CHAPTER 8

# Rotational Motion

CHAPTER 9

# Bodies in Equilibrium; Elasticity and Fracture

The rotational motion of an ideal rigid body, an object of fixed shape, describes the movement of each of its particles in a circle around a line called the **axis of rotation.**

## Angular Quantities

For any point on an object experiencing rotational motion, its distance from the axis of rotation is the radius, $r$, of the circle traveled by that point.

- The distance of rotation is given by the angle $\theta$ created by the movement with respect to a reference line. The arc traversed by the particle when it moves that angle is a measure of distance, $l$. The angle $\theta$ is measured in radians, relating the ratio of arc length and radius, such that $\theta = l/r$. There are $2\pi$ radians in a circle, so one radian is approximately equal to 57.3°.
- Angular displacement refers to the angular change due to rotation, so **average angular velocity** is defined as $\bar{\omega} = \Delta\theta/\Delta t$.
- **Instantaneous angular velocity** is given by

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}.$$

The units for angular velocity are radians per second. Angular velocity is uniform for each point on the object.
- **Average angular acceleration** is defined as $\bar{\alpha} = \Delta\omega/\Delta t$. **Instantaneous angular acceleration** is given by

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t}.$$

The units for angular acceleration are radians per second squared. Angular acceleration is uniform for each point on the object.

- Linear velocity at a point $r$ away from the center is related to angular velocity such that $v = r\omega$, so it is not uniform for each point on the object. The velocity is in the direction of motion, tangent to the radius at that point.
- **Tangential acceleration** is given by $a_{tan} = r\alpha$, and **radial acceleration** is given by $a_R = \omega^2 r$. Radial acceleration at a point is proportional to its distance from the center.
- The number of revolutions per second, called the **frequency,** is given by $f = \omega/2\pi$. The unit of frequency is the **hertz,** equivalent to revolutions per second. Consequently, the time in seconds for one revolution, $T$, is the reciprocal of frequency, $T = 1/f$.

## Kinematic Equations for Uniformly Accelerated Rotational Motion

Just as the equations for angular velocity and angular acceleration correspond to equations for linear velocity and linear acceleration, other kinematics equations correspond to rotational motion with constant angular acceleration, in which angular velocity is not constant. For angular velocity,

- $\omega = \omega_0 + \alpha t$
- $\theta = \omega_0 t + 1/2\alpha t^2$
- $\omega^2 = \omega_0^2 + 2\alpha\theta$
- $\overline{\omega} = (\omega + \omega_0)/2$

where $\omega_0$ represents the initial angular velocity.

## Rolling Motion

The rolling motion of an object depends on friction between the object and a surface, so that the object does not slip. The object has the interrelated motions of translation corresponding to the linear speed, $v$, of the center and of rotation corresponding to the angular velocity at the radius of a rotating circular or spherical object, such that $v = r\omega$.

## Torque

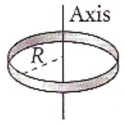
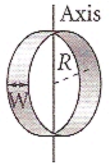
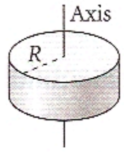
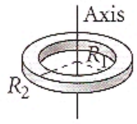
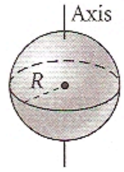
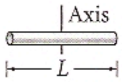
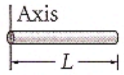
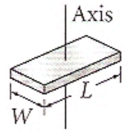
The forces involved in rotational motion are also analogous to those in linear kinematics. The perpendicular component of a force, $F_\perp$, applied to an object at a distance from the axis of rotation causes rotational motion to change.

- The distance from the axis of rotation, $r$, is called the **lever arm.** The perpendicular component of force applied to the lever arm is proportional to the angular acceleration it produces, and **torque** is defined as their product, $\tau = F_\perp \cdot r = F \cdot r_\perp$, whose units are N·m.
- The most efficient direction of force is perpendicular to a radius from the axis of rotation, and the least efficient direction is parallel to a radius from the axis of rotation.
- Torque is additive when force is applied in the same angular direction, and it is subtracted when applied in the opposite angular direction.
- Counterclockwise torque is, by convention, a positive quantity, whereas clockwise torque is considered a negative quantity.

## Rotational Dynamics; Torque and Rotational Inertia

The sum of torques, where each sign corresponds to a reference direction, is proportional to the angular acceleration. For a mass, $m$, rotating at a distance $r$, $\Sigma\tau = mr^2\alpha$.

- In this case, the value $mr^2$ is called the **moment of inertia,** which is the sum of the products of the mass of all particles in an object and the square of the distance $r$ at which they are rotating.
- Generally, $\Sigma\tau = I\alpha$, with $I$ representing the moment of inertia of a fixed rotating object, which depends on the size, shape, and density of an object with respect to the axis of rotation.
- Rotational inertia is proportional to the square of the distance of the concentration of mass from the axis of rotation.
- Calculating the moments of inertia for fixed bodies of uniform composition is outside the scope of the AP Physics-B exam. The moments of inertia for common objects are as follows:

| | Object | Location of axis | | Moment of inertia |
|---|---|---|---|---|
| (a) | Thin hoop of radius $R$ | Through center |   | $MR^2$ |
| (b) | Thin hoop of radius $R$ and width $W$ | Through central diameter |   | $\frac{1}{2}MR^2 + \frac{1}{12}MW^2$ |
| (c) | Solid cylinder of radius $R$ | Through center |   | $\frac{1}{2}MR^2$ |
| (d) | Hollow cylinder of inner radius $R_1$ and outer radius $R_2$ | Through center |   | $\frac{1}{2}M(R_1^2 + R_2^2)$ |
| (e) | Uniform sphere of radius $R$ | Through center |   | $\frac{2}{5}MR^2$ |
| (f) | Long uniform rod of length $L$ | Through center |   | $\frac{1}{12}ML^2$ |
| (g) | Long uniform rod of length $L$ | Through end |   | $\frac{1}{3}ML^2$ |
| (h) | Rectangular thin plate of length $L$ and width $W$ | Through center |   | $\frac{1}{12}M(L^2 + W^2)$ |

### Rotational Kinetic Energy

The kinetic energy of a rotating body is proportional to the moment of inertia, $I$, and the square of the angular velocity, $\omega$, $KE = 1/2(I\omega^2)$, whose units are in joules.

- The total kinetic energy of a body undergoing both translational and rotational motion is equal to the sum of the kinetic energy caused by each motion, **Total** $KE = 1/2(mv_{cm}^2) + 1/2(I_{cm}\omega^2)$, where $m$ is the mass of the object, $v_{cm}$ is the linear velocity of the object's center of mass, $I_{cm}$ is the moment of inertia, and $\omega$ is the angular velocity.
- Note that the friction required for rolling (rotational) motion is static, parallel to the surface, and that it does not figure into the energy equation.
- Work performed by torque over an angle is given by $W = \tau\Delta\theta$.

### Angular Momentum and Its Conservation

The **angular momentum** for a fixed body rotating around an axis is the product of the moment of inertia and the angular velocity, $L = I\omega$, with units in $kg \cdot m^2/s$.

- The rotational analog for Newton's second law of motion is given by $\Sigma\tau = I\alpha = \Delta L/\Delta t$, which equates torque with the rate of change of angular momentum.
- A consequence of this definition is the **conservation of angular momentum,** which states that the total angular momentum is conserved as long as there is no net torque acting on a rotating body. This suggests that on a frictionless surface, the angular velocity of a rotating object is inversely proportional to its moment of inertia.

### Vector Nature of Angular Quantities

Angular velocity, acceleration, and momentum are vector quantities whose direction is given by the **right-hand rule.** According to this rule, if the fingers of the right hand are wrapped around the axis of rotation in the direction of motion, the thumb points in the direction of the vector. The direction, like the axis of rotation, is perpendicular to the plane of motion.

### The Conditions for Equilibrium

A body at rest has a net zero vector force acting on it. A net zero force and a net zero torque are requirements for equilibrium.

- The **first condition for equilibrium** states that for a body at rest, $\Sigma F_x = 0$, $\Sigma F_y = 0$ and $\Sigma F_z = 0$. The signs of each component of force are relative to the reference frame.
- The **second condition for equilibrium** states that for a body at rest, $\Sigma\tau = 0$ about any axis.

*For Additional Review*

Derive the kinematic equations for constant angular acceleration from their corresponding linear analogs for a specific rotating body.

*Multiple-Choice Questions*

1. What is the angular velocity of a point on an object in uniform rotational motion that has a frequency of 45 revolutions per minute?
   (A) 0.75 rad/s
   (B) 1.5 rad/s
   (C) $2/3\pi$ rad/s
   (D) $1.5\pi$ rad/s
   (E) $90\pi$ rad/s

2. What is the magnitude of the total linear acceleration of a particle 12 cm from the axis of rotation and rotating with an angular velocity of 44 rad/s and an angular acceleration of 250 rad/s$^2$?
   (A) 35 m/s$^2$
   (B) 18 m/s$^2$
   (C) 223 m/s$^2$
   (D) 234 m/s$^2$
   (E) 512 m/s$^2$

3. What is the radial acceleration of an object at a point 25 m from the axis of rotation that has a radius and a period of 0.22 seconds?
   (A) 10,000 m/s$^2$
   (B) 20,000 m/s$^2$
   (C) 40,000 m/s$^2$
   (D) 160,000 m/s$^2$
   (E) 980,000 m/s$^2$

4. What is the ratio of angular velocities on a rotating body at a point half a radius away from its axis of rotation to a point a radius away?
   (A) 1:4
   (B) 1:2
   (C) 1:1
   (D) 2:1
   (E) 4:1

5. The units of hertz are equivalent to
   I. Radian/second
   II. Revolutions/second
   III. $s^{-1}$
   (A) I
   (B) II
   (C) III
   (D) I and II
   (E) II and III

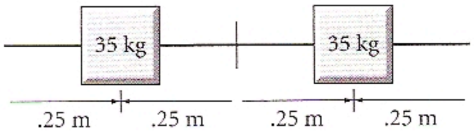
6. A point on a rotating disc has a frequency of 120 Hz. What is the angular acceleration of the point for it to be moving at 30 Hz in 1 minute?
   (A) $-1.5$ rad/s$^2$
   (B) $-2\pi/3$ rad/s$^2$
   (C) $-3\pi/2$ rad/s$^2$
   (D) $-3\pi$ rad/s$^2$
   (E) $-6$ rad/s$^2$

7. A spherical marble of radius 3 cm rolls from rest with an angular acceleration of 4 rad/s$^2$. How long will it take for it to roll 100 revolutions?
   (A) 3 s
   (B) 7 s
   (C) 9 s
   (D) 14 s
   (E) 18 s

8. What is the moment of inertia for four masses, located in the $(x, y)$ plane at (2 m, 0 m), ($-2$ m, 0 m), (0 m, $-2$ m), (0 m, 2 m), each of 15 kg, uniformly rotating around the origin?
   (A) 15 kg·m$^2$
   (B) 30 kg·m$^2$
   (C) 60 kg·m$^2$
   (D) 120 kg·m$^2$
   (E) 240 kg·m$^2$

9. For a pulley of radius 76 cm with a moment of inertia of 46.5 kg m$^2$, a rope wrapped around it is pulled with a force of 15 N. What is the magnitude of angular acceleration?
   (A) 0.013 rad/s$^2$
   (B) 0.245 rad/s$^2$
   (C) 4.08 rad/s$^2$
   (D) 11.4 rad/s$^2$
   (E) 530 rad/s$^2$

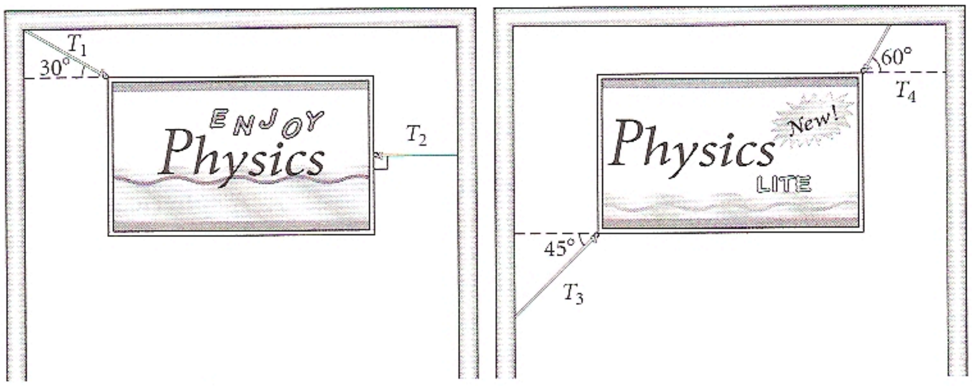
10. Two 35 kg weights are each initially halfway from the center to the endpoints of a massless 1-meter rod as shown, rotating around the rod's center at 12 rad/s. If the weights shift to the endpoints of the rod, what is the new angular velocity?




   (A) 3 rad/s
   (B) 6 rad/s
   (C) 9 rad/s
   (D) 12 rad/s
   (E) 15 rad/s

### *Free-Response Questions*

1. Two identical signs, each of mass $M$, are held aloft in adjacent archways. Following a large gust of wind, several ropes holding the signs snap, leaving only those shown below. If each sign is in translational equilibrium, find the tensions of the remaining ropes in terms of its mass and gravity, $g$.




2. A 13 kg solid cylinder with a 54 cm diameter rolls without slipping down a 30° incline from a height of 1.25 meters.
   (a) If a solid cylinder has a moment of inertia, $I = 1/2(MR^2)$, what will its speed be at the base of the incline?
   (b) What will its speed be if it rolls from a height of 1.25 meters down a 60° incline, and how do you account for this?

## ANSWERS AND EXPLANATIONS

### *Multiple-Choice Questions*

1. **(D) is correct.** The frequency is given by 45 revolutions per minute = (45 rev/min)(1 min/60 s) = 0.75 rev/s. This can be applied to angular velocity, where $\omega = 2\pi f = 2\pi(0.75) = 1.5\pi$ rad/s.

2. **(D) is correct.** Radial acceleration is given by $a_R = \omega^2 r = (44 \text{ rad/s})^2 (0.12 \text{ m}) = 232 \text{ m/s}^2$, and tangential acceleration is given by $a_{\tan} = r\alpha = (0.12 \text{ m})(250 \text{ rad/s}^2) = 30 \text{ m/s}^2$. The magnitude of total linear acceleration is given by $a = \sqrt{(a_{\tan}^2 + a_R^2)} = \sqrt{(232 \text{ m/s}^2)^2 + (30 \text{ m/s}^2)^2} = 234 \text{ m/s}^2$.

3. **(B) is correct.** If $T = 0.22$ seconds/revolution, the frequency is $f = 4.5$ rev/s. Radial acceleration, $a_R$ is given by $a_R = \omega^2 r$, where $\omega = 2\pi f$, so $a_R = (2\pi f)^2 r = [2\pi(4.5)]^2 (25 \text{ m}) = 20{,}000 \text{ m/s}^2$.

4. **(C) is correct.** All points on a rotating body have the same angular velocity. Thus both points have the same angular velocity, and the ratio is 1:1.

5. **(E) is correct.** II and III are equivalent units, whereas 1 hertz $= 2\pi$ radians/second, which are closely related but not equivalent.

6. **(D) is correct.** Frequency is given by $f = \omega/2\pi$, $(120 \text{ Hz})(2\pi) = 240\pi \text{ rad/s} = \omega_i$ and $(30 \text{ Hz})(2\pi) = 60\pi \text{ rad/s} = w_f$. From $\omega = \omega_0 + \alpha t$, $60\pi \text{ rad/s} = 240\pi \text{ rad/s} + \alpha(60 \text{ s})$, $-180\pi \text{ rad/s}/60 \text{ s} = \alpha = -3\pi \text{ rad/s}^2$.

7. **(E) is correct.** 100 revolutions means turning $200\pi$ radians. From $\theta = \omega_0 t + 1/2(\alpha t^2)$, $200\pi = 0t + 1/2[(4 \text{ rad/s})t^2]$, where $\omega_0 = 0$, since the marble is accelerated from rest. $t^2 = 100\pi = 18$ seconds.

8. **(E) is correct.** Moment of inertia must take into account each of the masses at its distance from the axis of rotation. So $I = \Sigma mr^2 = 15 \text{ kg } (2 \text{ m})^2 + 15 \text{ kg } (2 \text{ m})^2 + 15 \text{ kg } (2 \text{ m})^2 + 15 \text{ kg } (2 \text{ m})^2 = 4(15 \text{ kg}) (2 \text{ m})^2 = 240 \text{ kg·m}^2$.

9. **(B) is correct.** The relationship between moment of inertia and angular acceleration is given by $I\alpha = \tau$. To find torque, $\tau = F_\perp \cdot r = 15 \text{ N } (0.76 \text{ m}) = 11.4 \text{ N·m}$. Next, $\alpha = \tau/I = 11.4 \text{ N·m}/46.5 \text{ kg·m}^2 = 0.245 \text{ rad/s}^2$.

10. **(A) is correct.** From the conservation of linear momentum, $I_i\omega_i = I_f\omega_f$, where the initial angular velocity is 12 rad/s, the initial moment of inertia, $I_i$, is $I_i = \Sigma mr^2 = m_1 r_1^2 + m_2 r_2^2 = (35 \text{ kg})(0.25 \text{ m})^2 + (35 \text{ kg})(0.25 \text{ m})^2 = 4.375 \text{ kg·m}^2$, and the final moment of inertia, $I_f$, is $I_f = \Sigma mr^2 = m_1 r_1^2 + m^2 r_2^2 = (35 \text{ kg})(0.5 \text{ m})^2 + (35 \text{ kg})(0.5 \text{ m})^2 = 17.5 \text{ kg·m}^2$. Thus $\omega_f = I_i\omega_i/I_f = (4.375 \text{ kg·m}^2)(12 \text{ rad/s})/(17.5 \text{ kg·m}^2) = 3 \text{ rad/s}$.

### *Free-Response Questions*

1. For signs to be in translational equilibrium, each of the perpendicular force components must sum to zero, that is $\Sigma F_x = 0$ and $\Sigma F_y = 0$. For the first sign, we can separate the forces into components, so $\Sigma F_x = T_2 - T_1 \cos 30° = 0$ and $\Sigma F_y = T_1 \sin 30° - Mg = 0$. Using the latter, $T_1 = Mg/\sin 30° = 2\ Mg$. This can be applied to the former equation, so $T_2 = T_1 \cos 30° = 2\ Mg\ (\sqrt{3}/2) = \sqrt{3}\ Mg$. For the second sign, using the same procedure, $\Sigma F_x = T_4 \cos 60° - T_3 \cos 45° = 0$ and $\Sigma F_y = T_4 \sin 60° - T_3 \sin 45° - Mg = 0$. Using substitution to solve for two equations with two unknowns, one variable is solved for in terms of the other, so $T_4 = T_3 \cos 45°/\cos 60° = T_3\sqrt{2}$. This can be applied to the second equation, so $T_3\sqrt{2}\sqrt{3}/2 - T_3\sqrt{2}/2 = Mg$. Thus,
$T_3 = [\sqrt{2}\ Mg\,(\sqrt{3} + 1)]/2$ and $T_4 = Mg\,(\sqrt{3} + 1)$.

*This response correctly resolves the force vectors into perpendicular components so that the translational equilibrium definition can be applied. Note that the* *answers for $T_3$ and $T_4$ could be written in several equivalent forms. Here the denominators have been rationalized, and the answers have been put in simplest factored form.*

2. (a) This question makes use of conservation of energy. At the top of the incline, all energy is potential, whereas at the base of the incline, all is translational and rotational kinetic. The diameter (0.54 m) was given rather than the radius (0.27 m), so $U_i = KE_f$. $mgh = 1/2(mv_{cm}^2) + 1/2(I_{cm}\omega^2)$, where $\omega = v/0.27$ because the cylinder rolls without slipping. This becomes $(13 \text{ kg})(9.8 \text{ m/s}^2)(1.25 \text{ m}) =$
   $1/2(13 \text{ kg}\cdot v_{cm}^2) + 1/2(1/2)(13 \text{ kg})(0.27 \text{ m})^2(v/0.27 \text{ m})^2)$.
   So, $159 \text{ J} = 6.5 \text{ kg}\cdot v_{cm}^2 + 3.25 \text{ kg}\cdot v_{cm}^2 = 9.75 \text{ kg}\cdot v_{cm}^2$
   and $v_{cm}^2 = 16.31 \text{ m}^2/\text{s}^2$ and $4.0 \text{ m/s} = v$.

   (b) Its speed would be the same as in the previous item, because both sides of the energy equation depend only on the height and are independent of the angle of the incline.

*This response correctly analyzes the problem using conservation of energy. For the response to part a, the relationship between angular velocity and linear velocity, $\omega = v/r$, is used to reduce the single equation to the one variable being sought. For the response to part b, the independence of the energy equation from the angle is identified as the reason that parts a and b have the same answer.*