PUZZLER

Airbags have saved countless lives by reducing the forces exerted on vehicle occupants during collisions. How can airbags change the force needed to bring a person from a high speed to a complete stop? Why are they usually safer than seat belts alone? *(Courtesy of Saab)*

chapter 9

Linear Momentum and Collisions

Chapter Outline

Consider what happens when a golf ball is struck by a club. The ball is given a very large initial velocity as a result of the collision; consequently, it is able to travel more than 100 m through the air. The ball experiences a large acceleration. Furthermore, because the ball experiences this acceleration over a very short time interval, the average force exerted on it during the collision is very great. According to Newton's third law, the ball exerts on the club a reaction force that is equal in magnitude to and opposite in direction to the force exerted by the club on the ball. This reaction force causes the club to accelerate. Because the club is much more massive than the ball, however, the acceleration of the club is much less than the acceleration of the ball.

One of the main objectives of this chapter is to enable you to understand and analyze such events. As a first step, we introduce the concept of *momentum,* which is useful for describing objects in motion and as an alternate and more general means of applying Newton's laws. For example, a very massive football player is often said to have a great deal of momentum as he runs down the field. A much less massive player, such as a halfback, can have equal or greater momentum if his speed is greater than that of the more massive player. This follows from the fact that momentum is defined as the product of mass and velocity. The concept of momentum leads us to a second conservation law, that of conservation of momentum. This law is especially useful for treating problems that involve collisions between objects and for analyzing rocket propulsion. The concept of the center of mass of a system of particles also is introduced, and we shall see that the motion of a system of particles can be described by the motion of one representative particle located at the center of mass.

9.1 LINEAR MOMENTUM AND ITS CONSERVATION

In the preceding two chapters we studied situations too complex to analyze easily with Newton's laws. In fact, Newton himself used a form of his second law slightly different from $\Sigma\mathbf{F} = m\mathbf{a}$ (Eq. 5.2)—a form that is considerably easier to apply in complicated circumstances. Physicists use this form to study everything from subatomic particles to rocket propulsion. In studying situations such as these, it is often useful to know both something about the object and something about its motion. We start by defining a new term that incorporates this information:

Definition of linear momentum of a particle

The **linear momentum** of a particle of mass m moving with a velocity $\mathbf{v}$ is defined to be the product of the mass and velocity:

$$\mathbf{p} \equiv m\mathbf{v} \tag{9.1}$$

6.2

Linear momentum is a vector quantity because it equals the product of a scalar quantity m and a vector quantity $\mathbf{v}$. Its direction is along $\mathbf{v}$, it has dimensions ML/T, and its SI unit is kg · m/s.

If a particle is moving in an arbitrary direction, $\mathbf{p}$ must have three components, and Equation 9.1 is equivalent to the component equations

$$p_x = mv_x \qquad p_y = mv_y \qquad p_z = mv_z \tag{9.2}$$

As you can see from its definition, the concept of momentum provides a quantitative distinction between heavy and light particles moving at the same velocity. For example, the momentum of a bowling ball moving at 10 m/s is much greater than that of a tennis ball moving at the same speed. Newton called the product $m\mathbf{v}$

quantity of motion; this is perhaps a more graphic description than our present-day word *momentum,* which comes from the Latin word for movement.

Quick Quiz 9.1

Two objects have equal kinetic energies. How do the magnitudes of their momenta compare? (a) $p_1 < p_2$, (b) $p_1 = p_2$, (c) $p_1 > p_2$, (d) not enough information to tell.

Newton's second law for a particle

Using Newton's second law of motion, we can relate the linear momentum of a particle to the resultant force acting on the particle: **The time rate of change of the linear momentum of a particle is equal to the net force acting on the particle:**

$$\sum \mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d(m\mathbf{v})}{dt} \tag{9.3}$$

In addition to situations in which the velocity vector varies with time, we can use Equation 9.3 to study phenomena in which the mass changes. The real value of Equation 9.3 as a tool for analysis, however, stems from the fact that when the net force acting on a particle is zero, the time derivative of the momentum of the particle is zero, and therefore its linear momentum[1] is constant. Of course, if the particle is *isolated,* then by necessity $\Sigma \mathbf{F} = 0$ and $\mathbf{p}$ remains unchanged. This means that $\mathbf{p}$ is conserved. Just as the law of conservation of energy is useful in solving complex motion problems, the law of conservation of momentum can greatly simplify the analysis of other types of complicated motion.

Conservation of Momentum for a Two-Particle System

6.2 Consider two particles 1 and 2 that can interact with each other but are isolated from their surroundings (Fig. 9.1). That is, the particles may exert a force on each other, but no external forces are present. It is important to note the impact of Newton's third law on this analysis. If an internal force from particle 1 (for example, a gravitational force) acts on particle 2, then there must be a second internal force—equal in magnitude but opposite in direction—that particle 2 exerts on particle 1.

Suppose that at some instant, the momentum of particle 1 is $\mathbf{p}_1$ and that of particle 2 is $\mathbf{p}_2$. Applying Newton's second law to each particle, we can write

$$\mathbf{F}_{21} = \frac{d\mathbf{p}_1}{dt} \qquad \text{and} \qquad \mathbf{F}_{12} = \frac{d\mathbf{p}_2}{dt}$$

where $\mathbf{F}_{21}$ is the force exerted by particle 2 on particle 1 and $\mathbf{F}_{12}$ is the force exerted by particle 1 on particle 2. Newton's third law tells us that $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$ are equal in magnitude and opposite in direction. That is, they form an action–reaction pair $\mathbf{F}_{12} = -\mathbf{F}_{21}$. We can express this condition as

$$\mathbf{F}_{21} + \mathbf{F}_{12} = 0$$

or as

$$\frac{d\mathbf{p}_1}{dt} + \frac{d\mathbf{p}_2}{dt} = \frac{d}{dt}(\mathbf{p}_1 + \mathbf{p}_2) = 0$$

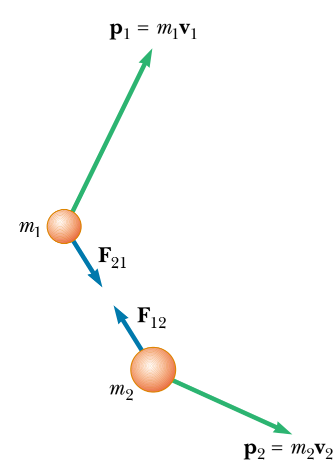


Figure 9.1 At some instant, the momentum of particle 1 is $\mathbf{p}_1 = m_1\mathbf{v}_1$ and the momentum of particle 2 is $\mathbf{p}_2 = m_2\mathbf{v}_2$. Note that $\mathbf{F}_{12} = -\mathbf{F}_{21}$. The total momentum of the system $\mathbf{p}_{\text{tot}}$ is equal to the vector sum $\mathbf{p}_1 + \mathbf{p}_2$.

[1] In this chapter, the terms *momentum* and *linear momentum* have the same meaning. Later, in Chapter 11, we shall use the term *angular momentum* when dealing with rotational motion.

Because the time derivative of the total momentum $\mathbf{p}_{\text{tot}} = \mathbf{p}_1 + \mathbf{p}_2$ is *zero,* we conclude that the *total* momentum of the system must remain constant:

$$\mathbf{p}_{\text{tot}} = \sum_{\text{system}} \mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2 = \text{constant} \tag{9.4}$$

or, equivalently,

$$\mathbf{p}_{1i} + \mathbf{p}_{2i} = \mathbf{p}_{1f} + \mathbf{p}_{2f} \tag{9.5}$$

where $\mathbf{p}_{1i}$ and $\mathbf{p}_{2i}$ are the initial values and $\mathbf{p}_{1f}$ and $\mathbf{p}_{2f}$ the final values of the momentum during the time interval dt over which the reaction pair interacts. Equation 9.5 in component form demonstrates that the total momenta in the x, y, and z directions are all independently conserved:

$$\sum_{\text{system}} p_{ix} = \sum_{\text{system}} p_{fx} \qquad \sum_{\text{system}} p_{iy} = \sum_{\text{system}} p_{fy} \qquad \sum_{\text{system}} p_{iz} = \sum_{\text{system}} p_{fz} \tag{9.6}$$

This result, known as the **law of conservation of linear momentum,** can be extended to any number of particles in an isolated system. It is considered one of the most important laws of mechanics. We can state it as follows:

Conservation of momentum

Whenever two or more particles in an isolated system interact, the total momentum of the system remains constant.

This law tells us that **the total momentum of an isolated system at all times equals its initial momentum.**

Notice that we have made no statement concerning the nature of the forces acting on the particles of the system. The only requirement is that the forces must be *internal* to the system.

Quick Quiz 9.2

Your physical education teacher throws a baseball to you at a certain speed, and you catch it. The teacher is next going to throw you a medicine ball whose mass is ten times the mass of the baseball. You are given the following choices: You can have the medicine ball thrown with (a) the same speed as the baseball, (b) the same momentum, or (c) the same kinetic energy. Rank these choices from easiest to hardest to catch.

Example 9.1 The Floating Astronaut

A SkyLab astronaut discovered that while concentrating on writing some notes, he had gradually floated to the middle of an open area in the spacecraft. Not wanting to wait until he floated to the opposite side, he asked his colleagues for a push. Laughing at his predicament, they decided not to help, and so he had to take off his uniform and throw it in one direction so that he would be propelled in the opposite direction. Estimate his resulting velocity.

Figure 9.2 A hapless astronaut has discarded his uniform to get somewhere.

Solution We begin by making some reasonable guesses of relevant data. Let us assume we have a 70-kg astronaut who threw his 1-kg uniform at a speed of 20 m/s. For convenience, we set the positive direction of the x axis to be the direction of the throw (Fig. 9.2). Let us also assume that the x axis is tangent to the circular path of the spacecraft.

We take the system to consist of the astronaut and the uniform. Because of the gravitational force (which keeps the astronaut, his uniform, and the entire spacecraft in orbit), the system is not really isolated. However, this force is directed perpendicular to the motion of the system. Therefore, momentum is constant in the x direction because there are no external forces in this direction.

The total momentum of the system before the throw is zero ($m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i} = 0$). Therefore, the total momentum after the throw must be zero; that is,

$$m_1\mathbf{v}_{1f} + m_2\mathbf{v}_{2f} = 0$$

With $m_1 = 70$ kg, $\mathbf{v}_{2f} = 20\mathbf{i}$ m/s, and $m_2 = 1$ kg, solving for $\mathbf{v}_{1f}$, we find the recoil velocity of the astronaut to be

$$\mathbf{v}_{1f} = -\frac{m_2}{m_1}\mathbf{v}_{2f} = -\left(\frac{1 \text{ kg}}{70 \text{ kg}}\right)(20\mathbf{i} \text{ m/s}) = \boxed{-0.3\mathbf{i} \text{ m/s}}$$

The negative sign for $\mathbf{v}_{1f}$ indicates that the astronaut is moving to the left after the throw, in the direction opposite the direction of motion of the uniform, in accordance with Newton's third law. Because the astronaut is much more massive than his uniform, his acceleration and consequent velocity are much smaller than the acceleration and velocity of the uniform.

EXAMPLE 9.2 Breakup of a Kaon at Rest

One type of nuclear particle, called the *neutral kaon* (K^0), breaks up into a pair of other particles called *pions* (π^+ and π^-) that are oppositely charged but equal in mass, as illustrated in Figure 9.3. Assuming the kaon is initially at rest, prove that the two pions must have momenta that are equal in magnitude and opposite in direction.

Solution The breakup of the kaon can be written

$$K^0 \longrightarrow \pi^+ + \pi^-$$

If we let $\mathbf{p}^+$ be the momentum of the positive pion and $\mathbf{p}^-$ the momentum of the negative pion, the final momentum of the system consisting of the two pions can be written

$$\mathbf{p}_f = \mathbf{p}^+ + \mathbf{p}^-$$

Because the kaon is at rest before the breakup, we know that $\mathbf{p}_i = 0$. Because momentum is conserved, $\mathbf{p}_i = \mathbf{p}_f = 0$, so that $\mathbf{p}^+ + \mathbf{p}^- = 0$, or

$$\mathbf{p}^+ = -\mathbf{p}^-$$

The important point behind this problem is that even though it deals with objects that are very different from those in the preceding example, the physics is identical: Linear momentum is conserved in an isolated system.

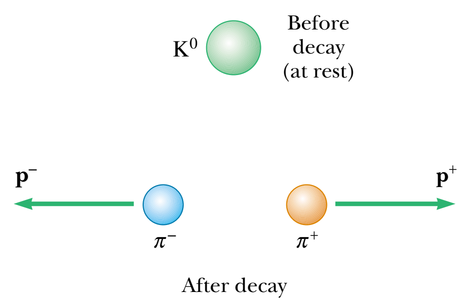


Figure 9.3 A kaon at rest breaks up spontaneously into a pair of oppositely charged pions. The pions move apart with momenta that are equal in magnitude but opposite in direction.

9.2 IMPULSE AND MOMENTUM

6.3 & 6.4

As we have seen, the momentum of a particle changes if a net force acts on the particle. Knowing the change in momentum caused by a force is useful in solving some types of problems. To begin building a better understanding of this important concept, let us assume that a single force $\mathbf{F}$ acts on a particle and that this force may vary with time. According to Newton's second law, $\mathbf{F} = d\mathbf{p}/dt$, or

$$d\mathbf{p} = \mathbf{F}\,dt \qquad \textbf{(9.7)}$$

We can integrate[2] this expression to find the change in the momentum of a particle when the force acts over some time interval. If the momentum of the particle

[2]Note that here we are integrating force with respect to time. Compare this with our efforts in Chapter 7, where we integrated force with respect to position to express the work done by the force.

changes from $\mathbf{p}_i$ at time t_i to $\mathbf{p}_f$ at time t_f, integrating Equation 9.7 gives

$$\Delta\mathbf{p} = \mathbf{p}_f - \mathbf{p}_i = \int_{t_i}^{t_f} \mathbf{F}\, dt \tag{9.8}$$

To evaluate the integral, we need to know how the force varies with time. The quantity on the right side of this equation is called the **impulse** of the force **F** acting on a particle over the time interval $\Delta t = t_f - t_i$. Impulse is a vector defined by

Impulse of a force

$$\mathbf{I} \equiv \int_{t_i}^{t_f} \mathbf{F}\, dt = \Delta\mathbf{p} \tag{9.9}$$

Impulse–momentum theorem

The impulse of the force **F** acting on a particle equals the change in the momentum of the particle caused by that force.

This statement, known as the **impulse–momentum theorem,**[3] is equivalent to Newton's second law. From this definition, we see that impulse is a vector quantity having a magnitude equal to the area under the force–time curve, as described in Figure 9.4a. In this figure, it is assumed that the force varies in time in the general manner shown and is nonzero in the time interval $\Delta t = t_f - t_i$. The direction of the impulse vector is the same as the direction of the change in momentum. Impulse has the dimensions of momentum—that is, ML/T. Note that impulse is *not* a property of a particle; rather, it is a measure of the degree to which an external force changes the momentum of the particle. Therefore, when we say that an impulse is given to a particle, we mean that momentum is transferred from an external agent to that particle.

Because the force imparting an impulse can generally vary in time, it is convenient to define a time-averaged force

$$\overline{\mathbf{F}} \equiv \frac{1}{\Delta t}\int_{t_i}^{t_f} \mathbf{F}\, dt \tag{9.10}$$

where $\Delta t = t_f - t_i$. (This is an application of the mean value theorem of calculus.) Therefore, we can express Equation 9.9 as

$$\mathbf{I} \equiv \overline{\mathbf{F}}\,\Delta t \tag{9.11}$$

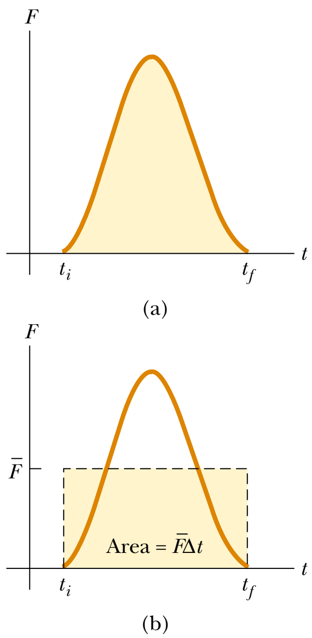


Figure 9.4 (a) A force acting on a particle may vary in time. The impulse imparted to the particle by the force is the area under the force versus time curve. (b) In the time interval Δt, the time-averaged force (horizontal dashed line) gives the same impulse to a particle as does the time-varying force described in part (a).

This time-averaged force, described in Figure 9.4b, can be thought of as the constant force that would give to the particle in the time interval Δt the same impulse that the time-varying force gives over this same interval.

In principle, if **F** is known as a function of time, the impulse can be calculated from Equation 9.9. The calculation becomes especially simple if the force acting on the particle is constant. In this case, $\overline{\mathbf{F}} = \mathbf{F}$ and Equation 9.11 becomes

$$\mathbf{I} = \mathbf{F}\,\Delta t \tag{9.12}$$

In many physical situations, we shall use what is called the **impulse approximation, in which we assume that one of the forces exerted on a particle acts for a short time but is much greater than any other force present.** This approximation is especially useful in treating collisions in which the duration of the

[3]Although we assumed that only a single force acts on the particle, the impulse–momentum theorem is valid when several forces act; in this case, we replace **F** in Equation 9.9 with $\Sigma\mathbf{F}$.

During the brief time the club is in contact with the ball, the ball gains momentum as a result of the collision, and the club loses the same amount of momentum.

collision is very short. When this approximation is made, we refer to the force as an *impulsive force.* For example, when a baseball is struck with a bat, the time of the collision is about 0.01 s and the average force that the bat exerts on the ball in this time is typically several thousand newtons. Because this is much greater than the magnitude of the gravitational force, the impulse approximation justifies our ignoring the weight of the ball and bat. When we use this approximation, it is important to remember that $\mathbf{p}_i$ and $\mathbf{p}_f$ represent the momenta *immediately* before and after the collision, respectively. Therefore, in any situation in which it is proper to use the impulse approximation, the particle moves very little during the collision.

QuickLab

If you can find someone willing, play catch with an egg. What is the best way to move your hands so that the egg does not break when you change its momentum to zero?

Quick Quiz 9.3

Two objects are at rest on a frictionless surface. Object 1 has a greater mass than object 2. When a force is applied to object 1, it accelerates through a distance d. The force is removed from object 1 and is applied to object 2. At the moment when object 2 has accelerated through the same distance d, which statements are true? (a) $p_1 < p_2$, (b) $p_1 = p_2$, (c) $p_1 > p_2$, (d) $K_1 < K_2$, (e) $K_1 = K_2$, (f) $K_1 > K_2$.

EXAMPLE 9.3 Teeing Off

A golf ball of mass 50 g is struck with a club (Fig. 9.5). The force exerted on the ball by the club varies from zero, at the instant before contact, up to some maximum value (at which the ball is deformed) and then back to zero when the ball leaves the club. Thus, the force–time curve is qualitatively described by Figure 9.4. Assuming that the ball travels 200 m, estimate the magnitude of the impulse caused by the collision.

Solution Let us use Ⓐ to denote the moment when the club first contacts the ball, Ⓑ to denote the moment when the club loses contact with the ball as the ball starts on its trajectory, and Ⓒ to denote its landing. Neglecting air resistance, we can use Equation 4.14 for the range of a projectile:

$$R = x_{\mathrm{C}} = \frac{{v_{\mathrm{B}}}^2}{g} \sin 2\theta_{\mathrm{B}}$$

Let us assume that the launch angle θ_{B} is 45°, the angle that provides the maximum range for any given launch velocity. This assumption gives $\sin 2\theta_{\mathrm{B}} = 1$, and the launch velocity of the ball is

$$v_B = \sqrt{x_C g} = \sqrt{(200\text{ m})(9.80\text{ m/s}^2)} = 44\text{ m/s}$$

Considering the time interval for the collision, $v_i = v_A = 0$ and $v_f = v_B$ for the ball. Hence, the magnitude of the impulse imparted to the ball is

$$I = \Delta p = mv_B - mv_A = (50 \times 10^{-3}\text{ kg})(44\text{ m/s}) - 0$$
$$= \boxed{2.2\text{ kg}\cdot\text{m/s}}$$

Exercise If the club is in contact with the ball for a time of 4.5×10^{-4} s, estimate the magnitude of the average force exerted by the club on the ball.

Answer 4.9×10^3 N, a value that is extremely large when compared with the weight of the ball, 0.49 N.

Figure 9.5 A golf ball being struck by a club. *(© Harold E. Edgerton/Courtesy of Palm Press, Inc.)*

EXAMPLE 9.4 How Good Are the Bumpers?

In a particular crash test, an automobile of mass 1 500 kg collides with a wall, as shown in Figure 9.6. The initial and final velocities of the automobile are $\mathbf{v}_i = -15.0\mathbf{i}$ m/s and $\mathbf{v}_f = 2.60\mathbf{i}$ m/s, respectively. If the collision lasts for 0.150 s, find the impulse caused by the collision and the average force exerted on the automobile.

Solution Let us assume that the force exerted on the car by the wall is large compared with other forces on the car so that we can apply the impulse approximation. Furthermore, we note that the force of gravity and the normal force exerted by the road on the car are perpendicular to the motion and therefore do not affect the horizontal momentum.

The initial and final momenta of the automobile are

$$\mathbf{p}_i = m\mathbf{v}_i = (1\,500\text{ kg})(-15.0\mathbf{i}\text{ m/s}) = -2.25 \times 10^4\mathbf{i}\text{ kg}\cdot\text{m/s}$$
$$\mathbf{p}_f = m\mathbf{v}_f = (1\,500\text{ kg})(2.60\mathbf{i}\text{ m/s}) = 0.39 \times 10^4\mathbf{i}\text{ kg}\cdot\text{m/s}$$

Hence, the impulse is

$$\mathbf{I} = \Delta\mathbf{p} = \mathbf{p}_f - \mathbf{p}_i = 0.39 \times 10^4\mathbf{i}\text{ kg}\cdot\text{m/s} - (-2.25 \times 10^4\mathbf{i}\text{ kg}\cdot\text{m/s})$$

$$\mathbf{I} = \boxed{2.64 \times 10^4\mathbf{i}\text{ kg}\cdot\text{m/s}}$$

The average force exerted on the automobile is

$$\overline{\mathbf{F}} = \frac{\Delta\mathbf{p}}{\Delta t} = \frac{2.64 \times 10^4\mathbf{i}\text{ kg}\cdot\text{m/s}}{0.150\text{ s}} = \boxed{1.76 \times 10^5\mathbf{i}\text{ N}}$$

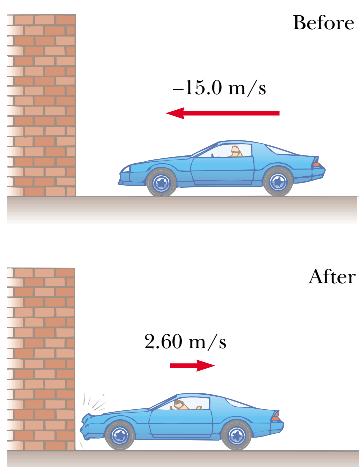


Figure 9.6 (a) This car's momentum changes as a result of its collision with the wall. (b) In a crash test, much of the car's initial kinetic energy is transformed into energy used to damage the car.

Note that the magnitude of this force is large compared with the weight of the car ($mg = 1.47 \times 10^4$ N), which justifies our initial assumption. Of note in this problem is how the signs of the velocities indicated the reversal of directions. What would the mathematics be describing if both the initial and final velocities had the same sign?

Quick Quiz 9.4

Rank an automobile dashboard, seatbelt, and airbag in terms of (a) the impulse and (b) the average force they deliver to a front-seat passenger during a collision.

9.3 COLLISIONS

6.5 & 6.6 In this section we use the law of conservation of linear momentum to describe what happens when two particles collide. We use the term **collision** to represent the event of two particles' coming together for a short time and thereby producing impulsive forces on each other. **These forces are assumed to be much greater than any external forces present.**

A collision may entail physical contact between two macroscopic objects, as described in Figure 9.7a, but the notion of what we mean by collision must be generalized because "physical contact" on a submicroscopic scale is ill-defined and hence meaningless. To understand this, consider a collision on an atomic scale (Fig. 9.7b), such as the collision of a proton with an alpha particle (the nucleus of a helium atom). Because the particles are both positively charged, they never come into physical contact with each other; instead, they repel each other because of the strong electrostatic force between them at close separations. When two particles 1 and 2 of masses m_1 and m_2 collide as shown in Figure 9.7, the impulsive forces may vary in time in complicated ways, one of which is described in Figure 9.8. If $\mathbf{F}_{21}$ is the force exerted by particle 2 on particle 1, and if we assume that no external forces act on the particles, then the change in momentum of particle 1 due to the collision is given by Equation 9.8:

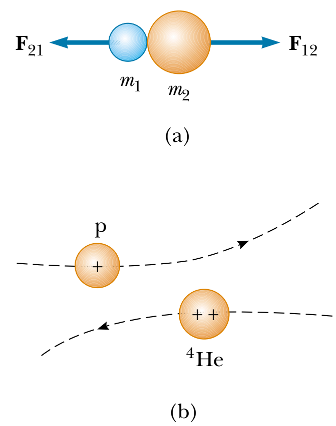


Figure 9.7 (a) The collision between two objects as the result of direct contact. (b) The "collision" between two charged particles.

$$\Delta \mathbf{p}_1 = \int_{t_i}^{t_f} \mathbf{F}_{21}\, dt$$

Likewise, if $\mathbf{F}_{12}$ is the force exerted by particle 1 on particle 2, then the change in momentum of particle 2 is

$$\Delta \mathbf{p}_2 = \int_{t_i}^{t_f} \mathbf{F}_{12}\, dt$$

From Newton's third law, we conclude that

$$\Delta \mathbf{p}_1 = -\Delta \mathbf{p}_2$$

$$\Delta \mathbf{p}_1 + \Delta \mathbf{p}_2 = 0$$

Because the total momentum of the system is $\mathbf{p}_{\text{system}} = \mathbf{p}_1 + \mathbf{p}_2$, we conclude that the *change* in the momentum of the system due to the collision is zero:

$$\mathbf{p}_{\text{system}} = \mathbf{p}_1 + \mathbf{p}_2 = \text{constant}$$

This is precisely what we expect because no external forces are acting on the system (see Section 9.2). Because the impulsive forces are internal, they do not change the total momentum of the system (only external forces can do that).

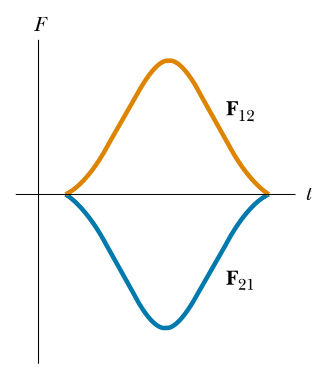


Figure 9.8 The impulse force as a function of time for the two colliding particles described in Figure 9.7a. Note that $\mathbf{F}_{12} = -\mathbf{F}_{21}$.

Momentum is conserved for any collision

Therefore, we conclude that **the total momentum of an isolated system just before a collision equals the total momentum of the system just after the collision.**

EXAMPLE 9.5 Carry Collision Insurance!

A car of mass 1800 kg stopped at a traffic light is struck from the rear by a 900-kg car, and the two become entangled. If the smaller car was moving at 20.0 m/s before the collision, what is the velocity of the entangled cars after the collision?

Solution We can guess that the final speed is less than 20.0 m/s, the initial speed of the smaller car. The total momentum of the system (the two cars) before the collision must equal the total momentum immediately after the collision because momentum is conserved in any type of collision. The magnitude of the total momentum before the collision is equal to that of the smaller car because the larger car is initially at rest:

$$p_i = m_1 v_{1i} = (900 \text{ kg})(20.0 \text{ m/s}) = 1.80 \times 10^4 \text{ kg}\cdot\text{m/s}$$

After the collision, the magnitude of the momentum of the entangled cars is

$$p_f = (m_1 + m_2)v_f = (2\,700 \text{ kg})v_f$$

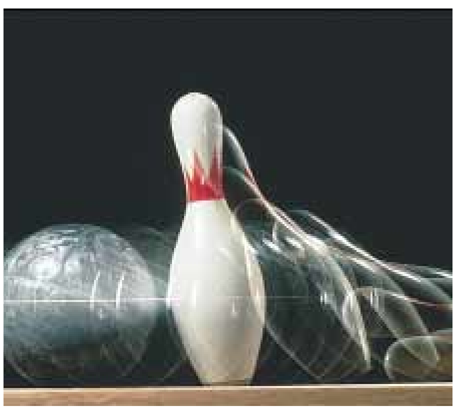
Equating the momentum before to the momentum after and solving for v_f, the final velocity of the entangled cars, we have

$$v_f = \frac{p_i}{m_1 + m_2} = \frac{1.80 \times 10^4 \text{ kg}\cdot\text{m/s}}{2\,700 \text{ kg}} = \boxed{6.67 \text{ m/s}}$$

The direction of the final velocity is the same as the velocity of the initially moving car.

Exercise What would be the final speed if the two cars each had a mass of 900 kg?

Answer 10.0 m/s.



When the bowling ball and pin collide, part of the ball's momentum is transferred to the pin. Consequently, the pin acquires momentum and kinetic energy, and the ball loses momentum and kinetic energy. However, the total momentum of the system (ball and pin) remains constant.

Quick Quiz 9.5

As a ball falls toward the Earth, the ball's momentum increases because its speed increases. Does this mean that momentum is not conserved in this situation?

Quick Quiz 9.6

A skater is using very low-friction rollerblades. A friend throws a Frisbee straight at her. In which case does the Frisbee impart the greatest impulse to the skater: (a) she catches the Frisbee and holds it, (b) she catches it momentarily but drops it, (c) she catches it and at once throws it back to her friend?

9.4 ELASTIC AND INELASTIC COLLISIONS IN ONE DIMENSION

As we have seen, momentum is conserved in any collision in which external forces are negligible. In contrast, kinetic energy may or may not be constant, depending on the type of collision. In fact, whether or not kinetic energy is the same before and after the collision is used to classify collisions as being either elastic or inelastic.

Elastic collision

An **elastic collision** between two objects is one in which *total kinetic energy (as well as total momentum) is the same before and after the collision*. Billiard-ball collisions and the collisions of air molecules with the walls of a container at ordinary temperatures are approximately elastic. Truly elastic collisions do occur, however, between atomic and subatomic particles. Collisions between certain objects in the macroscopic world, such as billiard-ball collisions, are only approximately elastic because some deformation and loss of kinetic energy take place.

An **inelastic collision** is one in which *total kinetic energy is not the same before and after the collision (even though momentum is constant).* Inelastic collisions are of two types. When the colliding objects stick together after the collision, as happens when a meteorite collides with the Earth, the collision is called **perfectly inelastic.** When the colliding objects do not stick together, but some kinetic energy is lost, as in the case of a rubber ball colliding with a hard surface, the collision is called **inelastic** (with no modifying adverb). For example, when a rubber ball collides with a hard surface, the collision is inelastic because some of the kinetic energy of the ball is lost when the ball is deformed while it is in contact with the surface.

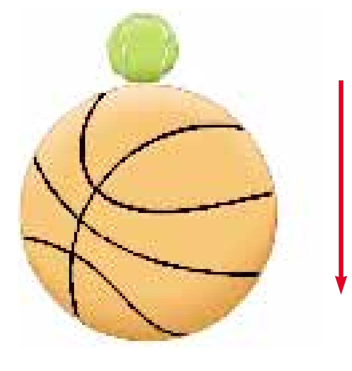
Inelastic collision

In most collisions, kinetic energy is *not* the same before and after the collision because some of it is converted to internal energy, to elastic potential energy when the objects are deformed, and to rotational energy. Elastic and perfectly inelastic collisions are limiting cases; most collisions fall somewhere between them.

In the remainder of this section, we treat collisions in one dimension and consider the two extreme cases—perfectly inelastic and elastic collisions. The important distinction between these two types of collisions is that **momentum is constant in all collisions, but kinetic energy is constant only in elastic collisions.**

QuickLab

Hold a Ping-Pong ball or tennis ball on top of a basketball. Drop them both at the same time so that the basketball hits the floor, bounces up, and hits the smaller falling ball. What happens and why?



Perfectly Inelastic Collisions

Consider two particles of masses m_1 and m_2 moving with initial velocities $\mathbf{v}_{1i}$ and $\mathbf{v}_{2i}$ along a straight line, as shown in Figure 9.9. The two particles collide head-on, stick together, and then move with some common velocity $\mathbf{v}_f$ after the collision. Because momentum is conserved in any collision, we can say that the total momentum before the collision equals the total momentum of the composite system after the collision:

$$m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i} = (m_1 + m_2)\mathbf{v}_f \tag{9.13}$$

$$\mathbf{v}_f = \frac{m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i}}{m_1 + m_2} \tag{9.14}$$

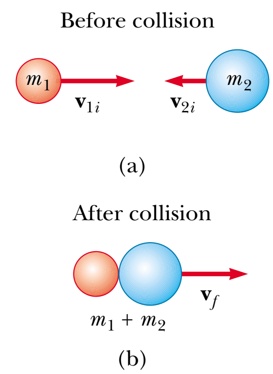


Figure 9.9 Schematic representation of a perfectly inelastic head-on collision between two particles: (a) before collision and (b) after collision.

Quick Quiz 9.7

Which is worse, crashing into a brick wall at 40 mi/h or crashing head-on into an oncoming car that is identical to yours and also moving at 40 mi/h?

Elastic Collisions

6.6 Now consider two particles that undergo an elastic head-on collision (Fig. 9.10). In this case, both momentum and kinetic energy are conserved; therefore, we have

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \tag{9.15}$$

$$\tfrac{1}{2}m_1 v_{1i}^2 + \tfrac{1}{2}m_2 v_{2i}^2 = \tfrac{1}{2}m_1 v_{1f}^2 + \tfrac{1}{2}m_2 v_{2f}^2 \tag{9.16}$$

Because all velocities in Figure 9.10 are either to the left or the right, they can be represented by the corresponding speeds along with algebraic signs indicating direction. We shall indicate v as positive if a particle moves to the right and negative

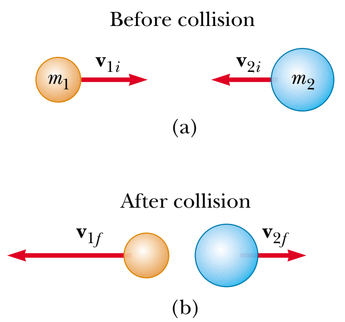


Figure 9.10 Schematic representation of an elastic head-on collision between two particles: (a) before collision and (b) after collision.

if it moves to the left. As has been seen in earlier chapters, it is common practice to call these values "speed" even though this term technically refers to the magnitude of the velocity vector, which does not have an algebraic sign.

In a typical problem involving elastic collisions, there are two unknown quantities, and Equations 9.15 and 9.16 can be solved simultaneously to find these. An alternative approach, however—one that involves a little mathematical manipulation of Equation 9.16—often simplifies this process. To see how, let us cancel the factor $\frac{1}{2}$ in Equation 9.16 and rewrite it as

$$m_1({v_{1i}}^2 - {v_{1f}}^2) = m_2({v_{2f}}^2 - {v_{2i}}^2)$$

and then factor both sides:

$$m_1(v_{1i} - v_{1f})(v_{1i} + v_{1f}) = m_2(v_{2f} - v_{2i})(v_{2f} + v_{2i}) \quad \textbf{(9.17)}$$

Next, let us separate the terms containing m_1 and m_2 in Equation 9.15 to get

$$m_1(v_{1i} - v_{1f}) = m_2(v_{2f} - v_{2i}) \quad \textbf{(9.18)}$$

To obtain our final result, we divide Equation 9.17 by Equation 9.18 and get

$$v_{1i} + v_{1f} = v_{2f} + v_{2i}$$

$$v_{1i} - v_{2i} = -(v_{1f} - v_{2f}) \quad \textbf{(9.19)}$$

This equation, in combination with Equation 9.15, can be used to solve problems dealing with elastic collisions. According to Equation 9.19, the relative speed of the two particles before the collision $v_{1i} - v_{2i}$ equals the negative of their relative speed after the collision, $-(v_{1f} - v_{2f})$.

Suppose that the masses and initial velocities of both particles are known. Equations 9.15 and 9.19 can be solved for the final speeds in terms of the initial speeds because there are two equations and two unknowns:

Elastic collision: relationships between final and initial velocities

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_{1i} + \left(\frac{2m_2}{m_1 + m_2}\right)v_{2i} \quad \textbf{(9.20)}$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right)v_{1i} + \left(\frac{m_2 - m_1}{m_1 + m_2}\right)v_{2i} \quad \textbf{(9.21)}$$

It is important to remember that the appropriate signs for v_{1i} and v_{2i} must be included in Equations 9.20 and 9.21. For example, if particle 2 is moving to the left initially, then v_{2i} is negative.

Let us consider some special cases: If $m_1 = m_2$, then $v_{1f} = v_{2i}$ and $v_{2f} = v_{1i}$. That is, the particles exchange speeds if they have equal masses. This is approximately what one observes in head-on billiard ball collisions—the cue ball stops, and the struck ball moves away from the collision with the same speed that the cue ball had.

If particle 2 is initially at rest, then $v_{2i} = 0$ and Equations 9.20 and 9.21 become

Elastic collision: particle 2 initially at rest

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_{1i} \quad \textbf{(9.22)}$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right)v_{1i} \quad \textbf{(9.23)}$$

If m_1 is much greater than m_2 and $v_{2i} = 0$, we see from Equations 9.22 and 9.23 that $v_{1f} \approx v_{1i}$ and $v_{2f} \approx 2v_{1i}$. That is, when a very heavy particle collides head-on with a very light one that is initially at rest, the heavy particle continues its motion unaltered after the collision, and the light particle rebounds with a speed equal to about twice the initial speed of the heavy particle. An example of such a collision would be that of a moving heavy atom, such as uranium, with a light atom, such as hydrogen.

If m_2 is much greater than m_1 and particle 2 is initially at rest, then $v_{1f} \approx -v_{1i}$ and $v_{2f} \approx v_{2i} = 0$. That is, when a very light particle collides head-on with a very heavy particle that is initially at rest, the light particle has its velocity reversed and the heavy one remains approximately at rest.

EXAMPLE 9.6 The Ballistic Pendulum

The ballistic pendulum (Fig. 9.11) is a system used to measure the speed of a fast-moving projectile, such as a bullet. The bullet is fired into a large block of wood suspended from some light wires. The bullet embeds in the block, and the entire system swings through a height h. The collision is perfectly inelastic, and because momentum is conserved, Equation 9.14 gives the speed of the system right after the collision, when we assume the impulse approximation. If we call the bullet particle 1 and the block particle 2, the total kinetic energy right after the collision is

$$(1) \qquad K_f = \tfrac{1}{2}(m_1 + m_2)v_f^2$$

With $v_{2i} = 0$, Equation 9.14 becomes

$$(2) \qquad v_f = \frac{m_1 v_{1i}}{m_1 + m_2}$$

Substituting this value of v_f into (1) gives

$$K_f = \frac{m_1^2 v_{1i}^2}{2(m_1 + m_2)}$$

Note that this kinetic energy immediately after the collision is less than the initial kinetic energy of the bullet. In all the energy changes that take place *after* the collision, however, the total amount of mechanical energy remains constant; thus, we can say that after the collision, the kinetic energy of the block and bullet at the bottom is transformed to potential energy at the height h:

$$\frac{m_1^2 v_{1i}^2}{2(m_1 + m_2)} = (m_1 + m_2)gh$$

Solving for v_{1i}, we obtain

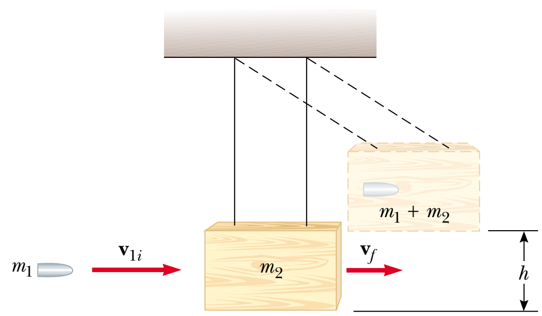
$$v_{1i} = \left(\frac{m_1 + m_2}{m_1}\right)\sqrt{2gh}$$

This expression tells us that it is possible to obtain the initial speed of the bullet by measuring h and the two masses.

Because the collision is perfectly inelastic, some mechanical energy is converted to internal energy and it would be *incorrect* to equate the initial kinetic energy of the incoming bullet to the final gravitational potential energy of the bullet–block combination.

Exercise In a ballistic pendulum experiment, suppose that $h = 5.00$ cm, $m_1 = 5.00$ g, and $m_2 = 1.00$ kg. Find (a) the initial speed of the bullet and (b) the loss in mechanical energy due to the collision.

Answer 199 m/s; 98.5 J.



(a)

(b)

Figure 9.11 (a) Diagram of a ballistic pendulum. Note that $\mathbf{v}_{1i}$ is the velocity of the bullet just before the collision and $\mathbf{v}_f = \mathbf{v}_{1f} = \mathbf{v}_{2f}$ is the velocity of the bullet + block system just after the perfectly inelastic collision. (b) Multiflash photograph of a ballistic pendulum used in the laboratory.

EXAMPLE 9.7 A Two-Body Collision with a Spring

A block of mass $m_1 = 1.60$ kg initially moving to the right with a speed of 4.00 m/s on a frictionless horizontal track collides with a spring attached to a second block of mass $m_2 = 2.10$ kg initially moving to the left with a speed of 2.50 m/s, as shown in Figure 9.12a. The spring constant is 600 N/m. (a) At the instant block 1 is moving to the right with a speed of 3.00 m/s, as in Figure 9.12b, determine the velocity of block 2.

Solution First, note that the initial velocity of block 2 is -2.50 m/s because its direction is to the left. Because momentum is conserved for the system of two blocks, we have

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

$$(1.60 \text{ kg})(4.00 \text{ m/s}) + (2.10 \text{ kg})(-2.50 \text{ m/s}) = (1.60 \text{ kg})(3.00 \text{ m/s}) + (2.10 \text{ kg}) v_{2f}$$

$$v_{2f} = \boxed{-1.74 \text{ m/s}}$$

The negative value for v_{2f} means that block 2 is still moving to the left at the instant we are considering.

(b) Determine the distance the spring is compressed at that instant.

Solution To determine the distance that the spring is compressed, shown as x in Figure 9.12b, we can use the concept of conservation of mechanical energy because no friction or other nonconservative forces are acting on the system. Thus, we have

$$\tfrac{1}{2} m_1 v_{1i}^2 + \tfrac{1}{2} m_2 v_{2i}^2 = \tfrac{1}{2} m_1 v_{1f}^2 + \tfrac{1}{2} m_2 v_{2f}^2 + \tfrac{1}{2} k x^2$$

Substituting the given values and the result to part (a) into this expression gives

$$x = \boxed{0.173 \text{ m}}$$

It is important to note that we needed to use the principles of both conservation of momentum and conservation of mechanical energy to solve the two parts of this problem.

Exercise Find the velocity of block 1 and the compression in the spring at the instant that block 2 is at rest.

Answer 0.719 m/s to the right; 0.251 m.

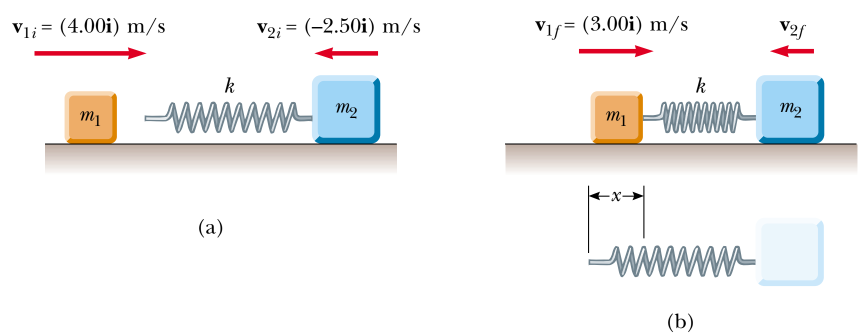


Figure 9.12

EXAMPLE 9.8 Slowing Down Neutrons by Collisions

In a nuclear reactor, neutrons are produced when a $^{235}_{92}U$ atom splits in a process called *fission.* These neutrons are moving at about 10^7 m/s and must be slowed down to about 10^3 m/s before they take part in another fission event. They are slowed down by being passed through a solid or liquid material called a *moderator.* The slowing-down process involves elastic collisions. Let us show that a neutron can lose most of its kinetic energy if it collides elastically with a moderator containing light nuclei, such as deuterium (in "heavy water," D_2O) or carbon (in graphite).

Solution Let us assume that the moderator nucleus of mass m_m is at rest initially and that a neutron of mass m_n and initial speed v_{ni} collides with it head-on.

Because these are elastic collisions, the first thing we do is recognize that both momentum and kinetic energy are constant. Therefore, Equations 9.22 and 9.23 can be applied to the head-on collision of a neutron with a moderator nucleus. We can represent this process by a drawing such as Figure 9.10.

The initial kinetic energy of the neutron is

$$K_{ni} = \tfrac{1}{2} m_n v_{ni}^2$$

After the collision, the neutron has kinetic energy $\frac{1}{2} m_n v_{nf}^2$, and we can substitute into this the value for v_{nf} given by Equation 9.22:

$$K_{nf} = \tfrac{1}{2} m_n v_{nf}^2 = \frac{m_n}{2}\left(\frac{m_n - m_m}{m_n + m_m}\right)^2 v_{ni}^2$$

Therefore, the fraction f_n of the initial kinetic energy possessed by the neutron after the collision is

$$(1) \qquad f_n = \frac{K_{nf}}{K_{ni}} = \left(\frac{m_n - m_m}{m_n + m_m}\right)^2$$

From this result, we see that the final kinetic energy of the neutron is small when m_m is close to m_n and zero when $m_n = m_m$.

We can use Equation 9.23, which gives the final speed of the particle that was initially at rest, to calculate the kinetic energy of the moderator nucleus after the collision:

$$K_{mf} = \tfrac{1}{2} m_m v_{mf}^2 = \frac{2 m_n^2 m_m}{(m_n + m_m)^2} v_{ni}^2$$

Hence, the fraction f_m of the initial kinetic energy transferred to the moderator nucleus is

$$(2) \qquad f_m = \frac{K_{mf}}{K_{ni}} = \frac{4 m_n m_m}{(m_n + m_m)^2}$$

Because the total kinetic energy of the system is conserved, (2) can also be obtained from (1) with the condition that $f_n + f_m = 1$, so that $f_m = 1 - f_n$.

Suppose that heavy water is used for the moderator. For collisions of the neutrons with deuterium nuclei in D_2O ($m_m = 2m_n$), $f_n = 1/9$ and $f_m = 8/9$. That is, 89% of the neutron's kinetic energy is transferred to the deuterium nucleus. In practice, the moderator efficiency is reduced because head-on collisions are very unlikely.

How do the results differ when graphite (^{12}C, as found in pencil lead) is used as the moderator?

Quick Quiz 9.8

An ingenious device that illustrates conservation of momentum and kinetic energy is shown in Figure 9.13a. It consists of five identical hard balls supported by strings of equal lengths. When ball 1 is pulled out and released, after the almost-elastic collision between it and ball 2, ball 5 moves out, as shown in Figure 9.13b. If balls 1 and 2 are pulled out and released, balls 4 and 5 swing out, and so forth. Is it ever possible that, when ball 1 is released, balls 4 and 5 will swing out on the opposite side and travel with half the speed of ball 1, as in Figure 9.13c?

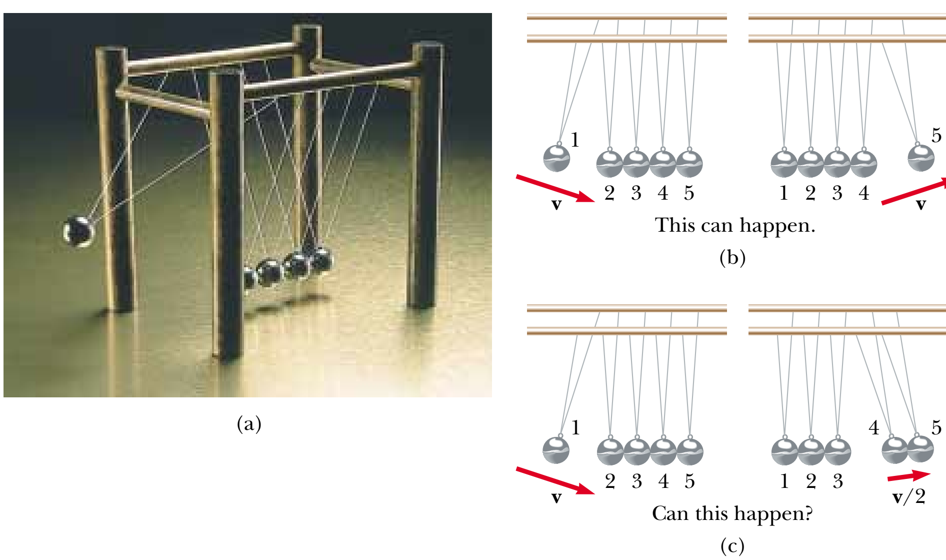


Figure 9.13 An executive stress reliever.

9.5 TWO-DIMENSIONAL COLLISIONS

In Sections 9.1 and 9.3, we showed that the momentum of a system of two particles is constant when the system is isolated. For any collision of two particles, this result implies that the momentum in each of the directions x, y, and z is constant. However, an important subset of collisions takes place in a plane. The game of billiards is a familiar example involving multiple collisions of objects moving on a two-dimensional surface. For such two-dimensional collisions, we obtain two component equations for conservation of momentum:

$$m_1v_{1ix} + m_2v_{2ix} = m_1v_{1fx} + m_2v_{2fx}$$

$$m_1v_{1iy} + m_2v_{2iy} = m_1v_{1fy} + m_2v_{2fy}$$

Let us consider a two-dimensional problem in which particle 1 of mass m_1 collides with particle 2 of mass m_2, where particle 2 is initially at rest, as shown in Figure 9.14. After the collision, particle 1 moves at an angle θ with respect to the horizontal and particle 2 moves at an angle ϕ with respect to the horizontal. This is called a *glancing* collision. Applying the law of conservation of momentum in component form, and noting that the initial y component of the momentum of the two-particle system is zero, we obtain

$$m_1v_{1i} = m_1v_{1f}\cos\theta + m_2v_{2f}\cos\phi \tag{9.24}$$

$$0 = m_1v_{1f}\sin\theta - m_2v_{2f}\sin\phi \tag{9.25}$$

where the minus sign in Equation 9.25 comes from the fact that after the collision, particle 2 has a y component of velocity that is downward. We now have two independent equations. As long as no more than two of the seven quantities in Equations 9.24 and 9.25 are unknown, we can solve the problem.

If the collision is elastic, we can also use Equation 9.16 (conservation of kinetic energy), with $v_{2i} = 0$, to give

$$\tfrac{1}{2}m_1{v_{1i}}^2 = \tfrac{1}{2}m_1{v_{1f}}^2 + \tfrac{1}{2}m_2{v_{2f}}^2 \tag{9.26}$$

Knowing the initial speed of particle 1 and both masses, we are left with four unknowns (v_{1f}, v_{2f}, θ, ϕ). Because we have only three equations, one of the four remaining quantities must be given if we are to determine the motion after the collision from conservation principles alone.

If the collision is inelastic, kinetic energy is *not* conserved and Equation 9.26 does *not* apply.

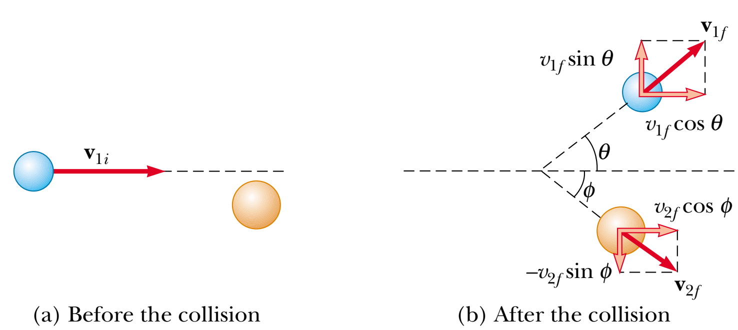


Figure 9.14 An elastic glancing collision between two particles.

Problem-Solving Hints

Collisions

The following procedure is recommended when dealing with problems involving collisions between two objects:

- Set up a coordinate system and define your velocities with respect to that system. It is usually convenient to have the x axis coincide with one of the initial velocities.
- In your sketch of the coordinate system, draw and label all velocity vectors and include all the given information.
- Write expressions for the x and y components of the momentum of each object before and after the collision. Remember to include the appropriate signs for the components of the velocity vectors.
- Write expressions for the total momentum in the x direction before and after the collision and equate the two. Repeat this procedure for the total momentum in the y direction. These steps follow from the fact that, because the momentum of the *system* is conserved in any collision, the total momentum along any direction must also be constant. Remember, it is the momentum of the *system* that is constant, not the momenta of the individual objects.
- If the collision is inelastic, kinetic energy is not conserved, and additional information is probably required. If the collision is perfectly inelastic, the final velocities of the two objects are equal. Solve the momentum equations for the unknown quantities.
- If the collision is elastic, kinetic energy is conserved, and you can equate the total kinetic energy before the collision to the total kinetic energy after the collision to get an additional relationship between the velocities.

Example 9.9 Collision at an Intersection

A 1 500-kg car traveling east with a speed of 25.0 m/s collides at an intersection with a 2 500-kg van traveling north at a speed of 20.0 m/s, as shown in Figure 9.15. Find the direction and magnitude of the velocity of the wreckage after the collision, assuming that the vehicles undergo a perfectly inelastic collision (that is, they stick together).

Solution Let us choose east to be along the positive x direction and north to be along the positive y direction. Before the collision, the only object having momentum in the x direction is the car. Thus, the magnitude of the total initial momentum of the system (car plus van) in the x direction is

$$\sum p_{xi} = (1\,500 \text{ kg})(25.0 \text{ m/s}) = 3.75 \times 10^4 \text{ kg}\cdot\text{m/s}$$

Let us assume that the wreckage moves at an angle θ and speed v_f after the collision. The magnitude of the total momentum in the x direction after the collision is

$$\sum p_{xf} = (4\,000 \text{ kg})v_f \cos\theta$$

Because the total momentum in the x direction is constant, we can equate these two equations to obtain

$$(1) \qquad 3.75 \times 10^4 \text{ kg}\cdot\text{m/s} = (4\,000 \text{ kg})v_f \cos\theta$$

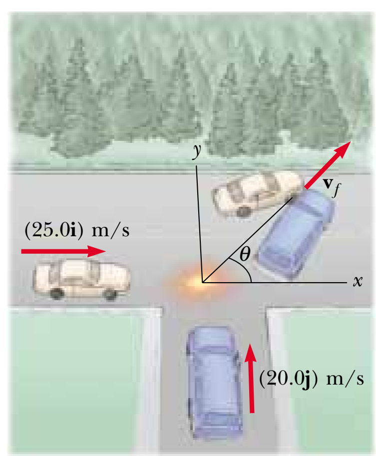


Figure 9.15 An eastbound car colliding with a northbound van.

Similarly, the total initial momentum of the system in the y direction is that of the van, and the magnitude of this momentum is $(2\,500 \text{ kg})(20.0 \text{ m/s})$. Applying conservation of momentum to the y direction, we have

$$\sum p_{yi} = \sum p_{yf}$$

$$(2\,500 \text{ kg})(20.0 \text{ m/s}) = (4\,000 \text{ kg})v_f \sin\theta$$

$$(2) \qquad 5.00 \times 10^4 \text{ kg}\cdot\text{m/s} = (4\,000 \text{ kg})v_f \sin\theta$$

If we divide (2) by (1), we get

$$\frac{\sin\theta}{\cos\theta} = \tan\theta = \frac{5.00 \times 10^4}{3.75 \times 10^4} = 1.33$$

$$\theta = \boxed{53.1°}$$

When this angle is substituted into (2), the value of v_f is

$$v_f = \frac{5.00 \times 10^4 \text{ kg}\cdot\text{m/s}}{(4\,000 \text{ kg})\sin 53.1°} = \boxed{15.6 \text{ m/s}}$$

It might be instructive for you to draw the momentum vectors of each vehicle before the collision and the two vehicles together after the collision.

EXAMPLE 9.10 Proton–Proton Collision

Proton 1 collides elastically with proton 2 that is initially at rest. Proton 1 has an initial speed of 3.50×10^5 m/s and makes a glancing collision with proton 2, as was shown in Figure 9.14. After the collision, proton 1 moves at an angle of 37.0° to the horizontal axis, and proton 2 deflects at an angle ϕ to the same axis. Find the final speeds of the two protons and the angle ϕ.

Solution Because both particles are protons, we know that $m_1 = m_2$. We also know that $\theta = 37.0°$ and $v_{1i} = 3.50 \times 10^5$ m/s. Equations 9.24, 9.25, and 9.26 become

$$v_{1f}\cos 37.0° + v_{2f}\cos\phi = 3.50 \times 10^5 \text{ m/s}$$

$$v_{1f}\sin 37.0° - v_{2f}\sin\phi = 0$$

$$v_{1f}^2 + v_{2f}^2 = (3.50 \times 10^5 \text{ m/s})^2$$

Solving these three equations with three unknowns simultaneously gives

$$v_{1f} = \boxed{2.80 \times 10^5 \text{ m/s}}$$

$$v_{2f} = \boxed{2.11 \times 10^5 \text{ m/s}}$$

$$\phi = \boxed{53.0°}$$

Note that $\theta + \phi = 90°$. This result is not accidental. **Whenever two equal masses collide elastically in a glancing collision and one of them is initially at rest, their final velocities are always at right angles to each other.** The next example illustrates this point in more detail.

EXAMPLE 9.11 Billiard Ball Collision

In a game of billiards, a player wishes to sink a target ball 2 in the corner pocket, as shown in Figure 9.16. If the angle to the corner pocket is 35°, at what angle θ is the cue ball 1 deflected? Assume that friction and rotational motion are unimportant and that the collision is elastic.

Solution Because the target ball is initially at rest, conservation of energy (Eq. 9.16) gives

$$\tfrac{1}{2}m_1 v_{1i}^2 = \tfrac{1}{2}m_1 v_{1f}^2 + \tfrac{1}{2}m_2 v_{2f}^2$$

But $m_1 = m_2$, so that

$$(1) \qquad v_{1i}^2 = v_{1f}^2 + v_{2f}^2$$

Applying conservation of momentum to the two-dimensional collision gives

$$(2) \qquad \mathbf{v}_{1i} = \mathbf{v}_{1f} + \mathbf{v}_{2f}$$

Note that because $m_1 = m_2$, the masses also cancel in (2). If we square both sides of (2) and use the definition of the dot

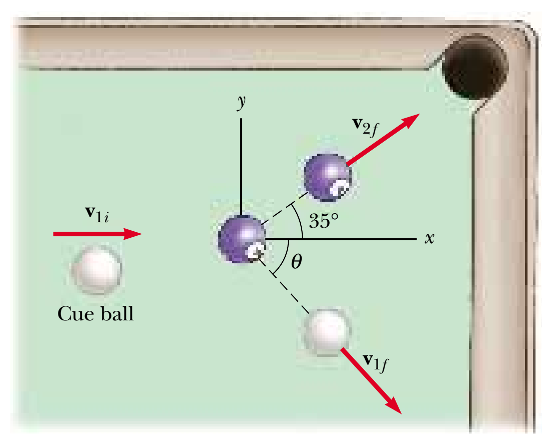


Figure 9.16

product of two vectors from Section 7.2, we get

$$v_{1i}^2 = (\mathbf{v}_{1f} + \mathbf{v}_{2f}) \cdot (\mathbf{v}_{1f} + \mathbf{v}_{2f}) = v_{1f}^2 + v_{2f}^2 + 2\mathbf{v}_{1f} \cdot \mathbf{v}_{2f}$$

Because the angle between $\mathbf{v}_{1f}$ and $\mathbf{v}_{2f}$ is $\theta + 35°$, $\mathbf{v}_{1f} \cdot \mathbf{v}_{2f} = v_{1f} v_{2f} \cos(\theta + 35°)$, and so

$$(3) \qquad v_{1i}^2 = v_{1f}^2 + v_{2f}^2 + 2v_{1f}v_{2f}\cos(\theta + 35°)$$

Subtracting (1) from (3) gives

$$0 = 2v_{1f}v_{2f}\cos(\theta + 35°)$$

$$0 = \cos(\theta + 35°)$$

$$\theta + 35° = 90° \qquad \text{or} \qquad \theta = \boxed{55°}$$

This result shows that whenever two equal masses undergo a glancing elastic collision and one of them is initially at rest, they move at right angles to each other after the collision. The same physics describes two very different situations, protons in Example 9.10 and billiard balls in this example.

9.6 THE CENTER OF MASS

6.7

In this section we describe the overall motion of a mechanical system in terms of a special point called the **center of mass** of the system. The mechanical system can be either a system of particles, such as a collection of atoms in a container, or an extended object, such as a gymnast leaping through the air. We shall see that the center of mass of the system moves as if all the mass of the system were concentrated at that point. Furthermore, if the resultant external force on the system is $\Sigma\mathbf{F}_{\text{ext}}$ and the total mass of the system is M, the center of mass moves with an acceleration given by $\mathbf{a} = \Sigma\mathbf{F}_{\text{ext}}/M$. That is, the system moves as if the resultant external force were applied to a single particle of mass M located at the center of mass. This behavior is independent of other motion, such as rotation or vibration of the system. This result was implicitly assumed in earlier chapters because many examples referred to the motion of extended objects that were treated as particles.

Consider a mechanical system consisting of a pair of particles that have different masses and are connected by a light, rigid rod (Fig. 9.17). One can describe the position of the center of mass of a system as being the *average position* of the system's mass. The center of mass of the system is located somewhere on the line joining the

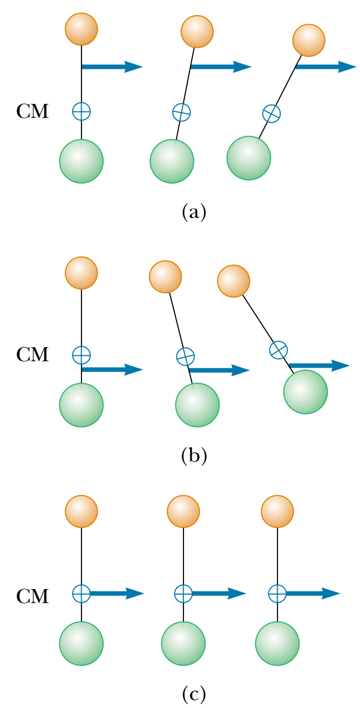
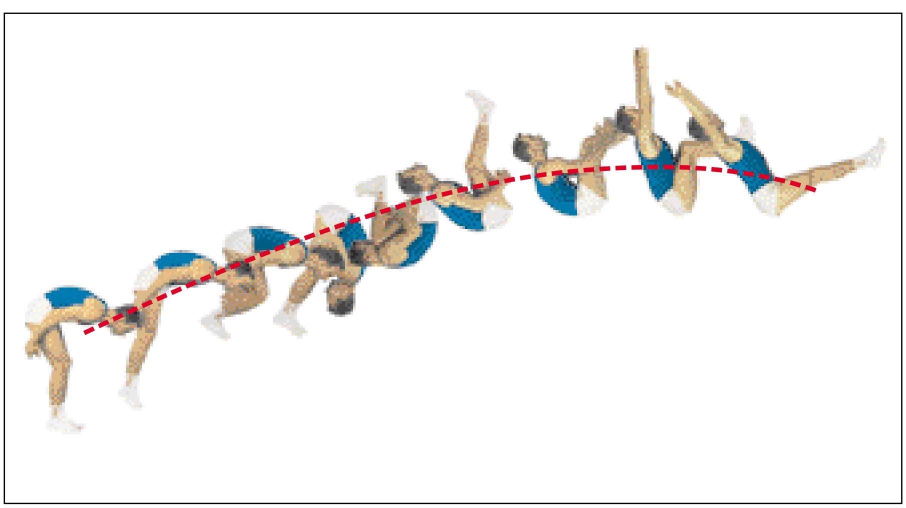


Figure 9.17 Two particles of unequal mass are connected by a light, rigid rod. (a) The system rotates clockwise when a force is applied between the less massive particle and the center of mass. (b) The system rotates counterclockwise when a force is applied between the more massive particle and the center of mass. (c) The system moves in the direction of the force without rotating when a force is applied at the center of mass.



This multiflash photograph shows that as the acrobat executes a somersault, his center of mass follows a parabolic path, the same path that a particle would follow.

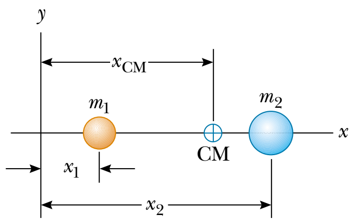


Figure 9.18 The center of mass of two particles of unequal mass on the x axis is located at x_{CM}, a point between the particles, closer to the one having the larger mass.

particles and is closer to the particle having the larger mass. If a single force is applied at some point on the rod somewhere between the center of mass and the less massive particle, the system rotates clockwise (see Fig. 9.17a). If the force is applied at a point on the rod somewhere between the center of mass and the more massive particle, the system rotates counterclockwise (see Fig. 9.17b). If the force is applied at the center of mass, the system moves in the direction of **F** without rotating (see Fig. 9.17c). Thus, the center of mass can be easily located.

The center of mass of the pair of particles described in Figure 9.18 is located on the x axis and lies somewhere between the particles. Its x coordinate is

$$x_{CM} \equiv \frac{m_1x_1 + m_2x_2}{m_1 + m_2} \tag{9.27}$$

For example, if $x_1 = 0$, $x_2 = d$, and $m_2 = 2m_1$, we find that $x_{CM} = \frac{2}{3}d$. That is, the center of mass lies closer to the more massive particle. If the two masses are equal, the center of mass lies midway between the particles.

We can extend this concept to a system of many particles in three dimensions. The x coordinate of the center of mass of n particles is defined to be

$$x_{CM} \equiv \frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots + m_nx_n}{m_1 + m_2 + m_3 + \cdots + m_n} = \frac{\sum_i m_ix_i}{\sum_i m_i} \tag{9.28}$$

where x_i is the x coordinate of the ith particle. For convenience, we express the total mass as $M \equiv \sum_i m_i$, where the sum runs over all n particles. The y and z coordinates of the center of mass are similarly defined by the equations

$$y_{CM} \equiv \frac{\sum_i m_iy_i}{M} \quad \text{and} \quad z_{CM} \equiv \frac{\sum_i m_iz_i}{M} \tag{9.29}$$

The center of mass can also be located by its position vector, $\mathbf{r}_{CM}$. The cartesian coordinates of this vector are x_{CM}, y_{CM}, and z_{CM}, defined in Equations 9.28 and 9.29. Therefore,

Vector position of the center of mass for a system of particles

$$\mathbf{r}_{CM} = x_{CM}\mathbf{i} + y_{CM}\mathbf{j} + z_{CM}\mathbf{k}$$
$$= \frac{\sum_i m_ix_i\mathbf{i} + \sum_i m_iy_i\mathbf{j} + \sum_i m_iz_i\mathbf{k}}{M}$$
$$\mathbf{r}_{CM} \equiv \frac{\sum_i m_i\mathbf{r}_i}{M} \tag{9.30}$$

where $\mathbf{r}_i$ is the position vector of the ith particle, defined by

$$\mathbf{r}_i \equiv x_i\mathbf{i} + y_i\mathbf{j} + z_i\mathbf{k}$$

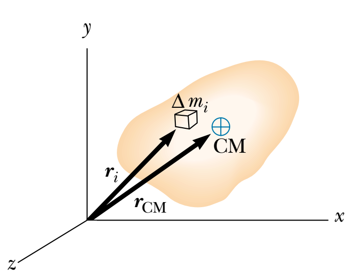


Figure 9.19 An extended object can be considered a distribution of small elements of mass Δm_i. The center of mass is located at the vector position $\mathbf{r}_{CM}$, which has coordinates x_{CM}, y_{CM}, and z_{CM}.

Although locating the center of mass for an extended object is somewhat more cumbersome than locating the center of mass of a system of particles, the basic ideas we have discussed still apply. We can think of an extended object as a system containing a large number of particles (Fig. 9.19). The particle separation is very small, and so the object can be considered to have a continuous mass distribution. By dividing the object into elements of mass Δm_i, with coordinates x_i, y_i, z_i, we see that the x coordinate of the center of mass is approximately

$$x_{CM} \approx \frac{\sum_i x_i\,\Delta m_i}{M}$$

with similar expressions for y_{CM} and z_{CM}. If we let the number of elements n approach infinity, then x_{CM} is given precisely. In this limit, we replace the sum by an integral and Δm_i by the differential element dm:

$$x_{\text{CM}} = \lim_{\Delta m_i \to 0} \frac{\sum_i x_i \Delta m_i}{M} = \frac{1}{M} \int x\, dm \tag{9.31}$$

Likewise, for y_{CM} and z_{CM} we obtain

$$y_{\text{CM}} = \frac{1}{M} \int y\, dm \quad \text{and} \quad z_{\text{CM}} = \frac{1}{M} \int z\, dm \tag{9.32}$$

We can express the vector position of the center of mass of an extended object in the form

$$\mathbf{r}_{\text{CM}} = \frac{1}{M} \int \mathbf{r}\, dm \tag{9.33}$$

which is equivalent to the three expressions given by Equations 9.31 and 9.32.

The center of mass of any symmetric object lies on an axis of symmetry and on any plane of symmetry.[4] For example, the center of mass of a rod lies in the rod, midway between its ends. The center of mass of a sphere or a cube lies at its geometric center.

One can determine the center of mass of an irregularly shaped object by suspending the object first from one point and then from another. In Figure 9.20, a wrench is hung from point *A*, and a vertical line *AB* (which can be established with a plumb bob) is drawn when the wrench has stopped swinging. The wrench is then hung from point *C*, and a second vertical line *CD* is drawn. The center of mass is halfway through the thickness of the wrench, under the intersection of these two lines. In general, if the wrench is hung freely from any point, the vertical line through this point must pass through the center of mass.

Because an extended object is a continuous distribution of mass, each small mass element is acted upon by the force of gravity. The net effect of all these forces is equivalent to the effect of a single force, $M\mathbf{g}$, acting through a special point, called the **center of gravity.** If $\mathbf{g}$ is constant over the mass distribution, then the center of gravity coincides with the center of mass. If an extended object is pivoted at its center of gravity, it balances in any orientation.

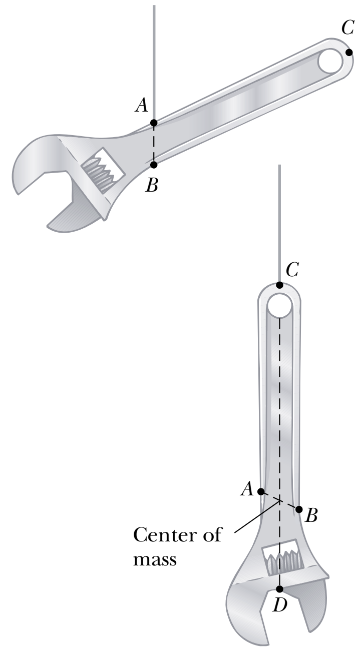


Figure 9.20 An experimental technique for determining the center of mass of a wrench. The wrench is hung freely first from point *A* and then from point *C*. The intersection of the two lines *AB* and *CD* locates the center of mass.

QuickLab

Cut a triangle from a piece of cardboard and draw a set of adjacent strips inside it, parallel to one of the sides. Put a dot at the approximate location of the center of mass of each strip and then draw a straight line through the dots and into the angle opposite your starting side. The center of mass for the triangle must lie on this bisector of the angle. Repeat these steps for the other two sides. The three angle bisectors you have drawn will intersect at the center of mass of the triangle. If you poke a hole anywhere in the triangle and hang the cardboard from a string attached at that hole, the center of mass will be vertically aligned with the hole.

Quick Quiz 9.9

If a baseball bat is cut at the location of its center of mass as shown in Figure 9.21, do the two pieces have the same mass?

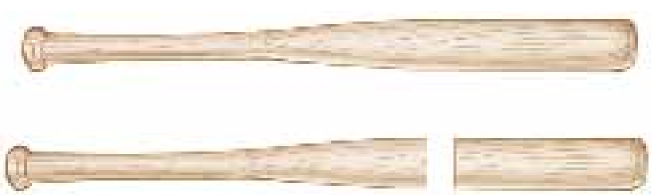


Figure 9.21 A baseball bat cut at the location of its center of mass.

[4]This statement is valid only for objects that have a uniform mass per unit volume.

EXAMPLE 9.12 The Center of Mass of Three Particles

A system consists of three particles located as shown in Figure 9.22a. Find the center of mass of the system.

Solution We set up the problem by labeling the masses of the particles as shown in the figure, with $m_1 = m_2 = 1.0$ kg and $m_3 = 2.0$ kg. Using the basic defining equations for the coordinates of the center of mass and noting that $z_{CM} = 0$, we obtain

$$
\begin{aligned}
x_{CM} &= \frac{\sum_i m_i x_i}{M} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3} \\
&= \frac{(1.0 \text{ kg})(1.0 \text{ m}) + (1.0 \text{ kg})(2.0 \text{ m}) + (2.0 \text{ kg})(0 \text{ m})}{1.0 \text{ kg} + 1.0 \text{ kg} + 2.0 \text{ kg}} \\
&= \frac{3.0 \text{ kg}\cdot\text{m}}{4.0 \text{ kg}} = 0.75 \text{ m}
\end{aligned}
$$

$$
\begin{aligned}
y_{CM} &= \frac{\sum_i m_i y_i}{M} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3} \\
&= \frac{(1.0 \text{ kg})(0) + (1.0 \text{ kg})(0) + (2.0 \text{ kg})(2.0 \text{ m})}{4.0 \text{ kg}} \\
&= \frac{4.0 \text{ kg}\cdot\text{m}}{4.0 \text{ kg}} = 1.0 \text{ m}
\end{aligned}
$$

The position vector to the center of mass measured from the origin is therefore

$$\mathbf{r}_{CM} = x_{CM}\mathbf{i} + y_{CM}\mathbf{j} = \boxed{0.75\mathbf{i} \text{ m} + 1.0\,\mathbf{j} \text{ m}}$$

We can verify this result graphically by adding together $m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3$ and dividing the vector sum by M, the total mass. This is shown in Figure 9.22b.

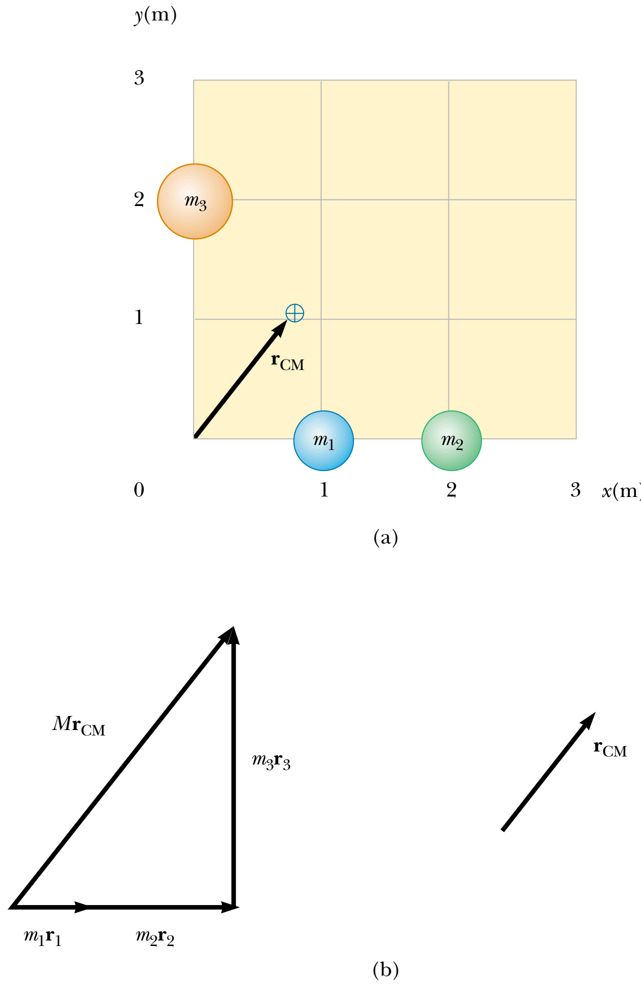


Figure 9.22 (a) Two 1-kg masses and a single 2-kg mass are located as shown. The vector indicates the location of the system's center of mass. (b) The vector sum of $m_i\mathbf{r}_i$.

EXAMPLE 9.13 The Center of Mass of a Rod

(a) Show that the center of mass of a rod of mass M and length L lies midway between its ends, assuming the rod has a uniform mass per unit length.

Solution The rod is shown aligned along the x axis in Figure 9.23, so that $y_{CM} = z_{CM} = 0$. Furthermore, if we call the mass per unit length λ (this quantity is called the *linear mass density*), then $\lambda = M/L$ for the uniform rod we assume here. If we divide the rod into elements of length dx, then the mass of each element is $dm = \lambda\, dx$. For an arbitrary element located a distance x from the origin, Equation 9.31 gives

$$x_{CM} = \frac{1}{M}\int x\, dm = \frac{1}{M}\int_0^L x\lambda\, dx = \frac{\lambda}{M}\left.\frac{x^2}{2}\right|_0^L = \frac{\lambda L^2}{2M}$$

Because $\lambda = M/L$, this reduces to

$$x_{CM} = \frac{L^2}{2M}\left(\frac{M}{L}\right) = \boxed{\frac{L}{2}}$$

One can also use symmetry arguments to obtain the same result.

(b) Suppose a rod is *nonuniform* such that its mass per unit length varies linearly with x according to the expression $\lambda = \alpha x$, where α is a constant. Find the x coordinate of the center of mass as a fraction of L.

Solution In this case, we replace dm by λdx where λ is not constant. Therefore, x_{CM} is

$$x_{CM} = \frac{1}{M}\int x\,dm = \frac{1}{M}\int_0^L x\lambda\,dx = \frac{1}{M}\int_0^L x\alpha x\,dx$$
$$= \frac{\alpha}{M}\int_0^L x^2\,dx = \frac{\alpha L^3}{3M}$$

We can eliminate α by noting that the total mass of the rod is related to α through the relationship

$$M = \int dm = \int_0^L \lambda\,dx = \int_0^L \alpha x\,dx = \frac{\alpha L^2}{2}$$

Substituting this into the expression for x_{CM} gives

$$x_{CM} = \frac{\alpha L^3}{3\alpha L^2/2} = \boxed{\frac{2}{3}L}$$

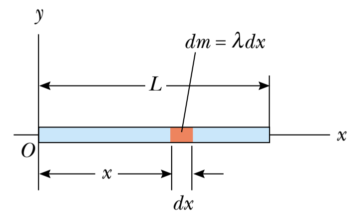


Figure 9.23 The center of mass of a uniform rod of length L is located at $x_{CM} = L/2$.

EXAMPLE 9.14 The Center of Mass of a Right Triangle

An object of mass M is in the shape of a right triangle whose dimensions are shown in Figure 9.24. Locate the coordinates of the center of mass, assuming the object has a uniform mass per unit area.

Solution By inspection we can estimate that the x coordinate of the center of mass must be past the center of the base, that is, greater than $a/2$, because the largest part of the triangle lies beyond that point. A similar argument indicates that its y coordinate must be less than $b/2$. To evaluate the x coordinate, we divide the triangle into narrow strips of width dx and height y as in Figure 9.24. The mass dm of each strip is

$$dm = \frac{\text{total mass of object}}{\text{total area of object}} \times \text{area of strip}$$
$$= \frac{M}{1/2\,ab}(y\,dx) = \left(\frac{2M}{ab}\right)y\,dx$$

Therefore, the x coordinate of the center of mass is

$$x_{CM} = \frac{1}{M}\int x\,dm = \frac{1}{M}\int_0^a x\left(\frac{2M}{ab}\right)y\,dx = \frac{2}{ab}\int_0^a xy\,dx$$

To evaluate this integral, we must express y in terms of x. From similar triangles in Figure 9.24, we see that

$$\frac{y}{x} = \frac{b}{a} \quad \text{or} \quad y = \frac{b}{a}x$$

With this substitution, x_{CM} becomes

$$x_{CM} = \frac{2}{ab}\int_0^a x\left(\frac{b}{a}x\right)dx = \frac{2}{a^2}\int_0^a x^2\,dx = \frac{2}{a^2}\left[\frac{x^3}{3}\right]_0^a$$
$$= \boxed{\frac{2}{3}a}$$

By a similar calculation, we get for the y coordinate of the center of mass

$$y_{CM} = \boxed{\frac{1}{3}b}$$

These values fit our original estimates.

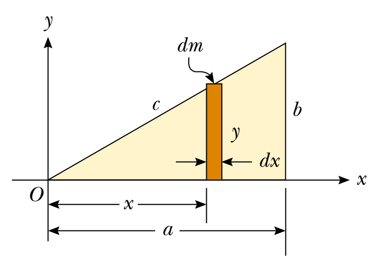


Figure 9.24

9.7 MOTION OF A SYSTEM OF PARTICLES

6.8 We can begin to understand the physical significance and utility of the center of mass concept by taking the time derivative of the position vector given by Equation 9.30. From Section 4.1 we know that the time derivative of a position vector is by definition a velocity. Assuming M remains constant for a system of particles, that is, no particles enter or leave the system, we get the following expression for the **velocity of the center of mass** of the system:

Velocity of the center of mass

$$\mathbf{v}_{\mathrm{CM}} = \frac{d\mathbf{r}_{\mathrm{CM}}}{dt} = \frac{1}{M}\sum_i m_i \frac{d\mathbf{r}_i}{dt} = \frac{\sum_i m_i \mathbf{v}_i}{M} \tag{9.34}$$

where $\mathbf{v}_i$ is the velocity of the ith particle. Rearranging Equation 9.34 gives

Total momentum of a system of particles

$$M\mathbf{v}_{\mathrm{CM}} = \sum_i m_i \mathbf{v}_i = \sum_i \mathbf{p}_i = \mathbf{p}_{\mathrm{tot}} \tag{9.35}$$

Therefore, we conclude that the **total linear momentum of the system** equals the total mass multiplied by the velocity of the center of mass. In other words, the total linear momentum of the system is equal to that of a single particle of mass M moving with a velocity $\mathbf{v}_{\mathrm{CM}}$.

If we now differentiate Equation 9.34 with respect to time, we get the **acceleration of the center of mass** of the system:

Acceleration of the center of mass

$$\mathbf{a}_{\mathrm{CM}} = \frac{d\mathbf{v}_{\mathrm{CM}}}{dt} = \frac{1}{M}\sum_i m_i \frac{d\mathbf{v}_i}{dt} = \frac{1}{M}\sum_i m_i \mathbf{a}_i \tag{9.36}$$

Rearranging this expression and using Newton's second law, we obtain

$$M\mathbf{a}_{\mathrm{CM}} = \sum_i m_i \mathbf{a}_i = \sum_i \mathbf{F}_i \tag{9.37}$$

where $\mathbf{F}_i$ is the net force on particle i.

The forces on any particle in the system may include both external forces (from outside the system) and internal forces (from within the system). However, by Newton's third law, the internal force exerted by particle 1 on particle 2, for example, is equal in magnitude and opposite in direction to the internal force exerted by particle 2 on particle 1. Thus, when we sum over all internal forces in Equation 9.37, they cancel in pairs and the net force on the system is caused *only* by external forces. Thus, we can write Equation 9.37 in the form

Newton's second law for a system of particles

$$\sum \mathbf{F}_{\mathrm{ext}} = M\mathbf{a}_{\mathrm{CM}} = \frac{d\mathbf{p}_{\mathrm{tot}}}{dt} \tag{9.38}$$

That is, the resultant external force on a system of particles equals the total mass of the system multiplied by the acceleration of the center of mass. If we compare this with Newton's second law for a single particle, we see that

The center of mass of a system of particles of combined mass M moves like an equivalent particle of mass M would move under the influence of the resultant external force on the system.

Finally, we see that if the resultant external force is zero, then from Equation 9.38 it follows that

$$\frac{d\mathbf{p}_{\mathrm{tot}}}{dt} = M\mathbf{a}_{\mathrm{CM}} = 0$$

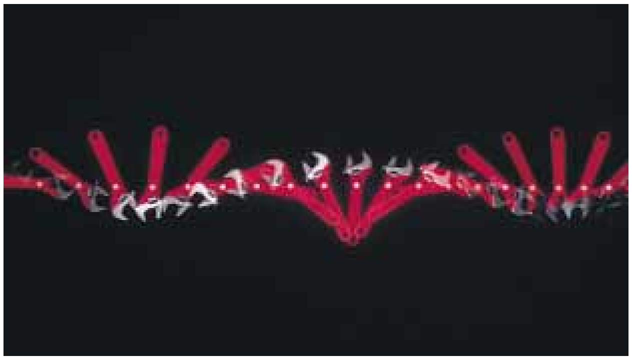


Figure 9.25 Multiflash photograph showing an overhead view of a wrench moving on a horizontal surface. The center of mass of the wrench moves in a straight line as the wrench rotates about this point, shown by the white dots.

so that

$$\mathbf{p}_{\text{tot}} = M\mathbf{v}_{\text{CM}} = \text{constant} \qquad (\text{when } \Sigma\mathbf{F}_{\text{ext}} = 0) \tag{9.39}$$

That is, the total linear momentum of a system of particles is conserved if no net external force is acting on the system. It follows that for an isolated system of particles, both the total momentum and the velocity of the center of mass are constant in time, as shown in Figure 9.25. This is a generalization to a many-particle system of the law of conservation of momentum discussed in Section 9.1 for a two-particle system.

Suppose an isolated system consisting of two or more members is at rest. The center of mass of such a system remains at rest unless acted upon by an external force. For example, consider a system made up of a swimmer standing on a raft, with the system initially at rest. When the swimmer dives horizontally off the raft, the center of mass of the system remains at rest (if we neglect friction between raft and water). Furthermore, the linear momentum of the diver is equal in magnitude to that of the raft but opposite in direction.

As another example, suppose an unstable atom initially at rest suddenly breaks up into two fragments of masses M_A and M_B, with velocities $\mathbf{v}_A$ and $\mathbf{v}_B$, respectively. Because the total momentum of the system before the breakup is zero, the total momentum of the system after the breakup must also be zero. Therefore, $M_A\mathbf{v}_A + M_B\mathbf{v}_B = 0$. If the velocity of one of the fragments is known, the recoil velocity of the other fragment can be calculated.

EXAMPLE 9.15 The Sliding Bear

Suppose you tranquilize a polar bear on a smooth glacier as part of a research effort. How might you estimate the bear's mass using a measuring tape, a rope, and knowledge of your own mass?

Solution Tie one end of the rope around the bear, and then lay out the tape measure on the ice with one end at the bear's original position, as shown in Figure 9.26. Grab hold of the free end of the rope and position yourself as shown, noting your location. Take off your spiked shoes and pull on the rope hand over hand. Both you and the bear will slide over the ice until you meet. From the tape, observe how far you have slid, x_p, and how far the bear has slid, x_b. The point where you meet the bear is the constant location of the center of mass of the system (bear plus you), and so you can determine the mass of the bear from $m_b x_b = m_p x_p$. (Unfortunately, you cannot get back to your spiked shoes and so are in big trouble if the bear wakes up!)

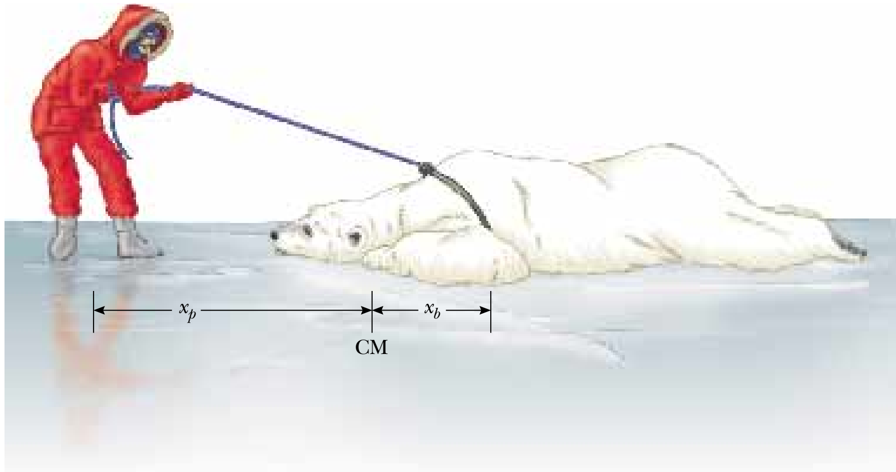
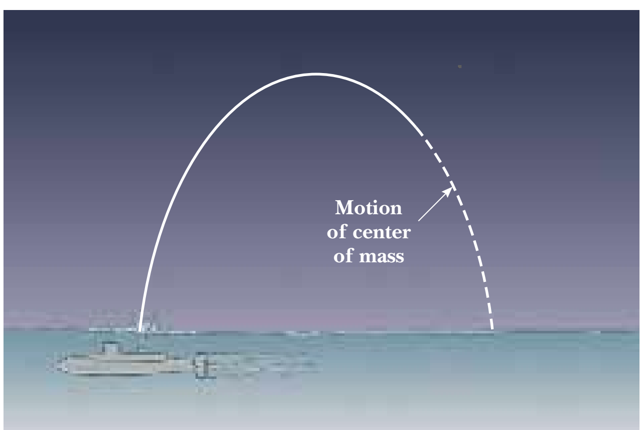


Figure 9.26 The center of mass of an isolated system remains at rest unless acted on by an external force. How can you determine the mass of the polar bear?

CONCEPTUAL EXAMPLE 9.16 Exploding Projectile

A projectile fired into the air suddenly explodes into several fragments (Fig. 9.27). What can be said about the motion of the center of mass of the system made up of all the fragments after the explosion?



Solution Neglecting air resistance, the only external force on the projectile is the gravitational force. Thus, if the projectile did not explode, it would continue to move along the parabolic path indicated by the broken line in Figure 9.27. Because the forces caused by the explosion are internal, they do not affect the motion of the center of mass. Thus, after the explosion the center of mass of the system (the fragments) follows the same parabolic path the projectile would have followed if there had been no explosion.

Figure 9.27 When a projectile explodes into several fragments, the center of mass of the system made up of all the fragments follows the same parabolic path the projectile would have taken had there been no explosion.

EXAMPLE 9.17 The Exploding Rocket

A rocket is fired vertically upward. At the instant it reaches an altitude of 1 000 m and a speed of 300 m/s, it explodes into three equal fragments. One fragment continues to move upward with a speed of 450 m/s following the explosion. The second fragment has a speed of 240 m/s and is moving east right after the explosion. What is the velocity of the third fragment right after the explosion?

Solution Let us call the total mass of the rocket M; hence, the mass of each fragment is $M/3$. Because the forces of the explosion are internal to the system and cannot affect its total momentum, the total momentum $\mathbf{p}_i$ of the rocket just before the explosion must equal the total momentum $\mathbf{p}_f$ of the fragments right after the explosion.

Before the explosion:

$$\mathbf{p}_i = M\mathbf{v}_i = M(300\mathbf{j}) \text{ m/s}$$

After the explosion:

$$\mathbf{p}_f = \frac{M}{3}(240\mathbf{i}) \text{ m/s} + \frac{M}{3}(450\mathbf{j}) \text{ m/s} + \frac{M}{3}\mathbf{v}_f$$

where $\mathbf{v}_f$ is the unknown velocity of the third fragment. Equating these two expressions (because $\mathbf{p}_i = \mathbf{p}_f$) gives

$$\frac{M}{3}\mathbf{v}_f + M(80\mathbf{i}) \text{ m/s} + M(150\mathbf{j}) \text{ m/s} = M(300\,\mathbf{j}) \text{ m/s}$$

$$\mathbf{v}_f = \boxed{(-240\mathbf{i} + 450\mathbf{j}) \text{ m/s}}$$

What does the sum of the momentum vectors for all the fragments look like?

Exercise Find the position of the center of mass of the system of fragments relative to the ground 3.00 s after the explosion. Assume the rocket engine is nonoperative after the explosion.

Answer The x coordinate does not change; $y_{CM} = 1.86$ km.

Optional Section

9.8 ROCKET PROPULSION

When ordinary vehicles, such as automobiles and locomotives, are propelled, the driving force for the motion is friction. In the case of the automobile, the driving force is the force exerted by the road on the car. A locomotive "pushes" against the tracks; hence, the driving force is the force exerted by the tracks on the locomotive. However, a rocket moving in space has no road or tracks to push against. Therefore, the source of the propulsion of a rocket must be something other than friction. Figure 9.28 is a dramatic photograph of a spacecraft at liftoff. **The operation of a rocket depends upon the law of conservation of linear momentum as applied to a system of particles, where the system is the rocket plus its ejected fuel.**

Rocket propulsion can be understood by first considering the mechanical system consisting of a machine gun mounted on a cart on wheels. As the gun is fired,

Figure 9.28 Liftoff of the space shuttle *Columbia*. Enormous thrust is generated by the shuttle's liquid-fuel engines, aided by the two solid-fuel boosters. Many physical principles from mechanics, thermodynamics, and electricity and magnetism are involved in such a launch.

The force from a nitrogen-propelled, hand-controlled device allows an astronaut to move about freely in space without restrictive tethers.

each bullet receives a momentum $m\mathbf{v}$ in some direction, where $\mathbf{v}$ is measured with respect to a stationary Earth frame. The momentum of the system made up of cart, gun, and bullets must be conserved. Hence, for each bullet fired, the gun and cart must receive a compensating momentum in the opposite direction. That is, the reaction force exerted by the bullet on the gun accelerates the cart and gun, and the cart moves in the direction opposite that of the bullets. If n is the number of bullets fired each second, then the average force exerted on the gun is $\mathbf{F}_{\text{av}} = nm\mathbf{v}$.

In a similar manner, as a rocket moves in free space, its linear momentum changes when some of its mass is released in the form of ejected gases. **Because the gases are given momentum when they are ejected out of the engine, the rocket receives a compensating momentum in the opposite direction.** Therefore, the rocket is accelerated as a result of the "push," or thrust, from the exhaust gases. In free space, the center of mass of the system (rocket plus expelled gases) moves uniformly, independent of the propulsion process.[5]

Suppose that at some time t, the magnitude of the momentum of a rocket plus its fuel is $(M + \Delta m)v$, where v is the speed of the rocket relative to the Earth (Fig. 9.29a). Over a short time interval Δt, the rocket ejects fuel of mass Δm, and so at the end of this interval the rocket's speed is $v + \Delta v$, where Δv is the change in speed of the rocket (Fig. 9.29b). If the fuel is ejected with a speed v_e relative to the rocket (the subscript "e" stands for *exhaust*, and v_e is usually called the *exhaust speed*), the velocity of the fuel relative to a stationary frame of reference is $v - v_e$. Thus, if we equate the total initial momentum of the system to the total final momentum, we obtain

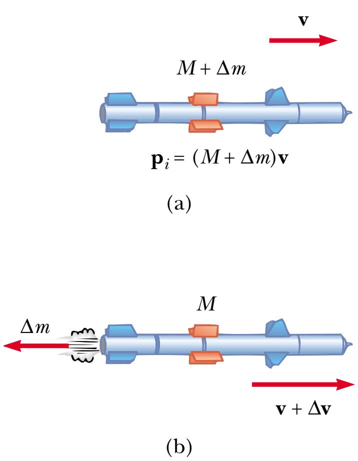


Figure 9.29 Rocket propulsion. (a) The initial mass of the rocket plus all its fuel is $M + \Delta m$ at a time t, and its speed is v. (b) At a time $t + \Delta t$, the rocket's mass has been reduced to M and an amount of fuel Δm has been ejected. The rocket's speed increases by an amount Δv.

$$(M + \Delta m)v = M(v + \Delta v) + \Delta m(v - v_e)$$

where M represents the mass of the rocket and its remaining fuel after an amount of fuel having mass Δm has been ejected. Simplifying this expression gives

$$M\,\Delta v = v_e\,\Delta m$$

We also could have arrived at this result by considering the system in the center-of-mass frame of reference, which is a frame having the same velocity as the center of mass of the system. In this frame, the total momentum of the system is zero; therefore, if the rocket gains a momentum $M\,\Delta v$ by ejecting some fuel, the exhausted fuel obtains a momentum $v_e\,\Delta m$ in the *opposite* direction, so that $M\,\Delta v - v_e\,\Delta m = 0$. If we now take the limit as Δt goes to zero, we get $\Delta v \rightarrow dv$ and $\Delta m \rightarrow dm$. Futhermore, the increase in the exhaust mass dm corresponds to an equal decrease in the rocket mass, so that $dm = -dM$. Note that dM is given a negative sign because it represents a decrease in mass. Using this fact, we obtain

$$M\,dv = v_e\,dm = -v_e\,dM \qquad \textbf{(9.40)}$$

Integrating this equation and taking the initial mass of the rocket plus fuel to be M_i and the final mass of the rocket plus its remaining fuel to be M_f, we obtain

$$\int_{v_i}^{v_f} dv = -v_e \int_{M_i}^{M_f} \frac{dM}{M}$$

Expression for rocket propulsion

$$v_f - v_i = v_e \ln\left(\frac{M_i}{M_f}\right) \qquad \textbf{(9.41)}$$

[5]It is interesting to note that the rocket and machine gun represent cases of the reverse of a perfectly inelastic collision: Momentum is conserved, but the kinetic energy of the system increases (at the expense of chemical potential energy in the fuel).

This is the basic expression of rocket propulsion. First, it tells us that the increase in rocket speed is proportional to the exhaust speed of the ejected gases, v_e. Therefore, the exhaust speed should be very high. Second, the increase in rocket speed is proportional to the natural logarithm of the ratio M_i/M_f. Therefore, this ratio should be as large as possible, which means that the mass of the rocket without its fuel should be as small as possible and the rocket should carry as much fuel as possible.

The **thrust** on the rocket is the force exerted on it by the ejected exhaust gases. We can obtain an expression for the thrust from Equation 9.40:

$$\text{Thrust} = M\frac{dv}{dt} = \left| v_e \frac{dM}{dt} \right| \tag{9.42}$$

This expression shows us that the thrust increases as the exhaust speed increases and as the rate of change of mass (called the *burn rate*) increases.

EXAMPLE 9.18 A Rocket in Space

A rocket moving in free space has a speed of 3.0×10^3 m/s relative to the Earth. Its engines are turned on, and fuel is ejected in a direction opposite the rocket's motion at a speed of 5.0×10^3 m/s relative to the rocket. (a) What is the speed of the rocket relative to the Earth once the rocket's mass is reduced to one-half its mass before ignition?

Solution We can guess that the speed we are looking for must be greater than the original speed because the rocket is accelerating. Applying Equation 9.41, we obtain

$$v_f = v_i + v_e \ln\left(\frac{M_i}{M_f}\right)$$

$$= 3.0 \times 10^3 \text{ m/s} + (5.0 \times 10^3 \text{ m/s})\ln\left(\frac{M_i}{0.5\, M_i}\right)$$

$$= \boxed{6.5 \times 10^3 \text{ m/s}}$$

(b) What is the thrust on the rocket if it burns fuel at the rate of 50 kg/s?

Solution

$$\text{Thrust} = \left| v_e \frac{dM}{dt} \right| = (5.0 \times 10^3 \text{ m/s})(50 \text{ kg/s})$$

$$= \boxed{2.5 \times 10^5 \text{ N}}$$

EXAMPLE 9.19 Fighting a Fire

Two firefighters must apply a total force of 600 N to steady a hose that is discharging water at 3 600 L/min. Estimate the speed of the water as it exits the nozzle.

Solution The water is exiting at 3 600 L/min, which is 60 L/s. Knowing that 1 L of water has a mass of 1 kg, we can say that about 60 kg of water leaves the nozzle every second. As the water leaves the hose, it exerts on the hose a thrust that must be counteracted by the 600-N force exerted on the hose by the firefighters. So, applying Equation 9.42 gives

$$\text{Thrust} = \left| v_e \frac{dM}{dt} \right|$$

$$600 \text{ N} = |v_e(60 \text{ kg/s})|$$

$$v_e = \boxed{10 \text{ m/s}}$$

Firefighting is dangerous work. If the nozzle should slip from their hands, the movement of the hose due to the thrust it receives from the rapidly exiting water could injure the firefighters.

Firefighters attack a burning house with a hose line.

SUMMARY

The **linear momentum p** of a particle of mass m moving with a velocity **v** is

$$\mathbf{p} \equiv m\mathbf{v} \tag{9.1}$$

The law of **conservation of linear momentum** indicates that the total momentum of an isolated system is conserved. If two particles form an isolated system, their total momentum is conserved regardless of the nature of the force between them. Therefore, the total momentum of the system at all times equals its initial total momentum, or

$$\mathbf{p}_{1i} + \mathbf{p}_{2i} = \mathbf{p}_{1f} + \mathbf{p}_{2f} \tag{9.5}$$

The **impulse** imparted to a particle by a force **F** is equal to the change in the momentum of the particle:

$$\mathbf{I} \equiv \int_{t_i}^{t_f} \mathbf{F}\, dt = \Delta\mathbf{p} \tag{9.9}$$

This is known as the **impulse–momentum theorem.**

Impulsive forces are often very strong compared with other forces on the system and usually act for a very short time, as in the case of collisions.

When two particles collide, the total momentum of the system before the collision always equals the total momentum after the collision, regardless of the nature of the collision. An **inelastic collision** is one for which the total kinetic energy is not conserved. A **perfectly inelastic collision** is one in which the colliding bodies stick together after the collision. An **elastic collision** is one in which kinetic energy is constant.

In a two- or three-dimensional collision, the components of momentum in each of the three directions (x, y, and z) are conserved independently.

The position vector of the center of mass of a system of particles is defined as

$$\mathbf{r}_{\text{CM}} \equiv \frac{\sum_i m_i \mathbf{r}_i}{M} \tag{9.30}$$

where $M = \sum_i m_i$ is the total mass of the system and $\mathbf{r}_i$ is the position vector of the ith particle.

The position vector of the center of mass of a rigid body can be obtained from the integral expression

$$\mathbf{r}_{\text{CM}} = \frac{1}{M}\int \mathbf{r}\, dm \tag{9.33}$$

The velocity of the center of mass for a system of particles is

$$\mathbf{v}_{\text{CM}} = \frac{\sum_i m_i \mathbf{v}_i}{M} \tag{9.34}$$

The total momentum of a system of particles equals the total mass multiplied by the velocity of the center of mass.

Newton's second law applied to a system of particles is

$$\sum \mathbf{F}_{\text{ext}} = M\mathbf{a}_{\text{CM}} = \frac{d\mathbf{p}_{\text{tot}}}{dt} \tag{9.38}$$

where $\mathbf{a}_{\text{CM}}$ is the acceleration of the center of mass and the sum is over all external forces. The center of mass moves like an imaginary particle of mass M under the influence of the resultant external force on the system. It follows from Equation 9.38 that the total momentum of the system is conserved if there are no external forces acting on it.

QUESTIONS

1. If the kinetic energy of a particle is zero, what is its linear momentum?
2. If the speed of a particle is doubled, by what factor is its momentum changed? By what factor is its kinetic energy changed?
3. If two particles have equal kinetic energies, are their momenta necessarily equal? Explain.
4. If two particles have equal momenta, are their kinetic energies necessarily equal? Explain.
5. An isolated system is initially at rest. Is it possible for parts of the system to be in motion at some later time? If so, explain how this might occur.
6. If two objects collide and one is initially at rest, is it possible for both to be at rest after the collision? Is it possible for one to be at rest after the collision? Explain.
7. Explain how linear momentum is conserved when a ball bounces from a floor.
8. Is it possible to have a collision in which all of the kinetic energy is lost? If so, cite an example.
9. In a perfectly elastic collision between two particles, does the kinetic energy of each particle change as a result of the collision?
10. When a ball rolls down an incline, its linear momentum increases. Does this imply that momentum is not conserved? Explain.
11. Consider a perfectly inelastic collision between a car and a large truck. Which vehicle loses more kinetic energy as a result of the collision?
12. Can the center of mass of a body lie outside the body? If so, give examples.
13. Three balls are thrown into the air simultaneously. What is the acceleration of their center of mass while they are in motion?
14. A meter stick is balanced in a horizontal position with the index fingers of the right and left hands. If the two fingers are slowly brought together, the stick remains balanced and the two fingers always meet at the 50-cm mark regardless of their original positions (try it!). Explain.
15. A sharpshooter fires a rifle while standing with the butt of the gun against his shoulder. If the forward momentum of a bullet is the same as the backward momentum of the gun, why is it not as dangerous to be hit by the gun as by the bullet?
16. A piece of mud is thrown against a brick wall and sticks to the wall. What happens to the momentum of the mud? Is momentum conserved? Explain.
17. Early in this century, Robert Goddard proposed sending a rocket to the Moon. Critics took the position that in a vacuum, such as exists between the Earth and the Moon, the gases emitted by the rocket would have nothing to push against to propel the rocket. According to *Scientific American* (January 1975), Goddard placed a gun in a vacuum and fired a blank cartridge from it. (A blank cartridge fires only the wadding and hot gases of the burning gunpowder.) What happened when the gun was fired?
18. A pole-vaulter falls from a height of 6.0 m onto a foam rubber pad. Can you calculate his speed just before he reaches the pad? Can you estimate the force exerted on him due to the collision? Explain.
19. Explain how you would use a balloon to demonstrate the mechanism responsible for rocket propulsion.
20. Does the center of mass of a rocket in free space accelerate? Explain. Can the speed of a rocket exceed the exhaust speed of the fuel? Explain.
21. A ball is dropped from a tall building. Identify the system for which linear momentum is conserved.
22. A bomb, initially at rest, explodes into several pieces. (a) Is linear momentum conserved? (b) Is kinetic energy conserved? Explain.
23. NASA often uses the gravity of a planet to "slingshot" a probe on its way to a more distant planet. This is actually a collision where the two objects do not touch. How can the probe have its speed increased in this manner?
24. The Moon revolves around the Earth. Is the Moon's linear momentum conserved? Is its kinetic energy conserved? Assume that the Moon's orbit is circular.
25. A raw egg dropped to the floor breaks apart upon impact. However, a raw egg dropped onto a thick foam rubber cushion from a height of about 1 m rebounds without breaking. Why is this possible? (If you try this experiment, be sure to catch the egg after the first bounce.)
26. On the subject of the following positions, state your own view and argue to support it: (a) The best theory of motion is that force causes acceleration. (b) The true measure of a force's effectiveness is the work it does, and the best theory of motion is that work on an object changes its energy. (c) The true measure of a force's effect is impulse, and the best theory of motion is that impulse injected into an object changes its momentum.

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging □ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics
□ = paired numerical/symbolic problems

Section 9.1 Linear Momentum and Its Conservation

1. A 3.00-kg particle has a velocity of $(3.00\mathbf{i} - 4.00\mathbf{j})$ m/s. (a) Find its x and y components of momentum. (b) Find the magnitude and direction of its momentum.
2. A 0.100-kg ball is thrown straight up into the air with an initial speed of 15.0 m/s. Find the momentum of the ball (a) at its maximum height and (b) halfway up to its maximum height.
3. A 40.0-kg child standing on a frozen pond throws a 0.500-kg stone to the east with a speed of 5.00 m/s. Neglecting friction between child and ice, find the recoil velocity of the child.
4. A pitcher claims he can throw a baseball with as much momentum as a 3.00-g bullet moving with a speed of 1 500 m/s. A baseball has a mass of 0.145 kg. What must be its speed if the pitcher's claim is valid?
5. How fast can you set the Earth moving? In particular, when you jump straight up as high as you can, you give the Earth a maximum recoil speed of what order of magnitude? Model the Earth as a perfectly solid object. In your solution, state the physical quantities you take as data and the values you measure or estimate for them.
6. Two blocks of masses M and $3M$ are placed on a horizontal, frictionless surface. A light spring is attached to one of them, and the blocks are pushed together with the spring between them (Fig. P9.6). A cord initially holding the blocks together is burned; after this, the block of mass $3M$ moves to the right with a speed of 2.00 m/s. (a) What is the speed of the block of mass M? (b) Find the original elastic energy in the spring if M = 0.350 kg.

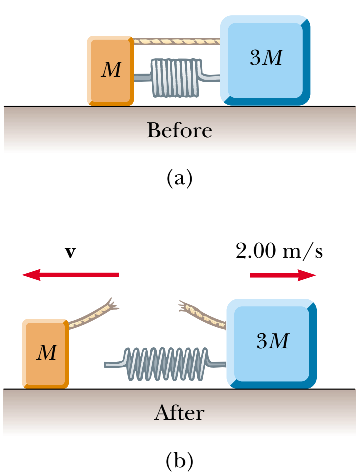


Figure P9.6

7. (a) A particle of mass m moves with momentum p. Show that the kinetic energy of the particle is given by $K = p^2/2m$. (b) Express the magnitude of the particle's momentum in terms of its kinetic energy and mass.

Section 9.2 Impulse and Momentum

8. A car is stopped for a traffic signal. When the light turns green, the car accelerates, increasing its speed from zero to 5.20 m/s in 0.832 s. What linear impulse and average force does a 70.0-kg passenger in the car experience?
9. An estimated force–time curve for a baseball struck by a bat is shown in Figure P9.9. From this curve, determine (a) the impulse delivered to the ball, (b) the average force exerted on the ball, and (c) the peak force exerted on the ball.

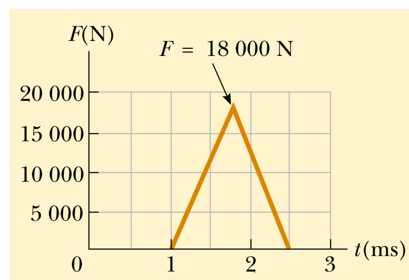


Figure P9.9

10. A tennis player receives a shot with the ball (0.060 0 kg) traveling horizontally at 50.0 m/s and returns the shot with the ball traveling horizontally at 40.0 m/s in the opposite direction. (a) What is the impulse delivered to the ball by the racket? (b) What work does the racket do on the ball?
11. WEB A 3.00-kg steel ball strikes a wall with a speed of 10.0 m/s at an angle of 60.0° with the surface. It bounces off with the same speed and angle (Fig. P9.11). If the ball is in contact with the wall for 0.200 s, what is the average force exerted on the ball by the wall?
12. In a slow-pitch softball game, a 0.200-kg softball crossed the plate at 15.0 m/s at an angle of 45.0° below the horizontal. The ball was hit at 40.0 m/s, 30.0° above the horizontal. (a) Determine the impulse delivered to the ball. (b) If the force on the ball increased linearly for 4.00 ms, held constant for 20.0 ms, and then decreased to zero linearly in another 4.00 ms, what was the maximum force on the ball?

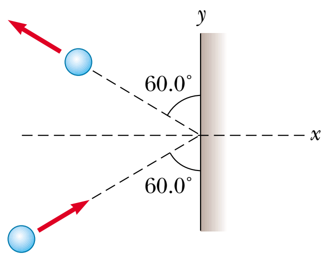


Figure P9.11

13. A garden hose is held in the manner shown in Figure P9.13. The hose is initially full of motionless water. What additional force is necessary to hold the nozzle stationary after the water is turned on if the discharge rate is 0.600 kg/s with a speed of 25.0 m/s?

Figure P9.13

14. A professional diver performs a dive from a platform 10 m above the water surface. Estimate the order of magnitude of the average impact force she experiences in her collision with the water. State the quantities you take as data and their values.

Section 9.3 Collisions

Section 9.4 Elastic and Inelastic Collisions in One Dimension

15. High-speed stroboscopic photographs show that the head of a golf club of mass 200 g is traveling at 55.0 m/s just before it strikes a 46.0-g golf ball at rest on a tee. After the collision, the club head travels (in the same direction) at 40.0 m/s. Find the speed of the golf ball just after impact.

16. A 75.0-kg ice skater, moving at 10.0 m/s, crashes into a stationary skater of equal mass. After the collision, the two skaters move as a unit at 5.00 m/s. Suppose the average force a skater can experience without breaking a bone is 4 500 N. If the impact time is 0.100 s, does a bone break?

$\boxed{\textbf{17.}}$ A 10.0-g bullet is fired into a stationary block of wood ($m = 5.00$ kg). The relative motion of the bullet stops inside the block. The speed of the bullet-plus-wood combination immediately after the collision is measured as 0.600 m/s. What was the original speed of the bullet?

18. As shown in Figure P9.18, a bullet of mass m and speed v passes completely through a pendulum bob of mass M. The bullet emerges with a speed of $v/2$. The pendulum bob is suspended by a stiff rod of length ℓ and negligible mass. What is the minimum value of v such that the pendulum bob will barely swing through a complete vertical circle?

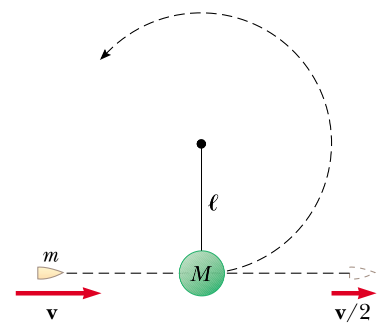


Figure P9.18

$\boxed{\textbf{19.}}$ A 45.0-kg girl is standing on a plank that has a mass of 150 kg. The plank, originally at rest, is free to slide on a frozen lake, which is a flat, frictionless supporting surface. The girl begins to walk along the plank at a constant speed of 1.50 m/s relative to the plank. (a) What is her speed relative to the ice surface? (b) What is the speed of the plank relative to the ice surface?

20. Gayle runs at a speed of 4.00 m/s and dives on a sled, which is initially at rest on the top of a frictionless snow-covered hill. After she has descended a vertical distance of 5.00 m, her brother, who is initially at rest, hops on her back and together they continue down the hill. What is their speed at the bottom of the hill if the total vertical drop is 15.0 m? Gayle's mass is 50.0 kg, the sled has a mass of 5.00 kg and her brother has a mass of 30.0 kg.

21. A 1 200-kg car traveling initially with a speed of 25.0 m/s in an easterly direction crashes into the rear end of a 9 000-kg truck moving in the same direction at 20.0 m/s (Fig. P9.21). The velocity of the car right after the collision is 18.0 m/s to the east. (a) What is the velocity of the truck right after the collision? (b) How much mechanical energy is lost in the collision? Account for this loss in energy.

22. A railroad car of mass 2.50×10^4 kg is moving with a speed of 4.00 m/s. It collides and couples with three other coupled railroad cars, each of the same mass as the single car and moving in the same direction with an initial speed of 2.00 m/s. (a) What is the speed of the four cars after the collision? (b) How much energy is lost in the collision?

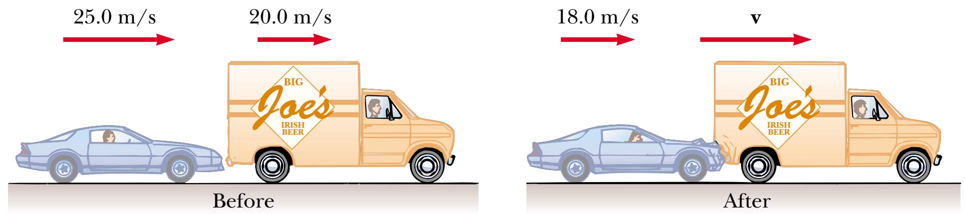


Figure P9.21

23. Four railroad cars, each of mass 2.50×10^4 kg, are coupled together and coasting along horizontal tracks at a speed of v_i toward the south. A very strong but foolish movie actor, riding on the second car, uncouples the front car and gives it a big push, increasing its speed to 4.00 m/s southward. The remaining three cars continue moving toward the south, now at 2.00 m/s. (a) Find the initial speed of the cars. (b) How much work did the actor do? (c) State the relationship between the process described here and the process in Problem 22.

24. A 7.00-kg bowling ball collides head-on with a 2.00-kg bowling pin. The pin flies forward with a speed of 3.00 m/s. If the ball continues forward with a speed of 1.80 m/s, what was the initial speed of the ball? Ignore rotation of the ball.

WEB **25.** A neutron in a reactor makes an elastic head-on collision with the nucleus of a carbon atom initially at rest. (a) What fraction of the neutron's kinetic energy is transferred to the carbon nucleus? (b) If the initial kinetic energy of the neutron is 1.60×10^{-13} J, find its final kinetic energy and the kinetic energy of the carbon nucleus after the collision. (The mass of the carbon nucleus is about 12.0 times greater than the mass of the neutron.)

26. Consider a frictionless track *ABC* as shown in Figure P9.26. A block of mass $m_1 = 5.00$ kg is released from *A*. It makes a head-on elastic collision at *B* with a block of mass $m_2 = 10.0$ kg that is initially at rest. Calculate the maximum height to which m_1 rises after the collision.

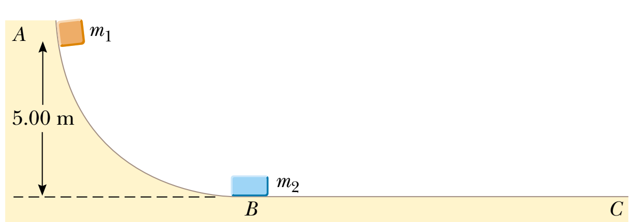


Figure P9.26

WEB **27.** A 12.0-g bullet is fired into a 100-g wooden block initially at rest on a horizontal surface. After impact, the block slides 7.50 m before coming to rest. If the coefficient of friction between the block and the surface is 0.650, what was the speed of the bullet immediately before impact?

28. A 7.00-g bullet, when fired from a gun into a 1.00-kg block of wood held in a vise, would penetrate the block to a depth of 8.00 cm. This block of wood is placed on a frictionless horizontal surface, and a 7.00-g bullet is fired from the gun into the block. To what depth will the bullet penetrate the block in this case?

Section 9.5 Two-Dimensional Collisions

29. A 90.0-kg fullback running east with a speed of 5.00 m/s is tackled by a 95.0-kg opponent running north with a speed of 3.00 m/s. If the collision is perfectly inelastic, (a) calculate the speed and direction of the players just after the tackle and (b) determine the energy lost as a result of the collision. Account for the missing energy.

30. The mass of the blue puck in Figure P9.30 is 20.0% greater than the mass of the green one. Before colliding, the pucks approach each other with equal and opposite momenta, and the green puck has an initial speed of 10.0 m/s. Find the speeds of the pucks after the collision if half the kinetic energy is lost during the collision.

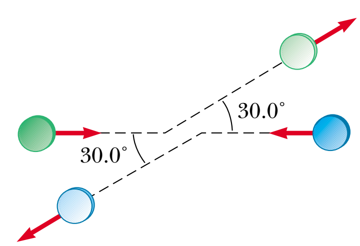


Figure P9.30

31. Two automobiles of equal mass approach an intersection. One vehicle is traveling with velocity 13.0 m/s toward the east and the other is traveling north with a speed of v_{2i}. Neither driver sees the other. The vehicles collide in the intersection and stick together, leaving parallel skid marks at an angle of 55.0° north of east. The speed limit for both roads is 35 mi/h, and the driver of the northward-moving vehicle claims he was within the speed limit when the collision occurred. Is he telling the truth?

32. A proton, moving with a velocity of $v_i\mathbf{i}$, collides elastically with another proton that is initially at rest. If the two protons have equal speeds after the collision, find (a) the speed of each proton after the collision in terms of v_i and (b) the direction of the velocity vectors after the collision.

33. A billiard ball moving at 5.00 m/s strikes a stationary ball of the same mass. After the collision, the first ball moves at 4.33 m/s and at an angle of 30.0° with respect to the original line of motion. Assuming an elastic collision (and ignoring friction and rotational motion), find the struck ball's velocity.

34. A 0.300-kg puck, initially at rest on a horizontal, frictionless surface, is struck by a 0.200-kg puck moving initially along the x axis with a speed of 2.00 m/s. After the collision, the 0.200-kg puck has a speed of 1.00 m/s at an angle of $\theta = 53.0°$ to the positive x axis (see Fig. 9.14). (a) Determine the velocity of the 0.300-kg puck after the collision. (b) Find the fraction of kinetic energy lost in the collision.

35. A 3.00-kg mass with an initial velocity of $5.00\mathbf{i}$ m/s collides with and sticks to a 2.00-kg mass with an initial velocity of $-3.00\mathbf{j}$ m/s. Find the final velocity of the composite mass.

36. Two shuffleboard disks of equal mass, one orange and the other yellow, are involved in an elastic, glancing collision. The yellow disk is initially at rest and is struck by the orange disk moving with a speed of 5.00 m/s. After the collision, the orange disk moves along a direction that makes an angle of 37.0° with its initial direction of motion, and the velocity of the yellow disk is perpendicular to that of the orange disk (after the collision). Determine the final speed of each disk.

37. Two shuffleboard disks of equal mass, one orange and the other yellow, are involved in an elastic, glancing collision. The yellow disk is initially at rest and is struck by the orange disk moving with a speed v_i. After the collision, the orange disk moves along a direction that makes an angle θ with its initial direction of motion, and the velocity of the yellow disk is perpendicular to that of the orange disk (after the collision). Determine the final speed of each disk.

38. During the battle of Gettysburg, the gunfire was so intense that several bullets collided in midair and fused together. Assume a 5.00-g Union musket ball was moving to the right at a speed of 250 m/s, 20.0° above the horizontal, and that a 3.00-g Confederate ball was moving to the left at a speed of 280 m/s, 15.0° above the horizontal. Immediately after they fuse together, what is their velocity?

WEB 39. An unstable nucleus of mass 17.0×10^{-27} kg initially at rest disintegrates into three particles. One of the particles, of mass 5.00×10^{-27} kg, moves along the y axis with a velocity of 6.00×10^6 m/s. Another particle, of mass 8.40×10^{-27} kg, moves along the x axis with a speed of 4.00×10^6 m/s. Find (a) the velocity of the third particle and (b) the total kinetic energy increase in the process.

Section 9.6 The Center of Mass

40. Four objects are situated along the y axis as follows: A 2.00-kg object is at $+3.00$ m, a 3.00-kg object is at $+2.50$ m, a 2.50-kg object is at the origin, and a 4.00-kg object is at -0.500 m. Where is the center of mass of these objects?

41. A uniform piece of sheet steel is shaped as shown in Figure P9.41. Compute the x and y coordinates of the center of mass of the piece.

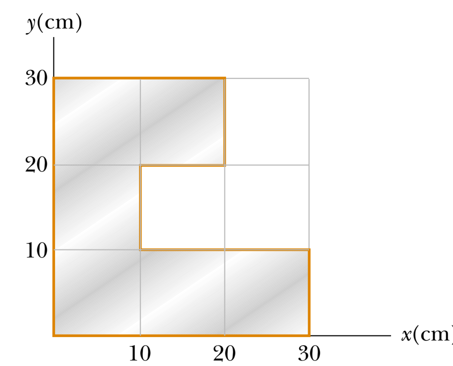


Figure P9.41

42. The mass of the Earth is 5.98×10^{24} kg, and the mass of the Moon is 7.36×10^{22} kg. The distance of separation, measured between their centers, is 3.84×10^8 m. Locate the center of mass of the Earth–Moon system as measured from the center of the Earth.

43. A water molecule consists of an oxygen atom with two hydrogen atoms bound to it (Fig. P9.43). The angle between the two bonds is 106°. If the bonds are 0.100 nm long, where is the center of mass of the molecule?

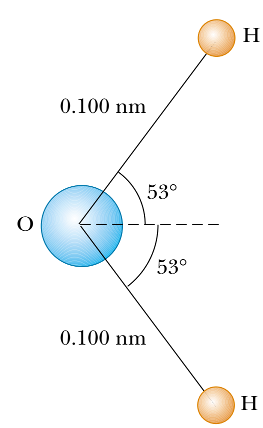


Figure P9.43

44. A 0.400-kg mass m_1 has position $\mathbf{r}_1 = 12.0\mathbf{j}$ cm. A 0.800-kg mass m_2 has position $\mathbf{r}_2 = -12.0\mathbf{i}$ cm. Another 0.800-kg mass m_3 has position $\mathbf{r}_3 = (12.0\mathbf{i} - 12.0\mathbf{j})$ cm. Make a drawing of the masses. Start from the origin and, to the scale 1 cm = 1 kg·cm, construct the vector $m_1\mathbf{r}_1$, then the vector $m_1\mathbf{r}_1 + m_2\mathbf{r}_2$, then the vector $m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3$, and at last $\mathbf{r}_{CM} = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3)/(m_1 + m_2 + m_3)$. Observe that the head of the vector $\mathbf{r}_{CM}$ indicates the position of the center of mass.

45. A rod of length 30.0 cm has linear density (mass-per-length) given by

$$\lambda = 50.0 \text{ g/m} + 20.0x \text{ g/m}^2$$

where x is the distance from one end, measured in meters. (a) What is the mass of the rod? (b) How far from the $x = 0$ end is its center of mass?

Section 9.7 Motion of a System of Particles

46. Consider a system of two particles in the xy plane: $m_1 = 2.00$ kg is at $\mathbf{r}_1 = (1.00\mathbf{i} + 2.00\mathbf{j})$ m and has velocity $(3.00\mathbf{i} + 0.500\mathbf{j})$ m/s; $m_2 = 3.00$ kg is at $\mathbf{r}_2 = (-4.00\mathbf{i} - 3.00\mathbf{j})$ m and has velocity $(3.00\mathbf{i} - 2.00\mathbf{j})$ m/s. (a) Plot these particles on a grid or graph paper. Draw their position vectors and show their velocities. (b) Find the position of the center of mass of the system and mark it on the grid. (c) Determine the velocity of the center of mass and also show it on the diagram. (d) What is the total linear momentum of the system?

47. Romeo (77.0 kg) entertains Juliet (55.0 kg) by playing his guitar from the rear of their boat at rest in still water, 2.70 m away from Juliet who is in the front of the boat. After the serenade, Juliet carefully moves to the rear of the boat (away from shore) to plant a kiss on Romeo's cheek. How far does the 80.0-kg boat move toward the shore it is facing?

48. Two masses, 0.600 kg and 0.300 kg, begin uniform motion at the same speed, 0.800 m/s, from the origin at $t = 0$ and travel in the directions shown in Figure P9.48. (a) Find the velocity of the center of mass in unit–vector notation. (b) Find the magnitude and direction of the velocity of the center of mass. (c) Write the position vector of the center of mass as a function of time.

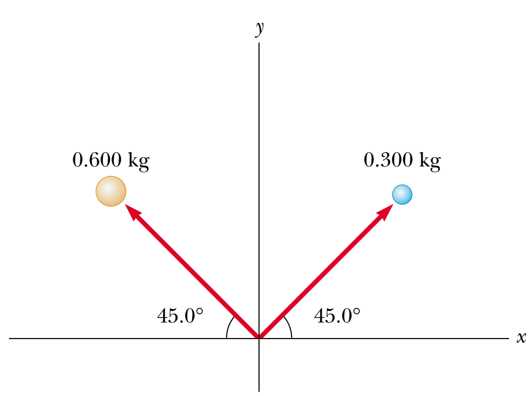


Figure P9.48

49. A 2.00-kg particle has a velocity of $(2.00\mathbf{i} - 3.00\mathbf{j})$ m/s, and a 3.00-kg particle has a velocity of $(1.00\mathbf{i} + 6.00\mathbf{j})$ m/s. Find (a) the velocity of the center of mass and (b) the total momentum of the system.

50. A ball of mass 0.200 kg has a velocity of $1.50\mathbf{i}$ m/s; a ball of mass 0.300 kg has a velocity of $-0.400\mathbf{i}$ m/s. They meet in a head-on elastic collision. (a) Find their velocities after the collision. (b) Find the velocity of their center of mass before and after the collision.

(Optional)

Section 9.8 Rocket Propulsion

WEB **51.** The first stage of a Saturn V space vehicle consumes fuel and oxidizer at the rate of 1.50×10^4 kg/s, with an exhaust speed of 2.60×10^3 m/s. (a) Calculate the thrust produced by these engines. (b) Find the initial acceleration of the vehicle on the launch pad if its initial mass is 3.00×10^6 kg. [*Hint:* You must include the force of gravity to solve part (b).]

52. A large rocket with an exhaust speed of $v_e = 3\,000$ m/s develops a thrust of 24.0 million newtons. (a) How much mass is being blasted out of the rocket exhaust per second? (b) What is the maximum speed the rocket can attain if it starts from rest in a force-free environment with $v_e = 3.00$ km/s and if 90.0% of its initial mass is fuel and oxidizer?

53. A rocket for use in deep space is to have the capability of boosting a total load (payload plus rocket frame and engine) of 3.00 metric tons to a speed of 10 000 m/s. (a) It has an engine and fuel designed to produce an exhaust speed of 2 000 m/s. How much fuel plus oxidizer is required? (b) If a different fuel and engine design could give an exhaust speed of 5 000 m/s, what amount of fuel and oxidizer would be required for the same task?

54. A rocket car has a mass of 2 000 kg unfueled and a mass of 5 000 kg when completely fueled. The exhaust velocity is 2 500 m/s. (a) Calculate the amount of fuel used to accelerate the completely fueled car from rest to 225 m/s (about 500 mi/h). (b) If the burn rate is constant at 30.0 kg/s, calculate the time it takes the car to reach this speed. Neglect friction and air resistance.

ADDITIONAL PROBLEMS

55. **Review Problem.** A 60.0-kg person running at an initial speed of 4.00 m/s jumps onto a 120-kg cart initially at rest (Fig. P9.55). The person slides on the cart's top surface and finally comes to rest relative to the cart. The coefficient of kinetic friction between the person and the cart is 0.400. Friction between the cart and ground can be neglected. (a) Find the final velocity of the person and cart relative to the ground. (b) Find the frictional force acting on the person while he is sliding across the top surface of the cart. (c) How long does the frictional force act on the person? (d) Find the change in momentum of the person and the change in momentum of the cart. (e) Determine the displacement of the person relative to the ground while he is sliding on the cart. (f) Determine the displacement of the cart relative to the ground while the person is sliding. (g) Find the change in kinetic energy of the person. (h) Find the change in kinetic energy of the cart. (i) Explain why the answers to parts (g) and (h) differ. (What kind of collision is this, and what accounts for the loss of mechanical energy?)

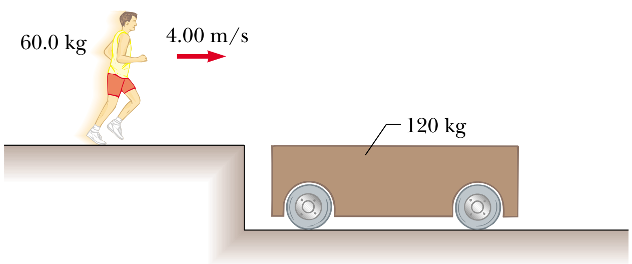


Figure P9.55

56. A golf ball ($m = 46.0$ g) is struck a blow that makes an angle of 45.0° with the horizontal. The ball lands 200 m away on a flat fairway. If the golf club and ball are in contact for 7.00 ms, what is the average force of impact? (Neglect air resistance.)

57. An 8.00-g bullet is fired into a 2.50-kg block that is initially at rest at the edge of a frictionless table of height 1.00 m (Fig. P9.57). The bullet remains in the block, and after impact the block lands 2.00 m from the bottom of the table. Determine the initial speed of the bullet.

58. A bullet of mass m is fired into a block of mass M that is initially at rest at the edge of a frictionless table of height h (see Fig. P9.57). The bullet remains in the block, and after impact the block lands a distance d from the bottom of the table. Determine the initial speed of the bullet.

59. An 80.0-kg astronaut is working on the engines of his ship, which is drifting through space with a constant velocity. The astronaut, wishing to get a better view of the Universe, pushes against the ship and much later finds himself 30.0 m behind the ship and at rest with respect to it. Without a thruster, the only way to return to the ship is to throw his 0.500-kg wrench directly away from the ship. If he throws the wrench with a speed of 20.0 m/s relative to the ship, how long does it take the astronaut to reach the ship?

60. A small block of mass $m_1 = 0.500$ kg is released from rest at the top of a curve-shaped frictionless wedge of mass $m_2 = 3.00$ kg, which sits on a frictionless horizontal surface, as shown in Figure P9.60a. When the block leaves the wedge, its velocity is measured to be 4.00 m/s to the right, as in Figure P9.60b. (a) What is the velocity of the wedge after the block reaches the horizontal surface? (b) What is the height h of the wedge?

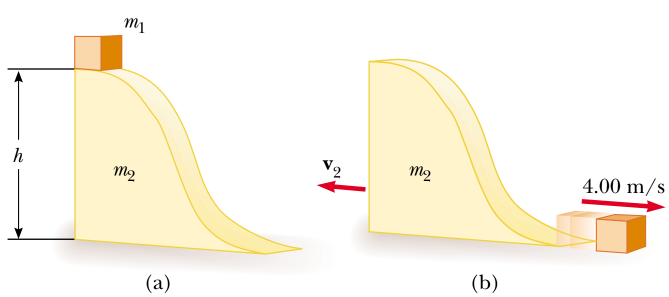


Figure P9.60

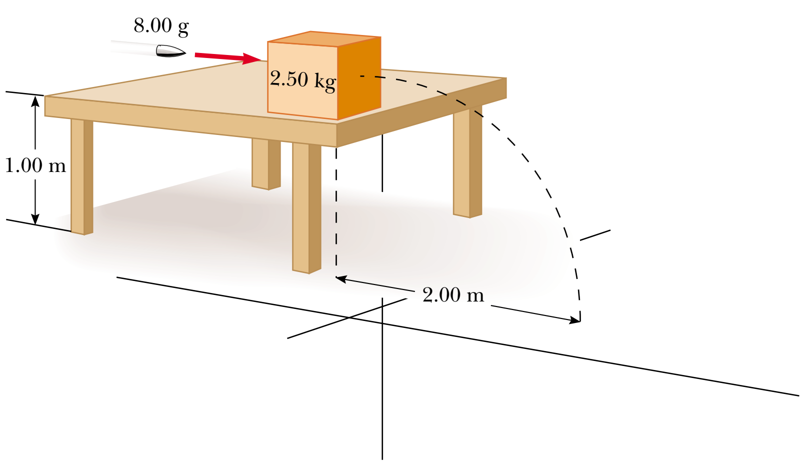


Figure P9.57 Problems 57 and 58.

61. Tarzan, whose mass is 80.0 kg, swings from a 3.00-m vine that is horizontal when he starts. At the bottom of his arc, he picks up 60.0-kg Jane in a perfectly inelastic collision. What is the height of the highest tree limb they can reach on their upward swing?

62. A jet aircraft is traveling at 500 mi/h (223 m/s) in horizontal flight. The engine takes in air at a rate of 80.0 kg/s and burns fuel at a rate of 3.00 kg/s. If the exhaust gases are ejected at 600 m/s relative to the aircraft, find the thrust of the jet engine and the delivered horsepower.

63. A 75.0-kg firefighter slides down a pole while a constant frictional force of 300 N retards her motion. A horizontal 20.0-kg platform is supported by a spring at the bottom of the pole to cushion the fall. The firefighter starts from rest 4.00 m above the platform, and the spring constant is 4 000 N/m. Find (a) the firefighter's speed just before she collides with the platform and (b) the maximum distance the spring is compressed. (Assume the frictional force acts during the entire motion.)

64. A cannon is rigidly attached to a carriage, which can move along horizontal rails but is connected to a post by a large spring, initially unstretched and with force constant $k = 2.00 \times 10^4$ N/m, as shown in Figure P9.64. The cannon fires a 200-kg projectile at a velocity of 125 m/s directed 45.0° above the horizontal. (a) If the mass of the cannon and its carriage is 5 000 kg, find the recoil speed of the cannon. (b) Determine the maximum extension of the spring. (c) Find the maximum force the spring exerts on the carriage. (d) Consider the system consisting of the cannon, carriage, and shell. Is the momentum of this system conserved during the firing? Why or why not?

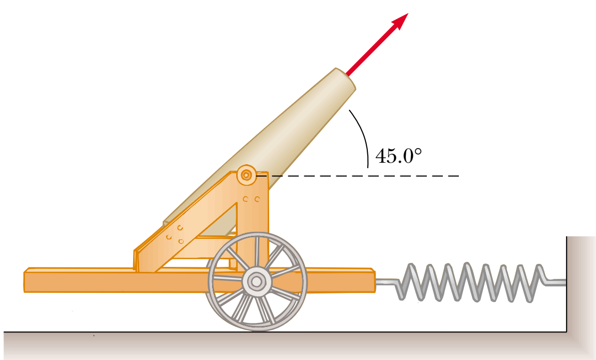


Figure P9.64

65. A chain of length L and total mass M is released from rest with its lower end just touching the top of a table, as shown in Figure P9.65a. Find the force exerted by the table on the chain after the chain has fallen through a distance x, as shown in Figure P9.65b. (Assume each link comes to rest the instant it reaches the table.)

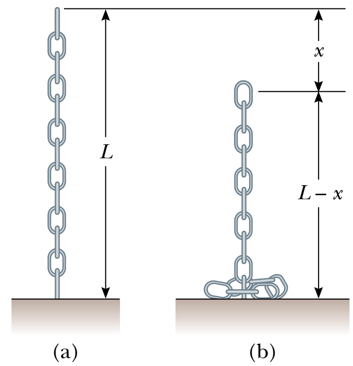


Figure P9.65

66. Two gliders are set in motion on an air track. A spring of force constant k is attached to the near side of one glider. The first glider of mass m_1 has a velocity of $\mathbf{v}_1$, and the second glider of mass m_2 has a velocity of $\mathbf{v}_2$, as shown in Figure P9.66 ($v_1 > v_2$). When m_1 collides with the spring attached to m_2 and compresses the spring to its maximum compression x_m, the velocity of the gliders is $\mathbf{v}$. In terms of $\mathbf{v}_1$, $\mathbf{v}_2$, m_1, m_2, and k, find (a) the velocity $\mathbf{v}$ at maximum compression, (b) the maximum compression x_m, and (c) the velocities of each glider after m_1 has lost contact with the spring.

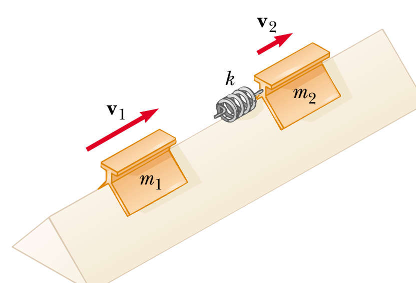


Figure P9.66

67. Sand from a stationary hopper falls onto a moving conveyor belt at the rate of 5.00 kg/s, as shown in Figure P9.67. The conveyor belt is supported by frictionless rollers and moves at a constant speed of 0.750 m/s under the action of a constant horizontal external force $\mathbf{F}_{ext}$ supplied by the motor that drives the belt. Find (a) the sand's rate of change of momentum in the horizontal direction, (b) the force of friction exerted by the belt on the sand, (c) the external force $\mathbf{F}_{ext}$, (d) the work done by $\mathbf{F}_{ext}$ in 1 s, and (e) the kinetic energy acquired by the falling sand each second due to the change in its horizontal motion. (f) Why are the answers to parts (d) and (e) different?

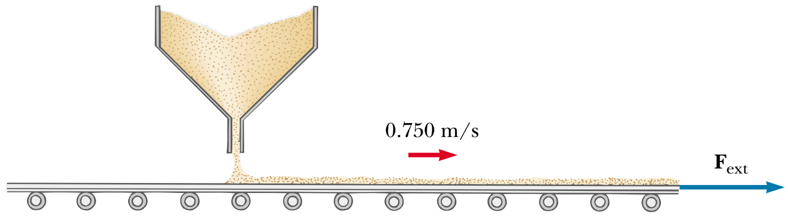


Figure P9.67

68. A rocket has total mass $M_i = 360$ kg, including 330 kg of fuel and oxidizer. In interstellar space it starts from rest, turns on its engine at time $t = 0$, and puts out exhaust with a relative speed of $v_e = 1\,500$ m/s at the constant rate $k = 2.50$ kg/s. Although the fuel will last for an actual burn time of 330 kg/(2.5 kg/s) = 132 s, define a "projected depletion time" as $T_p = M_i/k = 360\text{ kg}/(2.5\text{ kg/s}) = 144$ s. (This would be the burn time if the rocket could use its payload, fuel tanks, and even the walls of the combustion chamber as fuel.) (a) Show that during the burn the velocity of the rocket is given as a function of time by

$$v(t) = -v_e \ln(1 - t/T_p)$$

(b) Make a graph of the velocity of the rocket as a function of time for times running from 0 to 132 s. (c) Show that the acceleration of the rocket is

$$a(t) = v_e/(T_p - t)$$

(d) Graph the acceleration as a function of time. (e) Show that the displacement of the rocket from its initial position at $t = 0$ is

$$x(t) = v_e(T_p - t)\ln(1 - t/T_p) + v_e t$$

(f) Graph the displacement during the burn.

69. A 40.0-kg child stands at one end of a 70.0-kg boat that is 4.00 m in length (Fig. P9.69). The boat is initially 3.00 m from the pier. The child notices a turtle on a rock near the far end of the boat and proceeds to walk to that end to catch the turtle. Neglecting friction between the boat and the water, (a) describe the subsequent motion of the system (child plus boat). (b) Where is the child *relative to the pier* when he reaches the far end of the boat? (c) Will he catch the turtle? (Assume he can reach out 1.00 m from the end of the boat.)

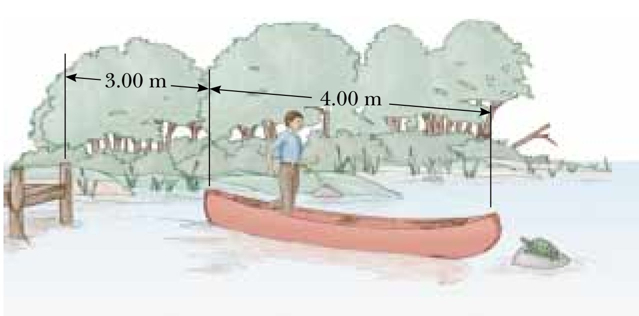


Figure P9.69

70. A student performs a ballistic pendulum experiment, using an apparatus similar to that shown in Figure 9.11b. She obtains the following average data: $h = 8.68$ cm, $m_1 = 68.8$ g, and $m_2 = 263$ g. The symbols refer to the quantities in Figure 9.11a. (a) Determine the initial speed v_{1i} of the projectile. (b) In the second part of her experiment she is to obtain v_{1i} by firing the same projectile horizontally (with the pendulum removed from the path) and measuring its horizontal displacement x and vertical displacement y (Fig. P9.70). Show that the initial speed of the projectile is related to x and y through the relationship

$$v_{1i} = \frac{x}{\sqrt{2y/g}}$$

What numerical value does she obtain for v_{1i} on the basis of her measured values of $x = 257$ cm and $y = 85.3$ cm? What factors might account for the difference in this value compared with that obtained in part (a)?

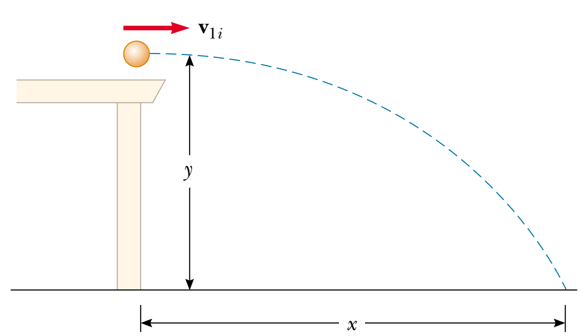


Figure P9.70

71. A 5.00-g bullet moving with an initial speed of 400 m/s is fired into and passes through a 1.00-kg block, as shown in Figure P9.71. The block, initially at rest on a frictionless, horizontal surface, is connected to a spring of force constant 900 N/m. If the block moves 5.00 cm to the right after impact, find (a) the speed at which the bullet emerges from the block and (b) the energy lost in the collision.

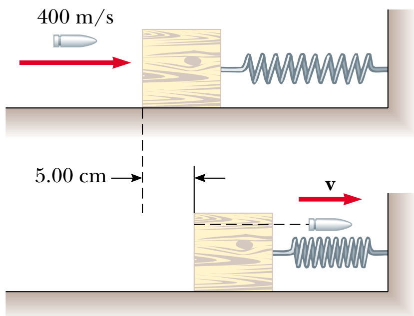


Figure P9.71

72. Two masses m and $3m$ are moving toward each other along the x axis with the same initial speeds v_i. Mass m is traveling to the left, while mass $3m$ is traveling to the right. They undergo a head-on elastic collision and each rebounds along the same line as it approached. Find the final speeds of the masses.

73. Two masses m and $3m$ are moving toward each other along the x axis with the same initial speeds v_i. Mass m is traveling to the left, while mass $3m$ is traveling to the right. They undergo an elastic glancing collision such that mass m is moving downward after the collision at right angles from its initial direction. (a) Find the final speeds of the two masses. (b) What is the angle θ at which the mass $3m$ is scattered?

74. Review Problem. There are (one can say) three co-equal theories of motion: Newton's second law, stating that the total force on an object causes its acceleration; the work–kinetic energy theorem, stating that the total work on an object causes its change in kinetic energy; and the impulse–momentum theorem, stating that the total impulse on an object causes its change in momentum. In this problem, you compare predictions of the three theories in one particular case. A 3.00-kg object has a velocity of 7.00$\mathbf{j}$ m/s. Then, a total force 12.0$\mathbf{i}$ N acts on the object for 5.00 s. (a) Calculate the object's final velocity, using the impulse–momentum theorem. (b) Calculate its acceleration from $\mathbf{a} = (\mathbf{v}_f - \mathbf{v}_i)/t$. (c) Calculate its acceleration from $\mathbf{a} = \Sigma\mathbf{F}/m$. (d) Find the object's vector displacement from $\mathbf{r} = \mathbf{v}_i t + \frac{1}{2}\mathbf{a}\,t^2$. (e) Find the work done on the object from $W = \mathbf{F}\cdot\mathbf{r}$. (f) Find the final kinetic energy from $\frac{1}{2}mv_f^2 = \frac{1}{2}m\mathbf{v}_f\cdot\mathbf{v}_f$. (g) Find the final kinetic energy from $\frac{1}{2}mv_i^2 + W$.

75. A rocket has a total mass of $M_i = 360$ kg, including 330 kg of fuel and oxidizer. In interstellar space it starts from rest. Its engine is turned on at time $t = 0$, and it puts out exhaust with a relative speed of $v_e = 1\,500$ m/s at the constant rate 2.50 kg/s. The burn lasts until the fuel runs out at time 330 kg/(2.5 kg/s) = 132 s. Set up and carry out a computer analysis of the motion according to Euler's method. Find (a) the final velocity of the rocket and (b) the distance it travels during the burn.

ANSWERS TO QUICK QUIZZES

9.1 (d). Two identical objects ($m_1 = m_2$) traveling in the same direction at the same speed ($v_1 = v_2$) have the same kinetic energies and the same momenta. However, this is not true if the two objects are moving at the same speed but in different directions. In the latter case, $K_1 = K_2$, but the differing velocity directions indicate that $\mathbf{p}_1 \neq \mathbf{p}_2$ because momentum is a vector quantity.

It also is possible for particular combinations of masses and velocities to satisfy $K_1 = K_2$ but not $p_1 = p_2$. For example, a 1-kg object moving at 2 m/s has the same kinetic energy as a 4-kg object moving at 1 m/s, but the two clearly do not have the same momenta.

9.2 (b), (c), (a). The slower the ball, the easier it is to catch. If the momentum of the medicine ball is the same as the momentum of the baseball, the speed of the medicine ball must be 1/10 the speed of the baseball because the medicine ball has 10 times the mass. If the kinetic energies are the same, the speed of the medicine ball must be $1/\sqrt{10}$ the speed of the baseball because of the squared speed term in the formula for K. The medicine ball is hardest to catch when it has the same speed as the baseball.

9.3 (c) and (e). Object 2 has a greater acceleration because of its smaller mass. Therefore, it takes less time to travel the distance d. Thus, even though the force applied to objects 1 and 2 is the same, the change in momentum is less for object 2 because Δt is smaller. Therefore, because the initial momenta were the same (both zero), $p_1 > p_2$. The work $W = Fd$ done on both objects is the same because both F and d are the same in the two cases. Therefore, $K_1 = K_2$.

9.4 Because the passenger is brought from the car's initial speed to a full stop, the change in momentum (the impulse) is the same regardless of whether the passenger is stopped by dashboard, seatbelt, or airbag. However, the dashboard stops the passenger very quickly in a front-end collision. The seatbelt takes somewhat more time. Used along with the seatbelt, the airbag can extend the passenger's stopping time further, notably for his head, which would otherwise snap forward. Therefore, the dashboard applies the greatest force, the seatbelt an intermediate force, and the airbag the least force. Airbags are designed to work in conjunction with seatbelts. Make sure you wear your seatbelt at all times while in a moving vehicle.

9.5 If we define the ball as our system, momentum is not conserved. The ball's speed—and hence its momentum—continually increase. This is consistent with the fact that the gravitational force is external to this chosen system. However, if we define our system as the ball and the Earth, momentum is conserved, for the Earth also has momentum because the ball exerts a gravitational force on it. As the ball falls, the Earth moves up to meet it (although the Earth's speed is on the order of 10^{25} times less than that of the ball!). This upward movement changes the Earth's momentum. The change in the Earth's momentum is numerically equal to the change in the ball's momentum but is in the opposite direction. Therefore, the total momentum of the Earth–ball system is conserved. Because the Earth's mass is so great, its upward motion is negligibly small.

9.6 (c). The greatest impulse (greatest change in momentum) is imparted to the Frisbee when the skater reverses its momentum vector by catching it and throwing it back. Since this is when the skater imparts the greatest impulse to the Frisbee, then this also is when the Frisbee imparts the greatest impulse to her.

9.7 Both are equally bad. Imagine watching the collision from a safer location alongside the road. As the "crush zones" of the two cars are compressed, you will see that the actual point of contact is stationary. You would see the same thing if your car were to collide with a solid wall.

9.8 No, such movement can never occur if we assume the collisions are elastic. The momentum of the system before the collision is mv, where m is the mass of ball 1 and v is its speed just before the collision. After the collision, we would have two balls, each of mass m and moving with a speed of $v/2$. Thus, the total momentum of the system after the collision would be $m(v/2) + m(v/2) = mv$. Thus, momentum is conserved. However, the kinetic energy just before the collision is $K_i = \frac{1}{2}mv^2$, and that after the collision is $K_f = \frac{1}{2}m(v/2)^2 + \frac{1}{2}m(v/2)^2 = \frac{1}{4}mv^2$. Thus, kinetic energy is *not* conserved. Both momentum and kinetic energy are conserved only when one ball moves out when one ball is released, two balls move out when two are released, and so on.

9.9 No they will not! The piece with the handle will have less mass than the piece made up of the end of the bat. To see why this is so, take the origin of coordinates as the center of mass before the bat was cut. Replace each cut piece by a small sphere located at the center of mass for each piece. The sphere representing the handle piece is farther from the origin, but the product of lesser mass and greater distance balances the product of greater mass and lesser distance for the end piece: