha}$, with $g_{\alpha}, h_{\alpha} \in \mathcal{O}\left(U_{\alpha}\right)$ relatively prime. We have $(f)=(g)-(h)$, with $(g)$ and $(h)$ effective divisors. The line bundle $[(f)]$ has transition functions

$$
g_{\alpha \beta}=\frac{g_{\alpha}}{g_{\beta}} \frac{h_{\beta}}{h_{\alpha}}=\frac{f_{\alpha}}{f_{\beta}}=1
$$

(since $f$ is a Čech cocycle) so that $[(f)]=\underline{\mathbb{C}}$, i.e. $[(f)]$ is the trivial line bundle.
Conversely, let $D$ be a divisor such that $[D]=\mathbb{C}$; then the transition functions of [ $D$ ] have the form

$$
g_{\alpha \beta}=\frac{h_{\alpha}}{h_{\beta}} \quad \text { with } \quad h_{\alpha} \in \mathcal{O}^{*}\left(U_{\alpha}\right) .
$$

Let $\left\{f_{\alpha}\right\}$ be meromorphic functions which define $D$, so that one also has $g_{\alpha \beta}=\frac{f_{\alpha}}{f_{\beta}}$, and

$$
\frac{f_{\alpha}}{h_{\alpha}}=g_{\alpha \beta} \frac{f_{\beta}}{h_{\alpha}}=\frac{f_{\beta}}{h_{\beta}} ;
$$

the quotients $\frac{f_{\alpha}}{h_{\alpha}}$ therefore determine a global nonzero meromorphic function, namely:
Proposition 6.2. The line bundle associated with a divisor $D$ is trivial if and only if $D$ is the divisor of a global meromorphic function.

In view of the identifications $\operatorname{Div}(S) \simeq H^{0}\left(S, \mathcal{M}^{*} / \mathcal{O}^{*}\right)$ and $\operatorname{Pic}(S) \simeq H^{1}\left(S, \mathcal{O}^{*}\right)$ this statement is equivalent to the exactness of the sequence

$$
H^{0}\left(S, \mathcal{M}^{*}\right) \rightarrow H^{0}\left(S, \mathcal{M}^{*} / \mathcal{O}^{*}\right) \rightarrow H^{1}\left(S, \mathcal{O}^{*}\right)
$$

Definition 6.3. Two divisors $D, D^{\prime} \in \operatorname{Div}(S)$ are linearly equivalent if $D^{\prime}=D+(f)$ for some $f \in H^{0}(S, \mathcal{M})$.

Quite evidently, $D$ and $D^{\prime}$ are linearly equivalent if and only if $[D] \simeq\left[D^{\prime}\right]$, so that there is an injective group homomorphism

$$
\operatorname{Div}(S) /\{\text { linear equivalence }\} \rightarrow \operatorname{Pic}(S) .
$$

1.3. Holomorphic and meromorphic sections of line bundles. If $L$ is a line bundle on $S$, we denote by $\mathcal{O}(L)$ the sheaf of its holomorphic sections, and by $\mathcal{M}(L)$ the sheaf of its meromorphic sections, the latter being defined as $\mathcal{M}(L)=\mathcal{O}(L) \otimes \mathcal{O} \mathcal{M}$. If $L$ has transition functions $g_{\alpha \beta}$ with respect to a cover $\left\{U_{\alpha}\right\}$ of $S$, then a global holomorphic section $s \in H^{0}(S, \mathcal{O}(L))$ of $L$ corresponds to a collection of functions $\left\{s_{\alpha} \in \mathcal{O}\left(U_{\alpha}\right)\right\}$ such that $s_{\alpha}=g_{\alpha \beta} s_{\beta}$ on $U_{\alpha} \cap U_{\beta}$. The same holds for meromorphic sections. A first consequence of this is that, if $s, s^{\prime} \in H^{0}(S, \mathcal{M}(L))$, we have

$$
\frac{s_{\alpha}}{s_{\alpha}^{\prime}}=\frac{g_{\alpha \beta} s_{\beta}}{g_{\alpha \beta} s_{\beta}^{\prime}}=\frac{s_{\beta}}{s_{\beta}^{\prime}} \quad \text { on } \quad U_{\alpha} \cap U_{\beta}
$$

so that the quotient of $s$ and $s^{\prime}$ is a well-defined global meromorphic function on $S$.
Let $s \in H^{0}(S, \mathcal{M}(L))$; we have

$$
\frac{s_{\alpha}}{s_{\beta}}=g_{\alpha \beta} \in \mathcal{O}^{*}\left(U_{\alpha} \cap U_{\beta}\right)
$$

so that

$$
\operatorname{ord}_{p} s_{\alpha}=\operatorname{ord}_{p} s_{\beta} \quad \text { for all } \quad p \in U_{\alpha} \cap U_{\beta}
$$

the quantity $\operatorname{ord}_{p} s$ is well defined, and we may associate with $s$ the divisor

$$
(s)=\sum_{p \in S} \operatorname{ord}_{p} s \cdot p
$$

By construction we have $[(s)] \simeq L$. Obviously, $s$ is holomorphic if and only if $(s)$ is effective.

So we have
Proposition 6.4. A line bundle $L$ is associated with a divisor $D$ (i.e. $L=[D]$ for some $D \in \operatorname{Div}(S)$ ) if and only if it has a global nontrivial meromorphic section. $L$ is the line bundle associated with an effective divisor if and only if it has a global nontrivial holomorphic section.

Proof. The "if" part has already been proven. For the "only if" part, let $L=[D]$ with $D$ a divisor with local equations $f_{\alpha}=0$. Then $f_{\alpha}=g_{\alpha \beta} f_{\beta}$, where the functions $g_{\alpha \beta}$ are transition functions for $L$; the functions $f_{\alpha}$ glue to yield a global meromorphic section $s$ of $L$. If $D$ is effective the functions $f_{\alpha}$ are holomorphic so that $s$ is holomorphic as well.

Corollary 6.5. The line bundle $[p]$ trivializes over the cover $\left\{U_{1}, U_{2}\right\}$, where $U_{1}=$ $S-\{p\}$ and $U_{2}$ is a neighbourhood of $p$, biholomorphic to a disc in $\mathbb{C}$.

Proof. Since $[p]$ is effective it has a global holomorphic section which vanishes only at $p$, so that $[p]$ is trivial on $U_{1}$. Of course it is trivial on $U_{2}$ as well.

So the same happens for the line bundles $[k p], k \in \mathbb{Z}$.

For the remainder of this section we assume that $S$ is compact. Let us define the degree of a divisor $D=\sum a_{i} p_{i}$ as the integer

$$
\operatorname{deg} D=\sum a_{i}
$$

For simplicity we shall write $\mathcal{O}(D)$ for $\mathcal{O}([D])$.
Corollary 6.6. If $\operatorname{deg} D<0$, then $H^{0}(S, \mathcal{O}(D))=0$.
If $L$ is a line bundle we denote by $\int_{S} c_{1}(L)$ the number obtained by integrating over $S$ a differential 2-form which via de Rham isomorphism represents ${ }^{1}$ the Čech cohomology class $c_{1}(L)$ regarded as an element in $H^{2}(S, \mathbb{R})$.

Proposition 6.7. For any $D \in \operatorname{Div}(S)$ one has

$$
\int_{S} c_{1}(D)=\operatorname{deg} D
$$

Before proving this result we need some preliminaries. We define a hermitian metric on a line bundle $L$ as an assignment of a hermitian scalar product in each $L_{p}$ which is $C^{\infty}$ in $p$; thus a hermitian metric is a $C^{\infty}$ section $h$ of the line bundle $L^{*} \otimes L^{*}$ such that each $h(p)$ is a hermitian scalar product in $L_{p}$. In terms of a local trivialization over an open cover $\left\{U_{\alpha}\right\}$ a hermitian metric is represented by nonvanishing real functions $h_{\alpha}$ on $U_{\alpha}$. On $U_{\alpha} \cap U_{\beta}$ one has $h_{\alpha}=\left|g_{\alpha \beta}\right|^{2} h_{\beta}$, so that the 2 -form $\frac{i}{2 \pi} \bar{\partial} \partial \log h_{\alpha}$ does not depend on $\alpha$, and defines a global closed 2 -form on $S$, which we denote by $\Theta$.

Lemma 6.8. The class of $\Theta$ is the image in $H_{D R}^{2}(S)$ of $c_{1}(L)$.
Proof. We need the explicit form of the de Rham correspondence. One has exact sequences

$$
\begin{equation*}
0 \rightarrow \mathbb{R} \rightarrow \mathcal{C}^{\infty} \rightarrow \mathcal{Z}^{1} \rightarrow 0, \quad 0 \rightarrow \mathcal{Z}^{1} \rightarrow \Omega^{1} \rightarrow \mathcal{Z}^{2} \rightarrow 0 \tag{6.2}
\end{equation*}
$$

(Here $\Omega^{1}$ is the sheaf of smooth real-valued 1 -forms.) From the long exact cohomology sequences of the second sequence we get

$$
H^{0}\left(S, \Omega^{1}\right) \rightarrow H^{0}\left(S, \mathcal{Z}^{2}\right) \rightarrow H^{1}\left(S, \mathcal{Z}^{1}\right) \rightarrow 0
$$

so that the connecting morphism $H^{0}\left(S, \mathcal{Z}^{2}\right) \rightarrow H^{1}\left(S, \mathcal{Z}^{1}\right)$ induces an isomorphism $H_{D R}^{2}(S) \rightarrow H^{1}\left(S, \mathcal{Z}^{1}\right)$. Since we may write $\Theta=\frac{i}{2 \pi} d \partial \log h_{\alpha}$ a cocycle representing the image of $[\Theta]$ in $H^{1}\left(S, \mathcal{Z}^{1}\right)$ is $\left\{\theta_{\alpha}-\theta_{\beta}\right\}$, with

$$
\theta_{\alpha}=\frac{i}{2 \pi} \partial \log h_{\alpha} .
$$

Notice that

$$
\theta_{\alpha}-\theta_{\beta}=\frac{i}{2 \pi} \partial\left(\log h_{\alpha}-\log h_{\beta}\right)=\frac{i}{2 \pi} d \log g_{\alpha \beta}
$$

so that $d\left(\theta_{\alpha}-\theta_{\beta}\right)=0$.

[^17]If we consider now the first of the sequences (6.2) we obtain from its long cohomology exact sequence a segment

$$
0 \rightarrow H^{1}\left(S, \mathcal{Z}^{1}\right) \rightarrow H^{2}(S, \mathbb{R}) \rightarrow 0
$$

so that the connecting morphism is now an isomorphism. If we apply it to the 1-cocycle $\left\{\theta_{\alpha}-\theta_{\beta}\right\}$ we get the 2 -cocycle of $\mathbb{R}$

$$
\frac{1}{2 \pi i} \log g_{\alpha \beta}+\frac{1}{2 \pi i} \log g_{\beta \gamma}+\frac{1}{2 \pi i} \log g_{\gamma \alpha}=\left(c_{1}(L)\right)_{\alpha \beta \gamma} .
$$

Proof of Proposition 6.7: Since $c_{1}$ and deg are both group homomorphisms, it is enough to consider the case $D=[p]$. Consider the open cover $\left\{U_{1}, U_{2}\right\}$, where $U_{1}=$ $S-\{p\}$, and $U_{2}$ is a small patch around $p$. Then

$$
\int_{S} c_{1}(D)=\int_{S} \Theta=\frac{i}{2 \pi} \lim _{\epsilon \rightarrow 0} \int_{S-B(\epsilon)} d \partial \log h_{1}
$$

where $B(\epsilon)$ is the disc $|z|<\epsilon$, with $z$ a local coordinate around $p$, and $z(p)=0$. Since $\bar{\partial} \partial=\frac{1}{2} d(\partial-\bar{\partial})$, and assuming that $h_{1 \mid U_{2}-B(\epsilon)}=|z|^{2}$, which can always be arranged, we have

$$
\int_{S} c_{1}(D)=\frac{1}{2 \pi i} \lim _{\epsilon \rightarrow 0} \int_{\partial B(\epsilon)} \partial \log z \bar{z}=\frac{1}{2 \pi i} \int_{\partial B(\epsilon)} \frac{d z}{z}=1
$$

having used Stokes' theorem and the residue theorem (note a change of sign due to a reversal of the orientation of $\partial B(\epsilon)$ ).

This result suggests to set

$$
\operatorname{deg} L=\int_{S} c_{1}(L)
$$

for all line bundles on $S$.
Corollary 6.9. If $\operatorname{deg} L<0$, then $H^{0}(S, \mathcal{O}(L))=0$.

Proof. If there is a nonzero $s \in H^{0}(S, \mathcal{O}(L))$, then $L=[D]$ with $D=(s)$. Since $\operatorname{deg} D<0$ by the previous Proposition, this contradicts Corollary 6.6.

Corollary 6.10. A global meromorphic function on a compact Riemann surface has the same number of zeroes and poles (both counted with their multiplicities).

Proof. If $f$ global meromorphic function, we must show that $\operatorname{deg}(f)=0$. But $f$ is a global meromorphic section of the trivial line bundle $\mathbb{C}$, whence

$$
\operatorname{deg}(f)=\int_{S} c_{1}(\underline{\mathbb{C}})=0
$$

1.4. The fundamental exact sequence of an effective divisor. Let us first define for all $p \in S$ the sheaf $k_{p}$ as the 1 -dimensional skyscraper sheaf concentrated at $p$, namely, the sheaf

$$
k_{p}(U)=\mathbb{C} \quad \text { if } \quad p \in U, \quad k_{p}(U)=0 \quad \text { if } \quad p \notin U .
$$

$k_{p}$ has stalk $\mathbb{C}$ at $p$ and stalk 0 elsewhere.
Let $D=\sum a_{i} p_{i}$ be an effective divisor. Then the line bundle $L=[D]$ has at least one section $s$; this allows one to define a morphism $\mathcal{O} \rightarrow \mathcal{O}(D)$ by letting $f \mapsto f s_{\mid U}$ for every $f \in \mathcal{O}(U)$. We also define the skyscraper sheaf $k_{D}=\sum_{i}\left(k_{p_{i}}\right)^{a_{i}}$ concentrated on D.

Proposition 6.11. The sequence

$$
\begin{equation*}
0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}(D) \rightarrow k_{D} \rightarrow 0 \tag{6.3}
\end{equation*}
$$

is exact.
Proof. We shall actually prove the exactness of the sequence

$$
\begin{equation*}
0 \rightarrow \mathcal{O}(-D) \rightarrow \mathcal{O} \rightarrow k_{D} \rightarrow 0 \tag{6.4}
\end{equation*}
$$

from which the previous sequence is obtained by tensoring by $\mathcal{O}(D) .{ }^{2}$ Notice also that $k_{D} \otimes_{\mathcal{O}} \mathcal{O}(D) \simeq k_{D}$ because in a neighbourhood of every point $p_{i}$ the sheaf $\mathcal{O}(D)$ is isomorphic to $\mathcal{O}$.

The exactness of the sequence (6.4) follows from the fact the any local holomorphic function can be represented around $p_{i}$ in the form (Taylor polynomial)

$$
f(z)=f\left(z_{0}\right)+\sum_{k=1}^{a_{i}-1} \frac{1}{k!} f^{(k)}\left(z_{0}\right)\left(z-z_{0}\right)^{k}+\left(z-z_{0}\right)^{a_{i}} g(z)
$$

where $z_{0}=z(p)$, and $g$ is a holomorphic function. The term $\left(z-z_{0}\right)^{a_{i}} g(z)$ is a section of $\mathcal{O}(-D)$, while the first two terms on the right single out a section of $k_{D}$.

The sheaf $\mathcal{O}(-D)$ can be regarded as the sheaf of holomorphic functions which at $p_{i}$ have a zero of order at least $a_{i}$. Since $\mathcal{O}(D) \simeq \mathcal{O}(-D)^{*}$, the $\mathcal{O}(D)$ may be identified with the sheaf of meromorphic functions which at $p_{i}$ have a pole of order at most $a_{i}$.

In particular one may write

$$
0 \rightarrow \mathcal{O}(-2 p) \rightarrow \mathcal{O} \rightarrow k_{p} \oplus T_{p}^{*} S \rightarrow 0
$$

where $T_{p}^{*} S$ is considered as a skyscraper sheaf concentrated at $p$ (indeed the quantity $f^{\prime}\left(z_{0}\right)$ determines an element in $\left.T_{p}^{*} S\right)$.

If $E$ is a holomorphic vector bundle on $S$, let us denote $E(D)=E \otimes[D]$. Then by tensoring the exact sequence (6.4) by $\mathcal{O}(E)$ we get

[^18]$$
0 \rightarrow \mathcal{O}(E(-D)) \rightarrow \mathcal{O}(E) \rightarrow E_{D} \rightarrow 0
$$
where $E_{D}=\oplus_{i} E_{p_{i}}^{\oplus a_{i}}$ is a skyscraper sheaf concentrated on $D$.

## 2. Divisors on higher-dimensional manifolds

We start with some preparatory material.
Definition 6.1. An analytic subvariety $V$ of a complex manifold $X$ is a subset of $X$ which is locally defined as the zero set of a finite collection of holomorphic functions.

An analytic subvariety $V$ is said to be reducible if $V=V_{1} \cup V_{2}$ with $V_{1}$ and $V_{2}$ properly contained in $V . V$ is said to be irreducible if it is not reducible.

A point $p \in V$ is a smooth point of $V$ if around $p$ the subvariety $V$ is a submanifold, namely, it can be written as $f_{1}\left(z^{1}, \ldots, z^{n}\right)=\ldots f_{k}\left(z^{1}, \ldots, z^{n}\right)=0$ with $\operatorname{rank} J=k$, where $\left\{z^{1}, \ldots, z^{n}\right\}$ is a local coordinate system for $X$ around $p$, and $J$ is the jacobian matrix of the functions $f_{1}, \ldots f_{k}$. The set of smooth points of $V$ is denoted by $V^{*}$; the set $V_{s}=V-V^{*}$ is the singular locus of $V$. The dimension of $V$ is by definition the dimension of $V^{*}$.

If $\operatorname{dim} V=\operatorname{dim} X-1, V$ will be called an analytic hypersurface.
Proposition 6.2. Any analytic subvariety $V$ can be expressed around a point $p \in V$ as the union of a finite number of analytic subvarieties $V_{i}$ which are irreducible around $p$, and are such that $V_{i} \not \subset V_{j}$.

Proof. This follows from the fact that the stalk $\mathcal{O}_{p}$ is a unique factorization domain ([9] page 12). ${ }^{3}$ Let us sketch the proof for hypersurfaces. In a neighbourhood of $p$ the hypersurface $V$ is given by $f=0$. Denoting by the same letter the germ of $f$ in $p$, since $\mathcal{O}_{p}$ (where $\mathcal{O}$ is the sheaf of holomorphic functions on $X$ ) is a unique factorization domain we have

$$
f=f_{1} \cdots \cdots f_{m}
$$

where the $f_{i}$ 's are irreducible in $\mathcal{O}_{p}$, and are defined up to multiplication by invertible elements in $\mathcal{O}_{p}$; if $V_{i}$ is the locus of zeroes of $f_{i}$, then $V=\cup_{i} V_{i}$. Since $f_{i}$ irreducible, $V_{i}$ is irreducible as well; since it is not true that $f_{j}=g f_{i}$ for some $g \in \mathcal{O}_{p}$ which vanishes at $p$, we also have $V_{i} \not \subset V_{j}$.

We may now give the general definition of divisor:

[^19]Definition 6.3. $A$ divisor $D$ on a complex manifold $X$ is a locally finite formal linear combination with integer coefficients $D=\sum a_{i} V_{i}$, where the $V_{i}$ 's are irreducible analytic hypersurfaces in $X$.

If $V \subset X$ is an analytic irreducible hypersurface, and $p \in V$, we may choose around $p$ a coordinate system $\left\{w, z^{2}, \ldots, z^{n}\right\}$ such that $V$ is given around $p$ by $w=0$. Given a function $f$ defined in a neighbourhood of $p$, let $a$ be the greatest integer such that

$$
f\left(w, z^{2}, \ldots, z^{n}\right)=w^{a} h\left(w, z^{2}, \ldots, z^{n}\right)
$$

with $h(p) \neq 0$. The function $f$ has the same representation in all nearby points of $V$, so that $a$ is constant on the connected components of $V$, namely, it is constant on $V$, so that we may define

$$
\operatorname{ord}_{V} f=a .
$$

With this proviso all the theory previously developed applies to this situation; the only definition which no longer applies is that of degree of a line bundle, in that $c_{1}(L)$ is still represented by a 2 -form, while the quantities that can be integrated on $X$ are the $2 n$-forms if $\operatorname{dim}_{\mathbb{C}} X=n$. Proposition 6.7 must now be reformulated as follows. Let $D=\sum a_{i} V_{i}$ be a divisor, and let $V_{i}^{*}$ be the smooth locus of $V_{i}$. We then have:

Proposition 6.4. For any divisor $D \in \operatorname{Div}(X)$ and any $(2 n-2)$-form $\phi$ on $X$,

$$
\int_{X} c_{1}(D) \wedge \phi=\sum_{i} a_{i} \int_{V_{i}^{*}} \phi .
$$

Proof. The proof is basically the same as in Proposition 6.7 (cf. [9] page 141).

## 3. Linear systems

In this section we consider a compact complex manifold $X$ of arbitrary dimension. Let $D=\sum a_{i} V_{i} \in \operatorname{Div}(X)$, and define $|D|$ as the set of all effective divisors linearly equivalent to $D$. We start by showing that there is an isomorphism

$$
\lambda: \mathbb{P} H^{0}(X, \mathcal{O}(D)) \rightarrow|D| .
$$

We fix a global meromorphic section $s_{0}$ of $[D]$, and set

$$
\begin{equation*}
s \in H^{0}(X, \mathcal{O}(D)) \mapsto\left(\frac{s}{s_{0}}\right)+D \in|D| ; \tag{6.5}
\end{equation*}
$$

one should notice that $\operatorname{ord}_{p_{i}}\left(\frac{s}{s_{0}}\right) \geq-a_{i}$ if $p_{i} \in V_{i}$ so that $\left(\frac{s}{s_{0}}\right)+D$ is indeed effective. If $s^{\prime}=\alpha s$ with $\alpha \in \mathbb{C}^{*}$ then $\left(\frac{s}{s_{0}}\right)=\left(\frac{s^{\prime}}{s_{0}}\right)$ so that equation (6.5) does define a map $\mathbb{P} H^{0}(X, \mathcal{O}(D)) \rightarrow|D|$. This map is
(i) injective because if $\lambda\left(s_{1}\right)=\lambda\left(s_{2}\right)$ then $s_{1} / s_{2}$ is a global nonvanishing holomorphic function, i.e. $s_{1}=\alpha s_{2}$ with $\alpha \in \mathbb{C}^{*}$.
(ii) Surjective because if $D_{1} \in|D|$ then $D_{1}=D+(f)$ for a global meromorphic function $f$ with $\operatorname{ord}_{p_{i}}(f) \geq-a_{i}$ if $p_{i} \in V_{i}$. So $f s_{0}$ is a global holomorphic section of $[D]$.

Definition 6.1. A linear system is the set of divisors corresponding to a linear subspace of $\mathbb{P} H^{0}(X, \mathcal{O}(D))$. A linear system is said to be complete if it corresponds to the whole of $\mathbb{P} H^{0}(X, \mathcal{O}(D))$.

So a linear system is of the form $E=\left\{D_{\lambda}\right\}_{\lambda \in \mathbb{P}_{m}}$ for some $m$. The number $m$ is called the dimension of $E$. A one-dimensional linear system is called a pencil, a twodimensional one a net, and a three-dimensional one a web. Since all divisors in a linear system have the same degree, one can associate a degree to a linear system.

REmARK 6.2. If the elements $\lambda_{0}, \ldots, \lambda_{m}$ are independent in $\mathbb{P}_{m}$ (which means that they are images of linearly independent elements in $\mathbb{C}^{m+1}$ ), and $E=\left\{D_{\lambda}\right\}_{\lambda \in \mathbb{P}_{m}}$ is a linear system, then

$$
D_{\lambda_{0}} \cap \cdots \cap D_{\lambda_{m}}=\bigcap_{\lambda \in \mathbb{P}_{m}} D_{\lambda}
$$

For instance, if $m=1$, and $D_{\lambda_{0}}$ and $D_{\lambda_{1}}$ have local equations $f=0$ and $g=0$, then $D_{\lambda}$ has local equation $c_{0} f+c_{1} g=0$ if $\lambda=c_{0} \lambda_{0}+c_{1} \lambda_{1}$. So $D_{\lambda_{0}} \cap D_{\lambda_{1}} \subset \cap_{\lambda \in \mathbb{P}_{1}} D_{\lambda}$, which implies $D_{\lambda_{0}} \cap D_{\lambda_{1}}=\cap_{\lambda \in \mathbb{P}_{1}} D_{\lambda}$.

Definition 6.3. If $E=\left\{D_{\lambda}\right\}_{\lambda \in \mathbb{P}_{m}}$ is a linear system, we define its base locus as $B(E)=\cap_{\lambda \in \mathbb{P}_{m}} D_{\lambda}$.

EXAMPLE 6.4. If $E=\left\{D_{\lambda}\right\}_{\lambda \in \mathbb{P}_{1}}$ is a pencil, every $p \in X-B(E)$ lies on a unique $D_{\lambda}$, so that there is a well-defined map $X-B(E) \rightarrow \mathbb{P}_{1}$. This map is holomorphic. We may indeed write a local equation for $D_{\lambda}$ in the form

$$
\begin{equation*}
f\left(z^{1}, \ldots, z^{n}\right)+\lambda g\left(z^{1}, \ldots, z^{n}\right)=0 \tag{6.6}
\end{equation*}
$$

where $f$ and $f$ are local defining functions for $D_{0}$ and $D_{\infty}$ (holomorphic because the divisors in $E$ are effective). $f$ and $g$ do not vanish simultaneously on $X-B(E)$, so that they do not vanish separately either. Then the above map is given by $\lambda=$ $-f\left(z^{1}, \ldots, z^{n}\right) / g\left(z^{1}, \ldots, z^{n}\right)$.

Example 6.5. Since $H^{1}\left(\mathbb{P}_{n}, \mathcal{O}\right)=H^{2}\left(\mathbb{P}_{n}, \mathcal{O}\right)=0$, the line bundles on $\mathbb{P}_{n}$ are classified by $H^{2}\left(\mathbb{P}_{n}, \mathbb{Z}\right) \simeq \mathbb{Z}$. Moreover, since $c_{1}(H)=1$ under this identification (i.e. $\operatorname{deg} H=1$ ), all divisors are linearly equivalent to multiples of $H$; in other terms, on $\mathbb{P}_{n}$ the only complete linear system of degree $d$ is $|d H|$.

Notice that $|H|$ is base-point free, i.e. $B(|H|)=\emptyset$.

A fundamental result in the theory of linear systems is the following.
Proposition 6.6. (Bertini's theorem) The generic element of a linear system is smooth away from the base locus.

By this we mean that the set of divisors in a linear system $E$ which have singular points outside the base locus form a subvariety of $E$ of dimension strictly smaller than that of $E$.

Proof. If $E$ is linear system, and $D \in E$ has singularities outside $B(E)$, Bertini's theorem would be violated by all pencils containing $D$. It is therefore sufficient to prove the theorem for pencils; in this case genericity means that the divisors having singularities out of the base locus are finite in number.

So let $E=\left\{D_{\lambda}\right\}_{\lambda \in \mathbb{P}_{1}}$ be a pencil, locally described by eq. (6.6), where the coordinates $\left\{z^{1}, \ldots, z^{n}\right\}$ can be defined on an open subset $\Delta \subset X$ whose image in $\mathbb{C}^{n}$ is a polydisc. Let $p_{\lambda}$ be a singular point of $D_{\lambda}$ which is not contained in the base locus. We have the conditions

$$
\begin{gather*}
f\left(p_{\lambda}\right)+\lambda g\left(p_{\lambda}\right)=0  \tag{6.7}\\
\frac{\partial f}{\partial z^{i}}\left(p_{\lambda}\right)+\lambda \frac{\partial f}{\partial z^{i}}\left(p_{\lambda}\right)=0, \quad i=1, \ldots, n  \tag{6.8}\\
f\left(p_{\lambda}\right), g\left(p_{\lambda}\right) \neq 0
\end{gather*}
$$

We then have $\lambda=-f\left(p_{\lambda}\right) / g\left(p_{\lambda}\right)$, so that

$$
\frac{\partial f}{\partial z^{i}}-\frac{f}{g} \frac{\partial g}{\partial z^{i}}=0 \quad \text { in } \quad p_{\lambda}
$$

and

$$
\begin{equation*}
\frac{\partial}{\partial z^{i}}\left(\frac{f}{g}\right)=0 \quad \text { in } \quad p_{\lambda} \tag{6.9}
\end{equation*}
$$

Let $Y$ be the locus in $\Delta \times \mathbb{P}_{1}$ cut out by the conditions (6.7) and (6.8); $Y$ is an analytic variety, so the same holds true for its image $V$ in $\Delta$. Actually $V$ is nothing but the locus of all singular points of the divisors $D_{\lambda}$. Equation (6.9) shows that $f / g$ is constant on the connected components of $V-B$, that is, every connected component of $V-B$ meets only one divisor of the pencil. Since the connected components of $V-B$ are finitely many by Proposition 6.2, the divisors which have singularities outside $B(E)$ are finite in number.

## 4. The adjunction formula

If $V$ is a smooth analytic hypersurface in a complex manifold $X$, we may relate the canonical bundles $K_{V}$ and $K_{X}$. We shall denote by $\iota_{V}: V \rightarrow X$ the inclusion; one has an injective morphism $T V \rightarrow \iota_{V}^{*} T X$ of bundles on $V$. If we choose around $p \in V$ a coordinate system $\left(z^{1}, \ldots, z^{n}\right)$ for $X$ such that $z^{1}=0$ locally describes $V$, then the vector field $\frac{\partial}{\partial z^{1}}$ restricted to $V$ locally generates the quotient sheaf $N_{V}=\iota_{V}^{*} T X / T V$, so that $N_{V}$ is the sheaf of sections of a line bundle, which is called the normal bundle to $V$.

The dual $N_{V}^{*}$, the conormal bundle to $V$, is the subbundle of $\iota_{V}^{*} T^{*} X$ whose sections are holomorphic 1-forms which are zero on vectors tangent to $V$.

We first prove the isomorphism

$$
\begin{equation*}
N_{V}^{*} \simeq \iota_{V}^{*}[-V] . \tag{6.10}
\end{equation*}
$$

We consider the exact sequence of vector bundles on $V$

$$
0 \rightarrow N_{V}^{*} \rightarrow \iota_{V}^{*} T^{*} X \rightarrow T^{*} V \rightarrow 0
$$

whence we get ${ }^{4}$

$$
\begin{equation*}
\iota_{V}^{*} K_{X} \simeq K_{V} \otimes N_{V}^{*} \tag{6.11}
\end{equation*}
$$

If, relative to an open cover $\left\{U_{\alpha}\right\}$ of $X$, the divisor $V$ is locally given by functions $f_{\alpha} \in \mathcal{O}\left(U_{\alpha}\right)$, the line bundle [ $V$ ] has transition functions $g_{\alpha \beta}=f_{\alpha} / f_{\beta}$. The 1-form $d f_{\alpha \mid V \cap U_{\alpha}}$ is a section of $N_{V \mid V \cap U_{\alpha}}^{*}$, which never vanishes because $V$ is smooth. On $U_{\alpha} \cap U_{\beta}$ we have

$$
d f_{\alpha}=d\left(g_{\alpha \beta} f_{\beta}\right)=d g_{\alpha \beta} f_{\beta}+g_{\alpha \beta} d f_{\beta}=g_{\alpha \beta} d f_{\beta}
$$

the last equality holding on $V \cap U_{\alpha} \cap U_{\beta}$. This equation shows that the 1 -forms $d f_{\alpha}$ do not glue to a global section of $N_{V}^{*}$, but rather to a global section of the line bundle $N_{V}^{*} \otimes \iota_{V}^{*}[V]$, so that this bundle is trivial, and the isomorphism (6.10) holds.

By combining the formula (6.10) with the isomorphism (6.11) we obtain the adjunction formula:

$$
\begin{equation*}
K_{V} \simeq \iota_{V}^{*}\left(K_{X} \otimes[V]\right) \tag{6.12}
\end{equation*}
$$

Sometimes an additive notation is used, and then the adjunction formula reads

$$
K_{V}=K_{X \mid V}+[V]_{\mid V}
$$

Example 6.1. Let $V$ be the divisor cut out from $\mathbb{P}_{3}$ by the quartic equation

$$
\begin{equation*}
w_{0}^{4}+w_{1}^{4}+w_{2}^{4}+w_{3}^{4}=0 \tag{6.13}
\end{equation*}
$$

where the $w_{i}$ 's are homogeneous coordinates in $\mathbb{P}_{3}$. It is easily shown the $V$ is smooth, and it is of course compact: so it is a 2-dimensional compact complex manifold, called the Fermat surface. By a nontrivial result, known as Lefschetz hyperplane theorem ( $[\mathbf{9}]$ p. 156) one has $H^{1}(V, \mathbb{R})=0$, so that $H^{1}\left(V, \mathcal{O}_{V}\right)=0$. Then the group $\operatorname{Pic}^{0}(V)$, which classifies the line bundles whose first Chern classes vanishes, is trivial: if a line bundle $L$ on $V$ is such that $c_{1}(L)=0$, then it is trivial, and every line bundle is fully classified by its first Chern class. (The same happens on $\mathbb{P}_{3}$, since $\left.H^{1}\left(\mathbb{P}_{3}, \mathcal{O}_{\mathbb{P}_{3}}\right)=0\right)$.

[^20]We also know that $K_{\mathbb{P}_{3}}=\mathcal{O}_{\mathbb{P}_{3}}(-4 H)$, where $H$ is any hyperplane in $\mathbb{P}_{3}$. Therefore $\iota_{V}^{*} K_{X} \simeq \mathcal{O}_{V}\left(-4 H_{V}\right)$, where $H_{V}=H \cap V$ is a divisor in $V$.

Let us compute $c_{1}\left([V]_{\mid V}\right)=\iota_{V}^{*} c_{1}([V])$. We use the following fact: if $D_{1}, D_{2}, D_{3}$ are irreducible divisors in $\mathbb{P}_{3}$, then we can move the divisors inside their linear equivalence classes in such a way that they intersects at a finite number of points. This number is computed by the integral

$$
\int_{\mathbb{P}_{3}} c_{1}\left(\left[D_{1}\right]\right) \wedge c_{1}\left(\left[D_{2}\right]\right) \wedge c_{1}\left(\left[D_{3}\right]\right)
$$

where one considers the Chern classes $c_{1}\left(\left[D_{i}\right]\right)$ as de Rham cohomology classes. If we take $D_{1}=V, D_{2}=D_{3}=H$ the number of intersection points is 4 , because such is the degree of the algebraic system formed by the equation (6.13) and by the equations of two (different) hyperplanes. Since the class $h$, where $h=c_{1}([H])$, generates $H^{2}\left(\mathbb{P}_{3}, \mathbb{Z}\right)$, we have $c_{1}([V])=4 h$, that is, $V \sim 4 H$. Then $[V]_{\mid V} \simeq \mathcal{O}_{V}\left(4 H_{V}\right)$.

From the adjunction formula we get $K_{V} \simeq \mathbb{C}$ : the canonical bundle of $V$ is trivial. Since we also have $H_{D R}^{1}(V)=0, V$ is an example of a $K 3$ surface.

## CHAPTER 7

## Algebraic curves I

The main purpose of this chapter is to show that compact Riemann surfaces can be imbedded into projective space (i.e. they are algebraic curves), and to study some of their basic properties.

## 1. The Kodaira embedding

We start by showing that any compact Riemann surface can be embedded as a smooth subvariety in a projective space $\mathbb{P}_{N}$; this is special instance of the so-called Kodaira's embedding theorem. Together with Chow's Lemma this implies that every compact Riemann surface is algebraic.

We recall that, given two complex manifolds $X$ and $Y$, we say that $(Y, \iota)$ is a submanifold of $X$ is $\iota$ is an injective holomorphic map $Y \rightarrow X$ whose differential $\iota_{* p}$ : $T_{p} Y \rightarrow T_{\iota(p)} X$ is of maximal rank (given by the dimension of $Y$ ) at all $p \in Y$. In other terms, $\iota$ maps isomorphically $Y$ onto a smooth subvariety of $X$.

Proposition 7.1. Any compact Riemann surface can be realized as a submanifold of $\mathbb{P}_{N}$ for some $N$.

Proof. Pick up a line bundle $L$ on $S$ such that $\operatorname{deg} L>\operatorname{deg} K+2$ (choose an effective divisor $D$ with enough points, and let $L=[D]$ ). By Serre duality we have

$$
\begin{equation*}
H^{1}(S, \mathcal{O}(L-2 p)) \simeq H^{0}\left(S, \mathcal{O}(L-2 p)^{-1} \otimes K\right)^{*}=0 \tag{7.1}
\end{equation*}
$$

for any $p \in S$, since $\operatorname{deg}(K-L+2 p)<0$ (here $L-2 p=L \otimes[-2 p]$ ). Consider now the exact sequence

$$
0 \rightarrow \mathcal{O}(L-2 p) \rightarrow \mathcal{O}(L) \xrightarrow{d_{p} \oplus \mathrm{ev}_{p}} T_{p}^{*} S \oplus L_{p} \rightarrow 0
$$

(the morphism $d_{p}$ is Cartan's differential followed by evaluation at $p$, while $\mathrm{ev}_{p}$ is the evaluation of sections at $p$ ). Due to (7.1) we get

$$
0 \rightarrow H^{0}(S, \mathcal{O}(L-2 p)) \rightarrow H^{0}(S, \mathcal{O}(L)) \xrightarrow{d_{p} \oplus \mathrm{ev}_{p}} T_{p}^{*} S \oplus L_{p} \rightarrow 0
$$

so that $\operatorname{dim}|D| \geq 1$. Let $N=\operatorname{dim}|D|$, and let $\left\{s_{0}, \ldots, s_{N}\right\}$ be a basis of $|D|$. If $U$ is an open neighbourhood of $p$, and $\phi: L_{\mid U} \rightarrow U \times \mathbb{C}$ is a local trivialization of $L$, the quantity

$$
\begin{equation*}
\left[\phi \circ s_{0}, \ldots, \phi \circ s_{N}\right] \in \mathbb{P}_{N} \tag{7.2}
\end{equation*}
$$

does not depend on the trivialization $\phi$; we have therefore established a (holomorphic) map $\iota_{L}: S \rightarrow \mathbb{P}_{N} .{ }^{1}$ We must prove that (1) $\iota_{L}$ is injective, and (2) the differential $\left(\iota_{L}\right)_{*}$ never vanishes. (1) It is enough to prove that, given any two points $p, q \in S$, there is a section $s \in H^{0}(S, \mathcal{O}(L))$ such that $s(p) \neq \lambda s(q)$ for all $\lambda \in \mathbb{C}^{*}$; this in turn implied by the surjectivity of the map

$$
H^{0}(S, \mathcal{O}(L)) \xrightarrow{r_{p, q}} L_{p} \oplus L_{q}, \quad s \mapsto s(p)+s(q)
$$

To show this we start from the exact sequence

$$
0 \rightarrow \mathcal{O}(L-p-q) \rightarrow \mathcal{O}(L) \xrightarrow{r_{p, q}} L_{p} \oplus L_{q} \rightarrow 0
$$

and note that in coholomology we have

$$
H^{0}(S, \mathcal{O}(L-p-q)) \xrightarrow{r_{p, q}} L_{p} \oplus L_{q} \rightarrow H^{1}(S, \mathcal{O}(L-p-q))=0
$$

since

$$
H^{1}(S, \mathcal{O}(L-p-q)) \simeq H^{0}\left(S, \mathcal{O}(L-p-q)^{-1} \otimes K\right)^{*}=0
$$

because $\operatorname{deg}(L-p-q)^{-1} \otimes K=\operatorname{deg} K-\operatorname{deg} L+2<0$.
(2) We shall actually show that the adjoint map $\left(\iota_{L}\right)^{*}: T_{\iota_{L}(p)}^{*} \mathbb{P}_{N} \rightarrow T_{p}^{*} S$ is surjective. The cotangent space $T_{p}^{*} S$ can be realized as the space of equivalence classes of holomorphic functions which have the same value at $p$ (e.g; the zero value) and have a first-order contact (i.e. they have the same differential at $p$ ). Let $\phi$ be a trivializing map for $L$ around $p$; we must find a section $s \in H^{0}(S, \mathcal{O}(L))$ such that $\phi \circ s(p)=0$ (i.e. $s(p)=0)$ and $(\phi \circ s)^{*}$ is surjective at $p$. This is equivalent to showing that the map $H^{0}(S, \mathcal{O}(L-p)) \xrightarrow{d_{p}} T_{p}^{*} S$ is surjective, since $\mathcal{O}(L-p)$ is the sheaf of holomorphic sections of $L$ vanishing at $p$. We consider the exact sheaf sequence

$$
0 \rightarrow \mathcal{O}(L-2 p) \rightarrow \mathcal{O}(L-p) \xrightarrow{d_{p}} T_{p}^{*} S \rightarrow 0
$$

by Serre duality,

$$
H^{1}(S, \mathcal{O}(L-2 p))^{*} \simeq H^{0}(S, O(-L+2 p+K))=0
$$

so that $H^{0}(S, \mathcal{O}(L-p)) \xrightarrow{d_{p}} T_{p}^{*} S$ is surjective.
Given any complex manifold $X$, one says that a line bundle $L$ on $X$ is very ample if the construction (7.2) defines an imbedding of $X$ into $\mathbb{P} H^{0}(X, \mathcal{O}(L))$. A line bundle $L$ is said to be ample if $L^{n}$ is very ample for some natural $n$. A sufficient condition for a line bundle to be ample may be stated as follows (cf. [9]).

Definition 7.2. A (1,1) form $\omega$ on a complex manifold is said to be positive if it can be locally written in the form

$$
\omega=i \omega_{i j} d z^{i} \wedge d \bar{z}^{j}
$$

[^21]with $\omega_{i j}$ a positive definite hermitian matrix.
Proposition 7.3. If the first Chern class of a line bundle $L$ on a complex manifold can be represented by a positive 2-form, then $L$ is ample.

While we have seen that any compact Riemann surface carries plenty of very ample line bundles, this in general is not the case: there are indeed complex manifolds which cannot be imbedded into any projective space.

A first consequence of the imbedding theorem expressed by Proposition 7.1 is that any line bundle on a compact Riemann surface comes from a divisor, i.e. $\operatorname{Div}(S) / \operatorname{linear}$ equivalence $\simeq \operatorname{Pic}(S)$.

Proposition 7.4. If $M$ is a smooth 1 -dimensional ${ }^{2}$ analytic submanifold of projective space $\mathbb{P}_{n}$ (i.e. $M$ is the imbedding of a compact Riemann surface into $\mathbb{P}_{n}$ ), and $L$ is a line bundle on $M$, there is a divisor $D$ on $M$ such that $L=[D]$.

Proof. We must find a global meromorphic section of $L$. Let $H_{M}$ be the restriction to $M$ of the hyperplane bundle $H$ of $\mathbb{P}_{n}$, and let $V$ be the intersection of $M$ with a hyperplane in $\mathbb{P}_{n}$ (so $[V] \simeq H_{M}$, and since $V$ is effective, $H_{M}$ has global holomorphic sections). We shall show that for a big enough integer $m$ the line bundle $L+m H_{M}$ $\left(=L \otimes H_{M}^{m}\right)$ has a global holomorphic section $s$; if $t$ is a holomorphic section of $H_{M}$, the required meromorphic section of $L$ is $s / t^{m}$.

We have an exact sequence

$$
0 \rightarrow \mathcal{O}_{M}\left(-H_{M}\right) \xrightarrow{s} \mathcal{O}_{M} \rightarrow k_{V} \rightarrow 0
$$

so that after tensoring by $L+m H_{M}$,

$$
\begin{equation*}
0 \rightarrow \mathcal{O}_{M}\left(L+(m-1) H_{M}\right) \xrightarrow{s} \mathcal{O}_{M}\left(L+m H_{M}\right) \rightarrow k_{V} \rightarrow 0 . \tag{7.3}
\end{equation*}
$$

(Here $\xrightarrow{s}$ denotes the morphism given by multiplication by $s$ ). The associated long cohomology exact sequence contains the segment

$$
H^{0}\left(M, \mathcal{O}_{M}\left(L+m H_{M}\right)\right) \xrightarrow{r} \mathbb{C}^{N} \rightarrow H^{1}\left(M, \mathcal{O}_{M}\left(L+(m-1) H_{M}\right)\right)
$$

where $N=\operatorname{deg} V$. But

$$
H^{1}\left(M, \mathcal{O}_{M}\left(L+(m-1) H_{M}\right)\right) \simeq H^{0}\left(M, K_{M} \otimes \mathcal{O}\left(-L-(m-1) H_{M}\right)\right)^{*}=0
$$

by Serre duality and the vanishing theorem (if $m$ is big enough, $\operatorname{deg} K_{M} \otimes \mathcal{O}(-L-$ $\left.\left.(m-1) H_{M}\right)<0\right)$. Therefore the morphism $r$ in (7.3) is surjective, and $H^{0}\left(M, \mathcal{O}_{M}(L+\right.$ $\left.\left.m H_{M}\right)\right) \neq 0$.

We shall now proceed to identify compact Riemann surfaces with (smooth) algebraic curves. Given a homogeneous polynomial $F$ on $\mathbb{C}^{n+1}$ the zero locus of $F$ in $\mathbb{P}_{n}$ is by definition the projection to $\mathbb{P}_{n}$ of the zero locus of $F$ in $\mathbb{C}^{n+1}$.

[^22]Definition 7.5. A (projective) algebraic variety is a subvariety of $\mathbb{P}_{n}$ which is the zero locus of a finite collection of homogeneous polynomials. We shall say that an algebraic variety is smooth if it is so as an analytic subvariety of $\mathbb{P}_{n}$.

The dimension of an algebraic variety is its dimension as an analytic subvariety of $\mathbb{P}_{n}$. A one-dimensional algebraic variety is called an algebraic curve.

The following fundamental result, called Chow's lemma, it is not hard to prove; we shall anyway omit its proof for the sake of brevity (cf. [9] page 167).

Proposition 7.6. (Chow's lemma) Any analytic subvariety of $\mathbb{P}_{n}$ is algebraic.
Exercise 7.7. Use Chow's lemma to show that $H^{0}\left(\mathbb{P}_{n}, H^{d}\right)$ - where $H$ is the hyperplane line bundle - can be identified with the space of homogeneous polynomials of degree $d$ on $\mathbb{C}^{n+1}$.

Using Chow's lemma together with the imbedding theorem (Proposition 7.1) we obtain

Corollary 7.8. Any compact Riemann surface is a smooth algebraic curve.
We switch from the terminology "compact Riemann surface" to "algebraic curve", understanding that we shall only consider smooth algebraic curves. ${ }^{3}$

We shall usually denote an algebraic curve by the letter $C$.

## 2. Riemann-Roch theorem

A fundamental result in the study of algebraic curves in the Riemann-Roch theorem. Let $C$ be an algebraic curve, and denote by $K$ its canonical bundle. ${ }^{4}$ We denote $g=$ $h^{0}(K)$, and call it the arithmetic genus of $C$ (this number will be shortly identified with the topological genus of $C$ ).

Proposition 7.1. (Riemann-Roch theorem) For any line bundle $L$ on $C$ one has

$$
h^{0}(L)-h^{1}(L)=\operatorname{deg} L-g+1 .
$$

Proof. If $L=\mathbb{C}$ is the trivial line bundle, the result holds obviously (notice that $H^{1}(C, \mathcal{O})^{*} \simeq H^{0}(C, K)$ by Serre duality). Exploiting the fact that $L=[D]$ for some divisor $D$, it is enough to prove that if the results hold for $L=[D]$, then it also holds for $L^{\prime}=[D+p]$ and $L^{\prime \prime}=[D-p]$.

In the first case we start from the exact sequence

$$
0 \rightarrow \mathcal{O}(D) \rightarrow \mathcal{O}(D+p) \rightarrow k_{p} \rightarrow 0
$$

[^23]which gives $\left(\right.$ since $\left.H^{1}\left(C, k_{p}\right)=0\right)$
$$
0 \rightarrow H^{0}(S, \mathcal{O}(D)) \rightarrow H^{0}(S, \mathcal{O}(D+p)) \rightarrow \mathbb{C} \rightarrow H^{1}(S, \mathcal{O}(D)) \rightarrow H^{1}(S, \mathcal{O}(D+p)) \rightarrow 0
$$
whence
$$
h^{0}\left(L^{\prime}\right)-h^{1}\left(L^{\prime}\right)=h^{0}(L)-h^{1}(L)+1=\operatorname{deg} L-g+2=\operatorname{deg} L^{\prime}-g+1 .
$$

Analogously for $L^{\prime \prime}$.
By using the Riemann-Roch theorem and Serre duality we may compute the degree of $K$, obtaining

$$
\operatorname{deg} K=2 g-2 .
$$

This is called the Riemann-Hurwitz formula. It allows us to identify $g$ with the topological genus $g_{\text {top }}$ of $C$ regarded as a compact oriented 2-dimensional real manifold $S$. To this end we may use the Gauss-Bonnet theorem, which states that the integral of the Euler class of the real tangent bundle to $S$ is the Euler characteristic of $S$, $\chi(S)=2-2 g_{\mathrm{top}}$. On the other hand the complex structure of $C$ makes the real tangent bundle into a complex holomorphic line bundle, isomorphic to the holomorphic tangent bundle $T C$, and under this identification the Euler class corresponds to the first Chern class of $T C$. Therefore we get $\operatorname{deg} K=2 g_{\text {top }}-2$, namely, ${ }^{5}$

$$
g=g_{\mathrm{top}} .
$$

## 3. Some general results about algebraic curves

Let us fix some notations and give some definitions.
3.1. The degree of a map. Let $C$ be an algebraic curve, and $\omega$ a smooth 2form on $C$, such that $\int_{C} \omega=1$; the de Rham cohomology class [ $\omega$ ] may be regarded as an element in $H^{2}(C, \mathbb{Z})$, and actually provides a basis of that space, allowing an identification $H^{2}(C, \mathbb{Z}) \simeq \mathbb{Z}$. If $f: C^{\prime} \rightarrow C$ is a nonconstant holomorphic map between two algebraic curves, then $f^{\sharp}[\omega]$ is a nonzero element in $H^{2}\left(C^{\prime}, \mathbb{Z}\right)$, and there is a well defined integer $n$ such that

$$
f^{\sharp}[\omega]=n\left[\omega^{\prime}\right],
$$

where $\omega^{\prime}$ is a smooth 2 -form on $C^{\prime}$ such that $\int_{C^{\prime}} \omega^{\prime}=1$. If $p \in C$ we have

$$
\operatorname{deg} f^{*}(p)=\int_{C^{\prime}} c_{1}\left(f^{*}[p]\right)=\int_{C^{\prime}} f^{\sharp} c_{1}([p])=n \int_{C} c_{1}([p])=n,
$$

so that the map $f$ takes the value $p$ exactly $n$ times, including multiplicities in the sense of divisors; we may say that $f$ covers $C n$ times. ${ }^{6}$ The integer $n$ is called the degree if $f$.

[^24]3.2. Branch points. Given again a nonconstant holomorphic map $f: C^{\prime} \rightarrow C$, we may find a coordinate $z$ around any $q \in C^{\prime}$ and a coordinate $w$ around $f(q)$ such that locally $f$ is described as
\[

$$
\begin{equation*}
w=z^{r} . \tag{7.4}
\end{equation*}
$$

\]

The number $r-1$ is called the ramification index of $f$ at $q$ (or at $p=f(q)$ ), and $p=f(q)$ is said to be a branch point if $r(p)>1$. The branch locus of $f$ is the divisor in $C^{\prime}$

$$
B^{\prime}=\sum_{q \in C^{\prime}}(r(q)-1) \cdot q
$$

or its image in $C$

$$
B=\sum_{q \in C^{\prime}}(r(q)-1) \cdot f(q) .
$$

For any $p \in C$ we have

$$
\begin{gathered}
f^{*}(p)=\sum_{q \in f^{-1}(p)} r(q) \cdot q \\
\operatorname{deg} f^{*}(p)=\sum_{q \in f^{-1}(p)} r(q)=n .
\end{gathered}
$$

From these formulae we may draw the following picture. If $p \in C^{\prime}$ does not lie in the branch locus, then exactly $n$ distinct points of $C^{\prime}$ are mapped to $f(p)$, which means that $f: C^{\prime}-B^{\prime} \rightarrow C-B$ is a covering map. ${ }^{7}$ It $p \in C^{\prime}$ is a branch point of ramification index $r-1$, at $p$ exactly $r$ sheets of the covering join together.

There is a relation between the canonical divisors of $C^{\prime}$ and $C$ and the branch locus. Let $\eta$ be a meromorphic 1-form on $C$, which can locally be written as

$$
\eta=\frac{g(w)}{h(w)} d w .
$$

From (7.4) we get

$$
f^{*} \eta=\frac{g\left(z^{r}\right)}{h\left(z^{r}\right)} d z^{r}=r z^{r-1} \frac{g\left(z^{r}\right)}{h\left(z^{r}\right)} d z
$$

so that

$$
\operatorname{ord}_{p} f^{*} \eta=\operatorname{ord}_{f(p)} \eta+r-1 .
$$

This implies the relation between divisors

$$
\left(f^{*} \eta\right)=f^{*}(\eta)+\sum_{p \in C^{\prime}}(r(p)-1) \cdot p
$$

On the other hand the divisor $(\eta)$ is just the canonical divisor of $C$, so that

$$
K_{C^{\prime}}=f^{*} K_{C}+B^{\prime}
$$

[^25]From this formula we may draw an interesting result. By taking degree we get

$$
\operatorname{deg} K_{C^{\prime}}=n \operatorname{deg} K_{C}+\sum_{p \in C^{\prime}}(r(p)-1)
$$

by using the Riemann-Hurwitz formula we obtain

$$
\begin{equation*}
g\left(C^{\prime}\right)=n(g(C)-1)+1+\frac{1}{2} \sum_{p \in C^{\prime}}(r(p)-1) \tag{7.5}
\end{equation*}
$$

ExERCISE 7.1. Prove that if $f: C^{\prime} \rightarrow C$ is nonconstant, then $f^{\sharp}: H^{0}\left(C, K_{C}\right) \rightarrow$ $H^{0}\left(C^{\prime}, K_{C^{\prime}}\right)$ is injective. (Hint: a nonzero element $\omega \in H^{0}\left(C, K_{C}\right)$ is a global holomorphic 1-form on $C$ which is different from zero at all points in an open dense subset of $C$. Write an explicit formula for $f^{*} \omega \ldots$.)

Both equation (7.5) and the previous Exercise imply

$$
g\left(C^{\prime}\right) \geq g(C)
$$

3.3. The genus formula for plane curves. An algebraic curve $C$ is said to be plane if it can be imbedded into $\mathbb{P}_{2}$. Its image in $\mathbb{P}_{2}$ is the zero locus of a homogeneous polynomial; the degree $d$ of this polynomial is by definition the degree of $C$. As a divisor, $C$ is linearly equivalent to $d H$ (indeed, since $\operatorname{Pic}\left(\mathbb{P}_{2}\right) \simeq \mathbb{Z}$, any divisor $D$ on $\mathbb{P}_{2}$ is linearly equivalent to $m H$ for some $m$; if $D$ is effective, $m$ is the number of intersection points between $D$ and a generic hyperplane in $\mathbb{P}_{2}$, and this is given by the degree of the polynomial cutting $D$ ). ${ }^{8}$

We want to show that for smooth plane curves the following relation between genus and degree holds:

$$
\begin{equation*}
g(C)=\frac{1}{2}(d-1)(d-2) \tag{7.6}
\end{equation*}
$$

(For singular plane curves this formula must be modified.) We may prove this equation by using the adjunction formula: $C$ is imbedded into $\mathbb{P}_{2}$ as a smooth analytic hypersurface, so that

$$
K_{C}=\iota^{*}\left(K_{\mathbb{P}_{2}}+C\right)
$$

where $\iota: C \rightarrow \mathbb{P}_{2}$. Recalling that $K_{\mathbb{P}_{2}}=-3 H$ we then have $K_{C}=(d-3) \iota^{*} H$.

[^26]To carry on the computation, we notice that, as a divisor on $C, \iota^{*} H=C \cap H$, so that

$$
\operatorname{deg} \iota^{*} H=d,
$$

and

$$
\operatorname{deg} K_{C}=d(d-3)=2 g-2
$$

whence the formula (7.6).
Example 7.2. Consider the affine curve in $\mathbb{C}^{2}$ having equation

$$
y^{2}=x^{6}-1
$$

By writing this equation in homogeneous coordinates one obtain a curve in $\mathbb{P}_{2}$ which is a double covering of $\mathbb{P}_{1}$ branched at 6 points. By the Riemann-Hurwitz formula we may compute the genus, obtaining $g=2$. Thus the formula (7.6), which would yield $g=10$, fails in this case. This happens because the curve is singular at the point at infinity.
3.4. The residue formula. A meromorphic 1 -form on an algebraic curve $C$ is a meromorphic section of the canonical bundle $K$. Given a point $p \in C$, and a local holomorphic coordinate $z$ such that $z(p)=0$, a meromorphic 1-form $\varphi$ is locally written around $p$ in the form $\varphi=f d z$, where $f$ is a meromorphic function. Let $a$ be coefficient of the $z^{-1}$ term in the Laurent expansion of $f$ around $p$, and let $B$ a small disc around $p$; by the Cauchy formula we have

$$
a=\int_{\partial B} \varphi
$$

so that the number $a$ does not depend on the representation of $\varphi$. We call it the residue of $\varphi$ at $p$, and denote it by $\operatorname{Res}_{p}(\varphi)$.

Given a meromorphic 1-form $\varphi$ its polar divisor is $D=\sum_{i} p_{i}$, where the $p_{i}$ 's are the points where the local representatives of $\varphi$ have poles of order 1 .

Proposition 7.3. Let $D=\sum_{i} p_{i}$ be the polar divisor of a meromorphic 1-form $\varphi$. Then $\sum_{i} \operatorname{Res}_{p_{i}}(\varphi)=0$.

Proof. Choose a small disc $B_{i}$ around each point $p_{i}$. Then

$$
\sum_{i} \operatorname{Res}_{p_{i}}(\varphi)=\int_{\partial \cup_{i} B_{i}} \varphi=-\int_{C-\cup_{i} B_{i}} d \varphi=0
$$

3.5. The $g=0$ case. We shall now show that all algebraic curves of genus zero are isomorphic to the Riemann sphere $\mathbb{P}_{1}$. Pick a point $p \in C$; the line bundle $[p]$ is trivial on $C-\{p\}$, and has a holomorphic section $s_{0}$ which is nonzero on $C-\{p\}$ and has a simple zero at $p$ (this means of course that $\left(s_{0}\right)=p$ ). On the other hand, since by Serre duality $h^{1}(\mathcal{O})=h^{0}(K)=0$, by taking the cohomology exact sequence associated with the sequence

$$
0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}(p) \rightarrow k_{p} \rightarrow 0
$$

we obtain the existence of a global section $s$ of $[p]$ which does not vanish at $p$. Of course $s$ vanishes at some other point $s_{0}$. Then the quotient $f=s / s_{0}$ is a global meromorphic function, with a simple pole at $p$ and a zero at $p_{0} .{ }^{9}$ By considering $\infty$ as the value of $f$ at $p$, we may think of $f$ as a holomorphic nonconstant map $f: C \rightarrow \mathbb{P}_{1}$; this map takes the value $\infty$ only once. Suppose that $f$ takes the same value $\alpha$ at two distinct points of $C$; then then function $f-\alpha$ has two zeroes and only one simple pole, which is not possible. Thus $f$ is injective. The following Lemma implies that $f$ is surjective as well, so that it is an isomorphism.

Lemma 7.4. Any holomorphic map between compact complex manifolds of the same dimension whose Jacobian determinant is not everywhere zero is surjective.

Proof. Let $f: X \rightarrow Y$ be such a map, and let $n=\operatorname{dim} X=\operatorname{dim} Y$. Let $\omega$ be a volume form on $Y$; since the Jacobian determinant of $f$ is not everywhere zero, and where it is not zero is positive, we have $\int_{X} f^{*} \omega>0$. Assume $q \neq \operatorname{Im} f$. Since $H^{2 n}(Y-\{q\}, \mathbb{R})=0$ (prove it by using a Mayer-Vietoris argument), we have $\omega=d \eta$ on $Y-\{q\}$. But then

$$
\int_{X} f^{*} \omega=\int_{\partial X} f^{*} \eta=0
$$

a contradiction.

[^27]
## CHAPTER 8

## Algebraic curves II

In this chapter we further study the geometry of algebraic curves. Topics covered include the Jacobian variety of an algebraic curve, some theory of elliptic curves, and the desingularization of nodal plane singular curves (this will involve the introduction of the notion of blowup of a complex surface at a point).

## 1. The Jacobian variety

A fundamental tool for the study of an algebraic curve $C$ is its Jacobian variety $J(C)$, which we proceed now to define. Let $V$ be an $m$-dimensional complex vector space, and think of it as an abelian group. A lattice $\Lambda$ in $V$ is a subgroup of $V$ of the form

$$
\begin{equation*}
\Lambda=\left\{\sum_{i=1}^{2 m} n_{i} v_{i}, \quad n_{i} \in \mathbb{Z}\right\} \tag{8.1}
\end{equation*}
$$

where $\left\{v_{i}\right\}_{i=1, \ldots, 2 m}$ is a basis of $V$ as a real vector space. The quotient space $T=V / \Lambda$ has a natural structure of complex manifold, and one of abelian group, and the two structures are compatible, i.e. $T$ is a compact abelian complex Lie group. We shall call $T$ a complex torus. Notice that by varying the lattice $\Lambda$ one gets another complex torus which may not be isomorphic to the previous one (the complex structure may be different), even though the two tori are obviously diffeomorphic as real manifolds.

Example 8.1. If $C$ is an algebraic curve of genus $g$, the group $\operatorname{Pic}^{0}(C)$, classifying the line bundles on $C$ with vanishing first Chern class, has a structure of complex torus of dimension $g$, since it can be represented as $H^{1}(C, \mathcal{O}) / H^{1}(C, \mathbb{Z})$, and $H^{1}(C, \mathbb{Z})$ is a lattice in $H^{1}(C, \mathcal{O})$. This is the Jacobian variety of $C$. In what follows we shall construct this variety in a more explicit way.

Consider now a smooth algebraic curve $C$ of genus $g \geq 1$. We shall call abelian differentials the global sections of $K$ (i.e. the global holomorphic 1 -forms). If $\omega$ in abelian differential, we have $d \omega=0$ and $\omega \wedge \omega=0$; this means that $\omega$ singles out a cohomology class $[\omega]$ in $H^{1}(C, \mathbb{C})$, and that

$$
\begin{equation*}
\int_{C} \omega \wedge \omega=0 . \tag{8.2}
\end{equation*}
$$

Moreover, since locally $\omega=f(z) d z$, we have

$$
\begin{equation*}
i \int_{C} \omega \wedge \bar{\omega}>0 \quad \text { if } \quad \omega \neq 0 \tag{8.3}
\end{equation*}
$$

If $\gamma$ is a smooth loop in $C$, and $\omega \in H^{0}(C, K)$, the number $\int_{\gamma} \omega$ depends only on the homology class of $\gamma$ and the cohomology class of $\omega$, and expresses the pairing $<,>$ between the Poincaré dual spaces $H_{1}(C, \mathbb{C})=H_{1}(C, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$ and $H^{1}(C, \mathbb{C})$.

Pick up a basis $\left\{\left[\gamma_{1}\right], \ldots,\left[\gamma_{2 g}\right]\right\}$ of the $2 g$-dimensional free $\mathbb{Z}$-module $H_{1}(C, \mathbb{Z})$, where the $\gamma_{i}$ 's are smooth loops in $C$, and a basis $\left\{\omega_{1}, \ldots, \omega_{g}\right\}$ of $H^{0}(C, K)$. We associate with these data the $g \times 2 g$ matrix $\Omega$ whose entries are the numbers

$$
\Omega_{i j}=\int_{\gamma_{j}} \omega_{i} .
$$

This is called the period matrix. Its columns $\Omega_{j}$ are linearly independent over $\mathbb{R}$ : if for all $i=1, \ldots g$

$$
0=\sum_{j=1}^{2 g} \lambda_{j} \Omega_{i j}=\sum_{j=1}^{2 g} \lambda_{j} \int_{\gamma_{j}} \omega_{i}
$$

then also $\sum_{j=1}^{2 g} \lambda_{j} \int_{\gamma_{j}} \bar{\omega}_{i}=0$. Since $\left\{\omega_{i}, \bar{\omega}_{i}\right\}$ is a basis for $H^{1}(C, \mathbb{C})$, this implies $\sum_{j=1}^{2 g} \lambda_{j}\left[\gamma_{j}\right]=0$, that is, $\lambda_{j}=0$. So the columns of the period matrix generate a lattice $\Lambda$ in $\mathbb{C}^{g}$. The quotient complex torus $J(C)=\mathbb{C}^{g} / \Lambda$ is the Jacobian variety of $C$.

Define now the intersection matrix $Q$ by letting $Q_{i j}^{-1}=\left[\gamma_{j}\right] \cap\left[\gamma_{i}\right]$ (this is the $\mathbb{Z}$ valued "cap" or "intersection" product in homology). Notice that $Q$ is antisymmetric. Intrinsically, $Q$ is an element in $\operatorname{Hom}_{\mathbb{Z}}\left(H^{1}(C, \mathbb{Z}), H_{1}(C, \mathbb{Z})\right)$. Since the cup product in cohomology is Poincaré dual to the cap product in homology, for any abelian differentials $\omega, \tau$ we have

$$
[\omega] \cup[\tau]=<Q[\omega],[\tau]>.
$$

The relations (8.2), (8.3) can then be written in the form

$$
\Omega Q \tilde{\Omega}=0, \quad i \Omega Q \Omega^{\dagger}>0
$$

(here ~ denotes transposition, and ${ }^{\dagger}$ hermitian conjugation). In this form they are called Riemann bilinear relations.

A way to check that the construction of the Jacobi variety does not depend on the choices we have made is to restate it invariantly. Integration over cycles defines a map

$$
i: H_{1}(C, \mathbb{Z}) \rightarrow H^{0}(C, K)^{*}, \quad i([\gamma])(\omega)=\int_{\gamma} \omega .
$$

This map is injective: if $i([\gamma])(\omega)=0$ for a given $\gamma$ and all $\omega$ then $\gamma$ is homologous to the constant loop. Then we have the representation $J(C)=H^{0}(C, K)^{*} / H_{1}(C, \mathbb{Z})$.

Exercise 8.2. By regarding $J(C)$ as $H^{0}(C, K)^{*} / H_{1}(C, \mathbb{Z})$, show that Serre and Poincaré dualities establish an isomorphism $J(C) \simeq \operatorname{Pic}^{0}(C)$.
1.1. The Abel map. After fixing a point $p_{0}$ in $C$ and a basis $\left\{\omega_{1}, \ldots, \omega_{g}\right\}$ in $H^{0}(C, K)$ we define a map

$$
\begin{equation*}
\mu: C \rightarrow J(C) \tag{8.4}
\end{equation*}
$$

by letting

$$
\mu(p)=\left(\int_{p_{0}}^{p} \omega_{1}, \ldots, \int_{p_{0}}^{p} \omega_{g}\right)
$$

Actually the value of $\mu(p)$ in $\mathbb{C}^{g}$ will depend on the choice of the path from $p_{0}$ to $p$; however, if $\delta_{1}$ and $\delta_{2}$ are two paths, the oriented sum $\delta_{1}-\delta_{2}$ will define a cycle in homology, the two values will differ by an element in the lattice, and $\mu(p)$ is a welldefined point in $J(C)$.

From (8.4) we may get a group homomorphism

$$
\begin{equation*}
\mu: \operatorname{Div}(C) \rightarrow J(C) \tag{8.5}
\end{equation*}
$$

by letting

$$
\mu(D)=\sum_{i} \mu\left(p_{i}\right)-\sum_{j} \mu\left(q_{j}\right) \quad \text { if } \quad D=\sum_{i} p_{i}-\sum_{j} q_{j}
$$

All of this depends on the choice of the base point $p_{0}$, note however that if $\operatorname{deg} D=0$ then the choice of $p_{0}$ is immaterial.

Proposition 8.3. (Abel's theorem) Two divisors $D, D^{\prime} \in \operatorname{Div}(C)$ are linearly equivalent if and only if $\mu(D)=\mu\left(D^{\prime}\right)$.

Proof. For a proof see [9] page 232.
Corollary 8.4. The Abel map $\mu: C \rightarrow J(C)$ is injective.
Proof. If $\mu(p)=\mu(q)$ by the previous Proposition $p \sim q$ as divisors, but since $g(C) \geq 1$ this implies $p=q$ (this follows from considerations analogous to those in subsection 7.3.5).

Abel's theorem may be stated in a fancier language as follows. Let $\operatorname{Div}_{d}(C)$ be the subset of $\operatorname{Div}(C)$ formed by the divisors of degree $d$, and let $\operatorname{Pic}^{d}(C)$ be the set of line bundles of degree $d .{ }^{1}$ One has a surjective map $\ell:=\operatorname{Div}_{d}(C) \rightarrow \operatorname{Pic}^{d}(C)$ whose kernel is isomorphic to $H^{0}\left(C, \mathcal{M}^{*}\right) / H^{0}\left(C, \mathcal{O}^{*}\right)$. Then $\mu$ filters through a morphism $\nu: \operatorname{Pic}^{d}(C) \rightarrow J(C)$, and one has a commutative diagram


[^28]moreover, the morphism $\nu$ is injective (if $\nu(L)=0$, set $L=\ell(D)$ (i.e. $L=[D]$ ); then $\mu(L)=0$, that is, $L$ is trivial).

We can actually say more about the morphism $\nu$, namely, that it is a bijection. It is enough to prove that $\nu$ is surjective for a fixed value of $d$ (cf. previous footnote).

Let $C^{d}$ be the $d$-fold cartesian product of $C$ with itself. The symmetric group $S_{d}$ of order $d$ acts on $C^{d}$; we call the quotient $\operatorname{Sym}^{d}(C)=C^{d} / S_{d}$ the $d$-fold symmetric product of $C . \operatorname{Sym}^{d}(C)$ can be identified with the set of effective divisors of $C$ of degree $d$. The map $\mu$ defines a map $\mu_{d}: \operatorname{Sym}^{d}(C) \rightarrow J(C)$.

Any local coordinate $z$ on $C$ yields a local coordinate system $\left\{z^{1}, \ldots, z^{d}\right\}$ on $C^{d}$,

$$
z^{i}\left(p_{1}, \ldots, p_{d}\right)=z\left(p_{i}\right)
$$

and the elementary symmetric functions of the coordinates $z^{i}$ yield a local coordinate system for $\operatorname{Sym}^{d}(C)$. Therefore the latter is a $d$-dimensional complex manifold. Moreover, the holomorphic map

$$
C^{d} \rightarrow J(C), \quad\left(p_{1}, \ldots, p_{d}\right) \mapsto \mu\left(p_{1}\right)+\cdots+\mu\left(p_{d}\right)
$$

is $S_{d}$-invariant, hence it descends to a map $\operatorname{Sym}^{d}(C) \rightarrow J(C)$, which coincides with $\mu_{d}$. So the latter is holomorphic.

ExERCISE 8.5. Prove that $\operatorname{Sym}^{d}\left(\mathbb{P}_{1}\right) \simeq \mathbb{P}_{d}$. (Hint: write explicitly a morphism in homogeneous coordinates.)

The surjectivity of $\nu$ follows from the following fact, usually called Jacobi inversion theorem.

Proposition 8.6. The map $\mu_{g}: \operatorname{Sym}^{g}(C) \rightarrow J(C)$ is surjective.
Proof. Let $D=\sum p_{i} \in \operatorname{Sym}^{g}(C)$, with all the $p_{i}$ 's distinct, and let $z^{i}$ be a local coordinate centred in $p_{i}$; then $\left\{z^{1}, \ldots, z^{g}\right\}$ is a local coordinate system around $D$. If $D^{\prime}$ is near $D$ we have

$$
\begin{equation*}
\frac{\partial}{\partial z^{i}}\left(\mu_{g}\left(D^{\prime}\right)\right)^{j}=\frac{\partial}{\partial z^{i}} \int_{p_{0}}^{p_{i}^{\prime}} \omega_{j}=h_{j i} \tag{8.6}
\end{equation*}
$$

where $h_{j i}$ is the component of $\omega_{j}$ on $d z^{i}$.
Consider now the matrix

$$
\left(\begin{array}{ccc}
\omega_{1}\left(p_{1}\right) & \ldots & \omega_{1}\left(p_{g}\right)  \tag{8.7}\\
\ldots & \ldots & \ldots \\
\omega_{g}\left(p_{1}\right) & \ldots & \omega_{g}\left(p_{g}\right)
\end{array}\right)
$$

We may choose $p_{1}$ so that $\omega_{1}\left(p_{1}\right) \neq 0$, and then subtracting a suitable multiple of $\omega_{1}$ from $\omega_{2}, \ldots, \omega_{g}$ we may arrange that $\omega_{2}\left(p_{1}\right)=\cdots=\omega_{g}\left(p_{1}\right)=0$. We next choose $p_{2}$ so that $\omega_{2}\left(p_{2}\right) \neq 0$, and arrange that $\omega_{3}\left(p_{2}\right)=\cdots=\omega_{g}\left(p_{3}\right)=0$, and so on. In this way the matrix (8.7) is upper triangular. With these choices of the abelian differentials $\omega_{i}$ and of
the points $p_{i}$ the Jacobian matrix $\left\{h_{j i}\right\}$ is upper triangular as well, and since $\omega_{i}\left(p_{i}\right) \neq 0$, its diagonal elements $h_{i i}$ are nonzero at $D$, so that at the point $D$ corresponding to our choices the Jacobian determinant is nonzero. This means that the determinant is not everywhere zero, and by Lemma 7.4 one concludes.

Proposition 8.7. The map $\mu_{g}$ is generically one-to-one.
Proof. Let $u \in J(C)$, and choose a divisor $D \in \mu_{g}^{-1}(u)$. By Abel's theorem the fibre $\mu_{g}^{-1}(u)$ is formed by all effective divisors linearly equivalent to $D$, hence it is a projective space. But since $\operatorname{dim} J(C)=\operatorname{dim} \operatorname{Sym}^{d}(C)$ the fibre of $\mu_{g}$ is generically 0 -dimensional, so that generically it is a point.

This means that $\mu_{g}$ establishes a biholomorphic correspondence between a dense subset of $\operatorname{Sym}^{d}(C)$ and a dense subset of $J(C)$; such maps are called birational.

Corollary 8.8. Every divisor of degree $\geq g$ on an algebraic curve of genus $g$ is linearly equivalent to an effective divisor.

Proof. Let $D \in \operatorname{Div}_{d}(C)$ with $d \geq g$. We may write $D=D^{\prime}+D^{\prime \prime}$ with $\operatorname{deg} D^{\prime}=g$ and $D^{\prime \prime} \geq 0$. By mapping $D^{\prime}$ to $J(C)$ by Abel's map and taking a counterimage in Sym $^{g}(C)$ we obtain an effective divisor $E$ linearly equivalent to $D^{\prime}$. Then $E+D^{\prime \prime}$ is effective and linearly equivalent to $D$.

Corollary 8.9. Every elliptic smooth algebraic curve (i.e. every smooth algebraic curve of genus 1 ) is of the form $\mathbb{C} / \Lambda$ for some lattice $\Lambda \subset \mathbb{C}$.

Proof. We have $J(C)=\mathbb{C} / \Lambda$, and the map $\mu_{1}$ concides with $\mu$,

$$
\mu(p)=\int_{p_{0}}^{p} \omega
$$

By Abel's theorem, $\mu(p)=\mu(q)$ if and only if there is on $C$ a meromorphic function $f$ such that $(f)=p-q$; but on $C$ there are no meromorphic functions with a single pole, so that $\mu$ is injective. $\mu$ is also surjective by Lemma 7.4 (this is a particular case of Jacobi inversion theorem), hence it is bijective.

Corollary 8.10. The canonical bundle of any elliptic curve is trivial.

Proof. We represent an elliptic curve $C$ as a quotient $\mathbb{C} / \Lambda$. The (trivial) tangent bundle to $\mathbb{C}$ is invariant under the action of $\Lambda$, therefore the tangent bundle to $C$ is trivial as well.

Another consequence is that if $C$ is an elliptic algebraic curve and one chooses a point $p \in C$, the curve has a structure of abelian group, with $p$ playing the role of the identity element.
1.2. Jacobian varieties are algebraic. According to our previous discussion, any 1-dimensional complex torus is algebraic. This is no longer true for higher dimensional tori. However, the Jacobian variety of an algebraic curve is always algebraic.

Let $\Lambda$ be a lattice in $\mathbb{C}^{n}$. Any point in the lattice singles out univoquely a cell in the lattice, and two opposite sides of the cell determine after identification a closed smooth loop in the quotient torus $T=\mathbb{C}^{n} / \Lambda$. This provides an identification $\Lambda \simeq H_{1}(T, \mathbb{Z})$.

Let now $\xi$ be a skew-symmetric $\mathbb{Z}$-bilinear form on $H_{1}(T, \mathbb{Z})$. Since $\operatorname{Hom}_{\mathbb{Z}}\left(\Lambda^{2} H_{1}(T\right.$, $\mathbb{Z}), \mathbb{Z}) \simeq H^{2}(T, \mathbb{Z})$ canonically (check this isomorphism as an exercise), $\xi$ may be regarded as a smooth complex-valued differential 2-form on $T$.

Proposition 8.11. The 2-form $\xi$ which on the basis $\left\{e_{j}\right\}$ is represented by the intersection matrix $Q^{-1}$ is a positive $(1,1)$ form.

Proof. If $\left\{e_{j}, j=1 \ldots 2 n\right\}$ are the real basis vectors in $\mathbb{C}^{n}$ generating the lattice, they can be regarded as basis in $H_{1}(T, \mathbb{Z})$. They also generate $2 n$ real vector fields on $T$ (after identifying $\mathbb{C}^{n}$ with its tangent space at 0 the $e_{j}$ yield tangent vectors to $T$ at the point corresponding to 0 ; by transporting them in all points of $T$ by left transport one gets $2 n$ vector fields, which we still denote by $e_{j}$ ). Let $\left\{z^{1}, \ldots, z^{n}\right\}$ be the natural local complex coordinates in $T$; the period matrix may be described as

$$
\Omega_{i j}=\int_{e_{j}} d z^{i}
$$

After writing $\xi$ on the basis $\left\{d z^{i}, d \bar{z}^{j}\right\}$ one can check that the stated properties of $\xi$ are equivalent to the Riemann bilinear relations. ${ }^{2}$

There exists on $J(C)$ a (in principle smooth) line bundle $L$ whose first Chern class is the cohomology class of $\xi$. This line bundle has a connection whose curvature is (cohomologous to) $\frac{2 \pi}{i} \xi$; since this form is of type (1,1), $L$ may be given a holomorphic structure. With this structure, it is ample by Proposition 7.3. ${ }^{3}$ This defines a projective imbedding of $J(C)$, so that the latter is algebraic.

## 2. Elliptic curves

Consider the curve $C^{\prime}$ in $\mathbb{C}^{2}$ given by an equation

$$
\begin{equation*}
y^{2}=P(x), \tag{8.8}
\end{equation*}
$$

[^29]where $x, y$ are the standard coordinates in $\mathbb{C}^{2}$, and $P(x)$ is a polynomial of degree 3 . By writing the equation (8.8) in homogeneous coordinates, $C^{\prime}$ may be completed to an algebraic curve $C$ imbedded in $\mathbb{P}_{2}$ - a cubic curve in $\mathbb{P}_{2}$. Let us assume that $C$ is smooth. By the genus formula we see that $C$ is an elliptic curve.

Exercise 8.1. Show that $\omega=d x / y$ is a nowhere vanishing abelian differential on $C$. After proving that all elliptic curves may be written in the form (8.8), this provides another proof of the triviality of the canonical bundle of an elliptic curve. (Hint: around each branch point, $z=\sqrt{P(x)}$ is a good local coordinate...)

The equation (8.8) moreover exhibits $C$ as a cover of $\mathbb{P}_{1}$, which is branched of order 2 at the points where $y=0$ and at the point at infinity. One also checks that the point at infinity is a smooth point. We want to show that every smooth elliptic curve can be realized in this way.

So let $C$ be a smooth elliptic curve. If we fix a point $p$ in $C$ and consider the exact sequence of sheaves on $C$

$$
0 \rightarrow \mathcal{O}(p) \rightarrow \mathcal{O}(2 p) \rightarrow k_{p} \rightarrow 0
$$

proceeding as usual (Serre duality and vanishing theorem) one shows that $H^{0}(C, \mathcal{O}(2 p))$ is nonzero. A nontrivial section $f$ can be regarded as a global meromorphic function holomorphic in $C-\{p\}$, having a double pole at $p$. Moreover we fix a nowhere vanishing holomorphic 1-form $\omega$ (which exists because $K$ is trivial). We have

$$
\operatorname{Res}_{p}(f \omega)=0
$$

We realize $C$ as $\mathbb{C} / \Lambda$; these singles out a complex coordinate $z$ on the open subset of $C$ corresponding to the fundamental cell of the lattice $\Lambda$. Then we may choose $\omega=d z$, and $f$ may be chosen in such a way that

$$
f(z)=\frac{1}{z^{2}}+O(z)
$$

On the other hand, the meromorphic function $d f / \omega$ is holomorphic outside $p$, and has a triple pole at $p$. We may choose constants $a, b, c$ such that

$$
\tilde{f}=a \frac{d f}{\omega}+b f+c=\frac{1}{z^{3}}+O(z) .
$$

The line bundle $\mathcal{O}(3 p)$ is very ample, i.e., its complete linear system realizes the Kodaira imbedding of $C$ into projective space. By Riemann-Roch and the vanishing theorem we have $h^{0}(3 p)=3$, so that $C$ is imbedded into $\mathbb{P}_{2}$. To realize explicitly the imbedding we may choose three global sections corresponding to the meromorphic functions $1, f, \tilde{f}$. We shall see that these are related by a polynomial identity, which then expresses the equation cutting out $C$ in $\mathbb{P}_{2}$.

We indeed have, for suitable constants $\alpha, \beta, \gamma$,

$$
\tilde{f}^{2}=\frac{1}{z^{6}}+\frac{\alpha}{z^{2}}+O\left(\frac{1}{z}\right), \quad f^{3}=\frac{1}{z^{6}}+\frac{\beta}{z^{3}}+\frac{\gamma}{z^{2}}+O\left(\frac{1}{z}\right),
$$

so that, setting $\delta=\alpha-\beta$,

$$
\tilde{f}^{2}+\beta \tilde{f}-f^{3}+\delta f=O\left(\frac{1}{z}\right)
$$

So the meromorphic function in the left-hand side is holomorphic away from $p$, and has at $p$ a simple pole. Such a function must be constant, otherwise it would provide an isomorphism of $C$ with the Riemann sphere.

Thus $C$ may be described as a locus in $\mathbb{P}_{2}$ whose equation in affine coordinates is

$$
\begin{equation*}
y^{2}+\beta y=x^{3}-\delta x+\epsilon \tag{8.9}
\end{equation*}
$$

for a suitable constant $\epsilon$. By a linear transformation on $y$ we may set $\beta=0$, and then by a linear transformation of $x$ we may set the two roots of the polynomial in the righthand side of (8.9) to 0 and 1 . So we express the elliptic curve $C$ in the standard form (Weierstraß representation) ${ }^{4}$

$$
\begin{equation*}
y^{2}=x(x-1)(x-\lambda) . \tag{8.10}
\end{equation*}
$$

Exercise 8.2. Determine for what values of the parameter $\lambda$ the curve (8.10) is smooth.

We want to elaborate on this construction. Having fixed the complex coordinate $z$, the function $f$ is basically fixed as well. We call it the Weierstraß $\mathcal{P}$-function. Its derivative is $\mathcal{P}^{\prime}=-2 \tilde{f}$. Notice that $\mathcal{P}$ cannot contain terms of odd degree in its Laurent expansion, otherwise $\mathcal{P}(z)-\mathcal{P}(-z)$ would be a nonconstant holomorphic function on C. So

$$
\begin{gathered}
\mathcal{P}(z)=\frac{1}{z^{2}}+a z^{2}+b z^{4}+O\left(z^{6}\right) \\
\mathcal{P}^{\prime}(z)=-\frac{2}{z^{3}}+2 a z+4 b z^{3}+O\left(z^{5}\right) \\
(\mathcal{P}(z))^{3}=\frac{1}{z^{6}}+\frac{3 a}{z^{2}}+3 b+O\left(z^{2}\right) \\
\left(\mathcal{P}^{\prime}(z)\right)^{2}=\frac{4}{z^{6}}-\frac{8 a}{z^{2}}-16 b+O(z)
\end{gathered}
$$

for suitable constants $a, b$. From this we see that $\mathcal{P}$ satisfies the condition

$$
\left(\mathcal{P}^{\prime}\right)^{2}-4 \mathcal{P}^{3}+20 a=\text { constant }^{\prime}
$$

one usually writes $g_{2}$ for $20 a$ and $g_{3}$ for the constant in the right-hand side.
In terms of this representation we may introduce a map $j: \mathcal{M}_{1} \rightarrow \mathbb{C}$, where $\mathcal{M}_{1}$ is the set of isomorphism classes of smooth elliptic curves (the moduli space of genus one

[^30]curves) ${ }^{5}$
$$
j(C)=\frac{1728 g_{2}^{3}}{g_{2}^{3}-27 g_{3}^{2}}
$$

One shows that this map is bijective; in particular $\mathcal{M}_{1}$ gets a structure of complex manifold. The number $j(C)$ is called the $j$-invariant of the curve $C$. We may therefore say that the moduli space $\mathcal{M}_{1}$ is isomorphic to $\mathbb{C}$. ${ }^{6}$

Exercise 8.3. Write the $j$-invariant as a function of the parameter $\lambda$ in equation (8.10). Do you think that $\lambda$ is a good coordinate on the moduli space $\mathcal{M}_{1}$ ?

The holomorphic map

$$
\psi: C \rightarrow \mathbb{P}_{2}, \quad z \mapsto\left[1, \mathcal{P}(z), \mathcal{P}^{\prime}(z)\right]
$$

imbeds $C$ into $\mathbb{P}_{2}$ as the cubic curve cut out by the polynomial

$$
F=y^{2}-4 x^{3}+g_{2} x+g_{3}
$$

(we use the same affine coordinates as in the previous representation). Since $\tilde{f}=d f / \omega$ we have

$$
\omega=\frac{d x}{y}
$$

and the inverse of $\psi$ is the Abel map, ${ }^{7}$

$$
\psi^{-1}(p)=\int_{p_{0}}^{p} \frac{d x}{y} \bmod \Lambda
$$

having chosen $p_{0}$ at the point at infinity, $p_{0}=\psi(0)=[0,0,1]$.
In terms of this construction we may give an elementary geometric visualization of the group law in an elliptic curve. Let us choose $p_{0}$ as the identity element in $C$. We shall denote by $\bar{p}$ the element $p \in C$ regarded as a group element (so $\bar{p}_{0}=0$ ). By Abel's theorem, Proposition 8.3, we have that

$$
\bar{p}_{1}+\bar{p}_{2}+\bar{p}_{3}=0 \quad \text { if and and only if } \quad p_{1}+p_{2}+p_{3} \sim 3 p_{0}
$$

(indeed one may think that $\bar{p}=\mu(p)$, and one has $\mu\left(p_{1}+p_{2}+p_{3}-3 p_{0}\right)=0$ ).
Let $M(x, y)=m x+n y+q$ be the equation of the line in $\mathbb{P}_{2}$ through the points $p_{1}, p_{2}$, and let $p_{4}$ be the further intersection of this line with $C \subset \mathbb{P}_{2}$. The function $M(z)=M\left(\mathcal{P}(z), \mathcal{P}^{\prime}(z)\right)$ on $C$ vanishes (of order one) only at the points $p_{1}, p_{2}, p_{4}$, and has a pole at $p_{0}$. This pole must be of order three, so that the divisor of $M(z)$ is $p_{1}+p_{2}+p_{4}-3 p_{0}$, i.e; $p_{1}+p_{2}+p_{4}-3 p_{0} \sim 0$.

[^31]If $p_{1}+p_{2}+p_{3} \sim 3 p_{0}$, then $p_{3} \sim p_{4}$, so that $p_{3}=p_{4}$, and $p_{1}, p_{2}, p_{3}$ are collinear. Vice versa, if $p_{1}, p_{2}, p_{3}$ are collinear, $p_{1}+p_{2}+p_{3}-3 p_{0}$ is the divisor of the meromorphic function $M$, so that $p_{1}+p_{2}+p_{3}-3 p_{0} \sim 0$. We have therefore shown that $\bar{p}_{1}+\bar{p}_{2}+\bar{p}_{3}=0$ if and only if $p_{1}, p_{2}, p_{3}$ are collinear points in $\mathbb{P}_{2}$.

Example 8.4. Let $C$ be an elliptic curve having a Weierstraß representation $y^{2}=$ $x^{3}-1 . C$ is a double cover of $\mathbb{P}_{1}$, branched at the three points

$$
p_{1}=(1,0), \quad p_{2}=(\alpha, 0), \quad p_{3}=\left(\alpha^{2}, 0\right)
$$

(where $\alpha=e^{2 \pi i / 3}$ ) and at the point at infinity $p_{0}$. The points $p_{1}, p_{2}, p_{3}$ are collinear, so that $\bar{p}_{1}+\bar{p}_{2}+\bar{p}_{3}=0$.

The two points $q_{1}=(0, i), q_{2}=(0,-i)$ lie on $C$. The line through $q_{1}, q_{2}$ intersects $C$ at the point at infinity, as one may check in homogeneous coordinates. So in this case the elements $\bar{q}_{1}, \bar{q}_{2}$ are one the inverse of the other, and $q_{1}+q_{2} \sim 2 p_{0}$. More generally, if $q \in C$ is such that $\bar{q}=-\bar{p}$, then $p+q \sim 2 p_{0}$, and $q$ is the further intersection of $C$ with the line going through $p, p_{0}$; if $p=(a, b)$, then $q=(a,-b)$. So the branch points $p_{i}$ are 2-torsion elements in the group, $2 \bar{p}_{i}=0$.

## 3. Nodal curves

In this section we show how (plane) curve singularities may be resolved by a procedure called blowup.
3.1. Blowup. Blowing up a point in a variety ${ }^{8}$ means replacing the point with all possible directions along which one can approach it while moving in the variety. We shall at first consider the blowup of $\mathbb{C}^{2}$ at the origin; since this space is 2-dimensional, the set of all possible directions is a copy of $\mathbb{P}_{1}$. Let $x, y$ be the standard coordinates in $\mathbb{C}^{2}$, and $w_{0}$, $w_{1}$ homogeneous coordinates in $\mathbb{P}_{1}$. The blowup of $\mathbb{C}^{2}$ at the origin is the subvariety $\Gamma$ of $\mathbb{C}^{2} \times \mathbb{P}_{1}$ defined by the equation

$$
x w_{1}-y w_{0}=0 .
$$

To show that $\Gamma$ is a complex manifold we cover $\mathbb{C}^{2} \times \mathbb{P}_{1}$ with two coordinate charts, $V_{0}=\mathbb{C}^{2} \times U_{0}$ and $V_{1}=\mathbb{C}^{2} \times U_{1}$, where $U_{0}, U_{1}$ are the standard affine charts in $\mathbb{P}_{1}$, with coordinates $\left(x, y, t^{0}=w_{1} / w_{0}\right)$ and $\left(x, y, t^{1}=w_{0} / w_{1}\right)$. $\Gamma$ is a smooth hypersurface in $\mathbb{C}^{2} \times \mathbb{P}_{1}$, hence it is a complex surface. On the other hand if we put homogeneous coordinates $\left(v_{0}, v_{1}, v_{2}\right)$ in $\mathbb{C}^{2}$, then $\Gamma$ can be regarded as a open subset of the quadric in $\mathbb{P}_{2} \times \mathbb{P}_{1}$ having equation $v_{1} w_{1}-v_{2} w_{0}=0$, so that $\Gamma$ is actually algebraic.

[^32]Since $\Gamma$ is a subset of $\mathbb{C}^{2} \times \mathbb{P}_{1}$ there are two projections

which are holomorphic. If $p \in \mathbb{C}^{2}-\{0\}$ then $\sigma^{-1}(p)$ is a point (which means that there is a unique line through $p$ and 0 ), so that

$$
\sigma: \Gamma-\sigma^{-1}(0) \rightarrow \mathbb{C}^{2}-\{0\}
$$

is a biholomorphism. ${ }^{9}$ On the contrary $\sigma^{-1}(0) \simeq \mathbb{P}_{1}$ is the set of lines through the origin in $\mathbb{C}^{2}$.

The fibre of $\pi$ over a point $\left(w_{0}, w_{1}\right) \in \mathbb{P}_{1}$ is the line $x w_{1}-y w_{0}=0$, so that $\pi$ makes $\Gamma$ into the total space of a line bundle over $\mathbb{P}_{1}$. This bundle trivializes over the cover $\left\{U_{0}, U_{1}\right\}$, and the transition function $g: U_{0} \cap U_{1} \rightarrow \mathbb{C}^{*}$ is $g\left(w_{0}, w_{1}\right)=w_{0} / w_{1}$, so that the line bundle is actually the tautological bundle $\mathcal{O}_{\mathbb{P}_{1}}(-1)$.

This construction is local in nature and therefore can be applied to any complex surface $X$ (two-dimensional complex manifold) at any point $p$. Let $U$ be a chart around $p$, with complex coordinates $(x, y)$. By repeating the same construction we get a complex manifold $U^{\prime}$ with projections

and

$$
\sigma: U^{\prime}-\sigma^{-1}(p) \rightarrow U-\{p\}
$$

is a biholomorphism, so that one can replace $U$ by $U^{\prime}$ inside $X$, and get a complex manifold $X^{\prime}$ with a projection $\sigma: X^{\prime} \rightarrow X$ which is a biholomorphism outside $\sigma^{-1}(p)$. The manifold $X^{\prime}$ is the blowup of $X$ at $p$. The inverse image $E=\sigma^{-1}(p)$ is a divisor in $X^{\prime}$, called the exceptional divisor, and is isomorphic to $\mathbb{P}_{1}$. The construction of the blowup $\Gamma$ shows that $X^{\prime}$ is algebraic if $X$ is.

Example 8.1. The blowup of $\mathbb{P}_{2}$ at a point is an algebraic surface $X_{1}$ (an example of a Del Pezzo surface); the manifold $\Gamma$, obtained by blowing up $\mathbb{C}^{2}$ at the origin, is biholomorphic to $X_{1}$ minus a projective line (so $X_{1}$ is a compactification of $\Gamma$ ).
3.2. Transforms of a curve. Let $C$ be a curve in $\mathbb{C}^{2}$ containing the origin. We denote as before $\Gamma$ the blowup of $\mathbb{C}^{2}$ at the origin and make reference to the diagram (8.11). Notice that the inverse image $\sigma^{-1}(C) \subset \Gamma$ contains the exceptional divisor $E$, and that $\sigma^{-1}(C) \backslash E$ is isomorphic to $C-\{0\}$.

[^33]Definition 8.2. The curve $\sigma^{-1}(C) \subset \Gamma$ is the total transform of $C$. The curve obtained by taking the topological closure of $\sigma^{-1}(C) \backslash E$ in $\Gamma$ is the strict transform of $C$.

We want to check what points are added to $\sigma^{-1}(C) \backslash E$ when taking the topological closure. To this end we must understand what are the sequences in $\mathbb{C}^{2}$ which converge to 0 that are lifted by $\sigma$ to convergent sequences. Let $\left\{p_{k}=\left(x_{k}, y_{k}\right)\right\}_{k \in \mathbb{N}}$ be a sequence of points in $\mathbb{C}^{2}$ converging to 0 ; then $\sigma^{-1}\left(x_{k}, y_{k}\right)$ is the point $\left(x_{k}, y_{k}, w_{0}, w_{1}\right)$ with $x_{k} w_{1}-$ $y_{k} w_{0}=0$. Assume that for $k$ big enough one has $w_{0} \neq 0$ (otherwise we would assume $w_{1} \neq 0$ and would make a similar argument). Then $w_{1} / w_{0}=y_{k} / x_{k}$, and $\left\{\sigma^{-1}\left(p_{k}\right)\right\}$ converges if and only if $\left\{y_{k} / x_{k}\right\}$ has a limit, say $h$; in that case $\left\{\sigma^{-1}\left(p_{k}\right)\right\}$ converges to the point $(0,0,1, h)$ of $E$. This means that the lines $r_{k}$ joining 0 to $p_{k}$ approach the limit line $r$ having equation $y=h k$. So a sequence $\left\{p_{k}=\left(x_{k}, y_{k}\right)\right\}$ convergent to 0 lifts to a convergent sequence in $\Gamma$ if and only if the lines $r_{k}$ admit a limit line $r$; in that case, the lifted sequence converges to the point of $E$ representing the line $r$.

The strict transform $C^{\prime}$ of $C$ meets the exceptional divisor in as many points as are the directions along which one can approach 0 on $C$, namely, as are the tangents at $C$ at 0 . So, if $C$ is smooth at 0 , its strict transform meets $E$ at one point. Every intersection point must be counted with its multiplicity: if at the point 0 the curve $C$ has $m$ coinciding tangents, then the strict transform meets the exceptional divisor at a point of multiplicity $m$.

Definition 8.3. Let the (affine plane) curve $C$ be given by the equation $f(x, y)=0$. We say that $C$ has multiplicity $m$ at 0 if the Taylor expansion of $f$ at 0 starts at degree $m$.

This means that the curve has $m$ tangents at the point 0 (but some of them might coincide). By choosing suitable coordinates one can apply this notion to any point of a plane curve.

Example 8.4. A curve is smooth at 0 if and only if its multiplicity at 0 is 1 . The curves $x y=0, y^{2}=x^{2}$ and $y^{2}=x^{3}$ have multiplicity 2 at 0 . The first two have two distinct tangents at 0 , the third has a double tangent.

If the curve $C$ has multiplicity $m$ at 0 than it has $m$ tangents at 0 , and its strict transform meets the exceptional divisor of $\Gamma$ at $m$ points (notice however that these points are all distinct only if the $m$ tangents are distincts).

Definition 8.5. A singular point of a plane curve $C$ is said to be nodal if at that point $C$ has multiplicity 2, and the two tangents to the curve at that point are distinct.

Exercise 8.6. With reference to equation (8.10), determine for what values of $\lambda$ the curve has a nodal singularity.

ExERCISE 8.7. Show that around a nodal singularity a curve is isomorphic to an open neighbourhood of the origin of the curve $x y=0$ in $\mathbb{C}^{2}$.

Example 8.8. (Blowing up a nodal singularity.) We consider the curve $C \subset \mathbb{C}^{2}$ having equation $x^{3}+x^{2}-y^{2}=0$. This curve has multiplicity 2 at the origin, and its two tangents at the origin have equations $y= \pm x$. So $C$ has a nodal singularity at the origin. We recall that $\Gamma$ is described as the locus

$$
\left\{\left(u, v, w_{0}, w_{1}\right) \in \mathbb{C}^{2} \times \mathbb{P}_{1} \mid u w_{0}=v w_{1}\right\}
$$

The projection $\sigma$ is described as

$$
\left\{\begin{array} { l } 
{ x = u }  \tag{8.12}\\
{ y = u w _ { 0 } / w _ { 1 } }
\end{array} \quad \left\{\begin{array}{l}
x=v w_{1} / w_{0} \\
y=v
\end{array}\right.\right.
$$

in $\Gamma \cap V_{1}$ and $\Gamma \cap V_{0}$, respectively. By substituting the first of the representations (8.12) into the equation of $C$ we obtain the equation of the restriction of the total transform to $\Gamma \cap U_{1}$ :

$$
u^{2}\left(u+1-t^{2}\right)=0
$$

where $t=w_{0} / w_{1} . u^{2}=0$ is the equation of the exceptional divisor, so that the equation of the strict transform is $u+1-t^{2}=0$. By letting $u=0$ we obtain the points $(0,0,1,1)$ and $(0,0,1,-1)$ as intersection points of the strict transform with the exceptional divisor. By substituting the second representation in eq. (8.12) we obtain the equation of the total transform in $\Gamma \cap U_{0}$; the strict transform now has equation $t^{3} v+t^{2}-1$, yielding the same intersection points.

The total transform is a reducible curve, with two irreducible components which meet at two points.

Exercise 8.9. Repeat the previous calculations for the nodal curve $x y=0$. In particular show that the total transform is a reducible curve, consisting of the exceptional divisor and two more genus zero components, each of which meets the exceptional divisor at a point.

Example 8.10. (The cusp) Let $C$ be curve with equation $y^{2}=x^{3}$. This curve has multiplicity 2 at the origin where it has a double tangent. ${ }^{10}$ Proceeding as in the previous example we get the equation $v t^{3}=1$ for $C^{\prime}$ in $\Gamma \cap V_{0}$, so that $C^{\prime}$ does not meet $E$ in this chart. In the other chart the equation of $C^{\prime}$ is $t^{2}=u$, so that $C^{\prime}$ meets $E$ at the point $(0,0,0,1)$; we have one intersection point because the two tangents to $C$ at the origin coincide.

The strict transform is an irreducible curve, and the total transform is a reducible curve with two components meeting at a (double) point.

[^34]3.3. Normalization of a nodal plane curve. It is clear from the previous examples that the strict transform of a plane nodal curve $C$ (i.e., a plane curve with only nodal singularities) is again a nodal curve, with one less singular point. Therefore after a finite number of blowups we obtain a smooth curve $N$, together with a birational morphism $\pi: N \rightarrow C . N$ is called the normalization of $C$.

Example 8.11. Let us consider the smooth curve $C_{0}$ in $\mathbb{C}^{2}$ having equation $y^{2}=$ $x^{4}-1$. Projection onto the $x$-axis makes $C_{0}$ into a double cover of $\mathbb{C}$, branched at the points $( \pm 1,0)$ and $( \pm i, 0)$. The curve $C_{0}$ can be completed to a projective curve simply by writing its equation in homogeneous coordinates $\left(w_{0}, w_{1}, w_{2}\right)$ and considering it as a curve $C$ in $\mathbb{P}_{2}$; we are thus compactifying $C_{0}$ by adding a point at infinity, which in this case is not a branch point. The equation of $C$ is

$$
w_{0}^{2} w_{2}^{2}-w_{1}^{4}+w_{0}^{4}=0 .
$$

This curve has genus 1 and is singular at infinity (as one could have alredy guessed since the genus formula for smooth plane curves does not work); indeed, after introducing affine coordinates $\xi=w_{0} / w_{2}, \eta=w_{1} / w_{2}$ (in this coordinates the point at infinity on the $x$-axis is $\eta=\xi=0$ ) we have the equation

$$
\xi^{2}=\eta^{4}-\xi^{4}
$$

showing that $C$ is indeed singular at infinity. One can redefine the coordinates $\xi, \eta$ so that $C$ has equation

$$
\left(\xi-\eta^{2}\right)\left(\xi+\eta^{2}\right)=0
$$

showing that $C$ is a nodal curve. Then it can be desingularized as in Example 8.8.
A genus formula. We give here, without proof, a formula which can be used to compute the genus of the normalization $N$ of a nodal curve $C$. Assume that $N$ has $t$ irreducible components $N_{1}, \ldots, N_{t}$, and that $C$ has $\delta$ singular points. Then:

$$
g(C)=\sum_{1}^{t} g\left(N_{i}\right)+1-t+\delta
$$

For instance, by applying this formula to Example 8.8, we obtain that the normalization is a projective line.

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[^0]:    ${ }^{1}$ In intrinsic notation this means that

    $$
    \Omega^{k}(X \times \mathbb{R}) \simeq C^{\infty}(X \times \mathbb{R}) \otimes_{C^{\infty}(X)}\left[\Omega^{k}(X) \oplus \Omega^{k-1}(X)\right]
    $$

    ${ }^{2}$ The reader may consult e.g. [3], §I.4.

[^1]:    ${ }^{3}$ Sometimes this term is used for another cohomology complex, cf. [3].

[^2]:    ${ }^{4}$ The Mayer-Vietoris sequence foreshadows the Čech cohomology we shall study in Chapter 3.

[^3]:    ${ }^{5}$ For the fact that $F$ can be taken smooth cf. [3].

[^4]:    ${ }^{6} F$ is the group whose elements are words $x_{1}^{\epsilon_{1}} x_{2} \ldots x_{n}^{\epsilon}$ or the empty word, where the letters $x_{i}$ are either in $G_{1}$ or $G_{2}, \epsilon_{i}= \pm 1$, and the product is given by juxtaposition.
    ${ }^{7}$ The first relation tells that the product of letters in the words of $F$ are the product either in $G_{1}$ or $G_{2}$, when this makes sense. The second relation kind of "glues" $G_{1}$ and $G_{2}$ along the images of $G$.

[^5]:    ${ }^{1}$ Again, we understand the choice of a coefficient ring $R$.

[^6]:    ${ }^{1}$ This rather naive terminology can be made more precise by saying that a presheaf on $X$ is a contravariant functor from the category $\mathfrak{D}_{X}$ of open subsets of $X$ to the category of Abelian groups. $\mathfrak{O}_{X}$ is defined as the category whose objects are the open subsets of $X$ while the morphisms are the inclusions of open sets.

[^7]:    ${ }^{2}$ A function is locally constant on $U$ if it is constant on any connected component of $U$.

[^8]:    ${ }^{3}$ Let $I$ be a directed set. A subset $J$ of $I$ is said to be cofinal if for any $i \in I$ there is a $j \in J$ such that $i<j$. By the definition of direct limit we see that, given a directed family of Abelian groups $\left\{G_{i}\right\}_{i \in I}$, if $\left\{G_{j}\right\}_{j \in J}$ is the subfamily indexed by $J$, then

    $$
    \underset{i \in I}{\lim } G_{i} \simeq \underset{j \in J}{\lim } G_{j}
    $$

    that is, direct limits can be taken over cofinal subsets of the index set.
    ${ }^{4}$ Since we are dealing with Abelian groups, i.e. with $\mathbb{Z}$-modules, the Hom modules and tensor products are taken over $\mathbb{Z}$.

[^9]:    ${ }^{5}$ For a definition of fibred product see e.g. [15].

[^10]:    ${ }^{6}$ We are cheating a little bit, since the sheaf of rings $\mathcal{E} \operatorname{nd}\left(\mathcal{S}^{0}(\mathcal{F})\right)$ is not commutative. However a closer inspection of the proof would show that it works anyways.

[^11]:    ${ }^{1}$ The choice of having $K_{p}=K$ for $p \leq 0$ is due to notational convenience.

[^12]:    ${ }^{2}$ This assumption is made here for simplicity but one could let $p, q$ range over the integers; however some of the results we are going to give would be no longer valid.

[^13]:    ${ }^{3}$ Here a notational conflict arises, so that we shall denote by $D$ the differential of the total complex $T$.

[^14]:    ${ }^{1}$ This allows us also to define the first Chern class of a vector bundle $E$ of any rank by letting $c_{1}(E)=c_{1}(\operatorname{det} E)$.
    ${ }^{2}$ Here we use the fact that if $X$ is a complex manifold of dimension $n$, then $H^{k}(X, \mathcal{O})=0$ for all $k>n$.

[^15]:    ${ }^{3}$ The symmetric functions are defined as

    $$
    \sigma_{i}\left(x_{1}, \ldots, x_{k}\right)=\sum_{1 \leq j_{1}<\cdots<j_{i} \leq n} x_{j_{1}} \cdots \cdots x_{j_{i}}
    $$

    Thus, for instance,

    $$
    \begin{aligned}
    \sigma_{1}\left(x_{1}, \ldots, x_{k}\right) & =x_{1}+\cdots+x_{k} \\
    \sigma_{2}\left(x_{1}, \ldots, x_{k}\right) & =x_{1} x_{2}+x_{1} x_{3}+\cdots+x_{k-1} x_{k} \\
    & \cdots \\
    \sigma_{k}\left(x_{1}, \ldots, x_{k}\right) & =x_{1} \cdots x_{k}
    \end{aligned}
    $$

[^16]:    ${ }^{4}$ This map is well defined because different representatives of $[\omega]$ differ by an exact form, whose integral over $X$ vanishes.

[^17]:    ${ }^{1}$ The reader should check that the integral does not depend on the choice of the representative.

[^18]:    ${ }^{2}$ Here we use the fact that tensoring all elements of an exact sequence by the sheaf of sections of a vector bundle preserves exactness. This is quite obvious because by the local triviality of $E$ the stalk of $\mathcal{O}(E)$ at $p$ is $\mathcal{O}_{p}^{k}$, with $k$ the rank of $E$.

[^19]:    ${ }^{3}$ Let us recall this notion: one says that a ring $R$ is an integral domain if $u v=0$ implies that either $u=0$ or $v=0$. An element $u \in R$ in an integral domain is said to be irreducible if $u=v w$ implies that $v$ or $w$ is a unit; $R$ is a unique factorization domain if any element $u$ can be written as a product $u=u_{1} \cdots \ldots u_{m}$, where the $u_{i}$ are irreducible and unique up to multiplication by units.

[^20]:    ${ }^{4}$ We use the fact that whenever

    $$
    0 \rightarrow E \rightarrow F \rightarrow G \rightarrow 0
    $$

    is an exact sequence of vector bundles, then $\operatorname{det} F \simeq \operatorname{det} E \otimes \operatorname{det} G$, as one can prove by using transition functions.

[^21]:    ${ }^{1}$ This map actually depends on the choice of a basis of $|D|$; however, different choices correspond to an action of the group $\mathbb{P} G l(N+1, \mathbb{C})$ on $\mathbb{P}_{N}$ and therefore produce isomorphic subvarieties of $\mathbb{P}_{N}$.

[^22]:    ${ }^{2}$ This result is actually true whatever is the dimension of $M$, cf. [9].

[^23]:    ${ }^{3}$ Strictly speaking an algebraic curve consists of more data than a compact Riemann surface $S$, since the former requires an imbedding of $S$ into a projective space, i.e. the choice of an ample line bundle.
    ${ }^{4}$ We introduce the following notation: if $\mathcal{E}$ is a sheaf of $\mathcal{O}_{C}$-modules, then $h^{i}(\mathcal{E})=\operatorname{dim} H^{i}(C, \mathcal{E})$.

[^24]:    ${ }^{5}$ This need not be true if the algebraic curve $C$ is singular. However the Riemann-Roch theorem is still true (provided we know what a line bundle on a singular curve is!) with $g$ the arithmetic genus.
    ${ }^{6}$ Since two holomorphic functions of one variable which agree on a nondiscrete set are identical, and since $C^{\prime}$ is compact, the number of points in $f^{-1}(p)$ is always finite.

[^25]:    ${ }^{7}$ A (holomorphic) covering map $f: X \rightarrow Y$, with $X$ connected, is a map such that each $p \in Y$ has a connected neighbourhood $U$ such that $f^{-1}(U)=\cup_{\alpha} U_{\alpha}$ is the disjoint union of open subsets of $X$ which are biholomorphic to $U$ via $f$.

[^26]:    ${ }^{8}$ We are actually using here a piece of intersection theory. The fact is that any $k$-dimensional analytic subvariety $V$ of an $n$-dimensional complex manifold $X$ determines a homology class $[V]$ in the homology group $H_{2 k}(X, \mathbb{Z})$. Assume that $X$ is compact, and let $W$ be an $(n-k)$-dimensional analytic subvariety of $X$; the homology cap product $H_{2 k}(X, \mathbb{Z}) \cap H_{2 n-2 k}(X, \mathbb{Z}) \rightarrow \mathbb{Z}$, which is dual to the cup product in cohomology, associates the integer number $[V] \cap[W]$ with the two subvarieties. One may pick up different representatives $V^{\prime}$ and $W^{\prime}$ of $[V]$ and $[W]$ such that $V^{\prime}$ and $W^{\prime}$ meet transversally, i.e. they meet at a finite number of points; then the the number $[V] \cap[W]$ counts the intersection points (cf. [9] page 49).

    In our case the number of intersection points is given by the number of solutions to an algebraic system, given by the equation of $C$ in $\mathbb{P}_{2}$ (which has degree $d$ ) and the linear equation of a hyperplane. For a generic choice of the hyperplane, the number of solutions is $d$.

[^27]:    ${ }^{9}$ Otherwise one can directly identify the sections of $L$ with meromorphic functions having (only) a single pole at $p$, since such functions can be developed around $p$ in the form

    $$
    f(z)=\frac{a}{z}+g(z),
    $$

    where $g$ is a holomorphic function. $a \in \mathbb{C}$ should be indentified with the projection of $f$ onto $k_{p}$. (Here $z$ is a local complex coordinate such that $z(p)=0$.)

[^28]:    ${ }^{1}$ Notice that $\operatorname{Pic}^{d}(C) \simeq \operatorname{Pic}^{d^{\prime}}(C)$ as sets for all values of $d$ and $d^{\prime}$.

[^29]:    ${ }^{2}$ So we are not only proving that the Jacobian variety of an algebraic curve is algebraic, but, more generally, that any complex torus satisfying the Riemann bilinear relations is algebraic.
    ${ }^{3}$ We are using the fact that if a smooth complex vector bundle $E$ on a complex manifold $X$ has a connection whose curvature has no $(0,2)$ part, then the complex structure of $X$ can be "lifted" to $E$. Cf. [17]. Otherwise, we may use the fact that the image of the map $c_{1}$ in $H^{2}(J(C), \mathbb{Z})$ (the Néron-Severi group of $J(C)$, cf. subsection 5.5.1) may be represented as $H^{2}(J(C), \mathbb{Z}) \cap H^{1,1}(J(C), \mathbb{Z})$, i.e., as the group of integral 2-classes that are of Hodge type (1,1). The class of $\xi$ is clearly of this type.

[^30]:    ${ }^{4}$ Even though the Weierstraß representation only provides the equation of the affine part of an elliptic curve, the latter is nevertheless completely characterized. It is indeed true that any affine plane curve can be uniquely extended to a compact curve by adding points at infinity, as one can check by elementary considerations.

[^31]:    ${ }^{5}$ The fancy coefficient 1728 comes from arithmetic geometry, where the theory is tailored to work also for fields of characteristic 2 and 3.
    ${ }^{6}$ By uniformization theory one can also realize this moduli space as a quotient $\mathbb{H} / S l(2, \mathbb{Z})$, where $\mathbb{H}$ is the upper half complex plane. This is not contradictory in that the quotient $\mathbb{H} / S l(2, \mathbb{Z})$ is biholomorphic to $\mathbb{C}!$ (Notice that on the contrary, $\mathbb{H}$ and $\mathbb{C}$ are not biholomorphic). Cf. [10].
    ${ }^{7}$ One should bear in mind that we have identified $C$ with a quotient $\mathbb{C} / \Lambda$.

[^32]:    ${ }^{8}$ Our treatment of the blowup of an algebraic variety is basically taken from $[\mathbf{1}]$.

[^33]:    ${ }^{9}$ So, according to a terminology we have introduce in a previous chapter, the map $\sigma$ is a birational morphism.

[^34]:    ${ }^{10}$ Indeed this curve can be regarded as the limit for $\alpha \rightarrow 0$ of the family of nodal curves $x^{3}+\alpha^{2} x^{2}-$ $y^{2}=0$, which at the origin are tangent to the two lines $y= \pm \alpha x$.

