Difference Equations to **D**ifferential Equations

Section 1.1

Calculus: Areas And Tangents

The study of calculus begins with questions about change. What happens to the velocity of a swinging pendulum as its position changes? What happens to the position of a planet as time changes? What happens to a population of owls as its rate of reproduction changes? Mathematically, one is interested in learning to what extent changes in one quantity affect the value of another related quantity. Through the study of the way in which quantities change we are able to understand more deeply the relationships between the quantities themselves. For example, changing the angle of elevation of a projectile affects the distance it will travel; by considering the effect of a change in angle on distance, we are able to determine, for example, the angle which will maximize the distance.

Related to questions of change are problems of approximation. If we desire to approximate a quantity which cannot be computed directly (for example, the area of some planar region), we may develop a technique for approximating its value. The accuracy of our technique will depend on how many computations we are willing to make; calculus may then be used to answer questions about the relationship between the accuracy of the approximation and the number of calculations used. If we double the number of computations, how much do we gain in accuracy? As we increase the number of computations, do the approximations approach some limiting value? And if so, can we use our approximating method to arrive at an exact answer? Note that once again we are asking questions about the effects of change.

Two fundamental concepts for studying change are sequences and limits of sequences. For our purposes, a sequence is nothing more than a list of numbers. For example,

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$$

might represent the beginning of a sequence, where the ellipsis indicates that the list is to continue on indefinitely in some pattern. For example, the 5th term in this sequence might be

$$\frac{1}{16} = \frac{1}{2^4},$$

the 8th term

$$\frac{1}{128} = \frac{1}{2^7},$$

and, in general, the nth term

$$\frac{1}{2^{n-1}}$$

where $n = 1, 2, 3, \ldots$. Notice that the sequence is completely specified only when we have given the general form of a term in the sequence. Also note that this list of numbers is approaching 0, which we would call the limit of the sequence. In the next section of this chapter we will consider in some detail the basic question of determining the limit of a sequence.

The following two examples consider these ideas in the context of the two fundamental problems of calculus. The first of these is to determine the area of a region in the plane; the other is to find the line tangent to a curve at a given point on the curve. As the course progresses, we will find that general methods for solving these two problems are at the heart of the techniques used in calculus. Moreover, we will see that these two problems are, surprisingly, closely related, with the area problem actually being the inverse of the tangent problem. This intimate connection was one of the great discoveries of Isaac Newton (1642-1727) and Gottfried Leibniz (1646-1716), although anticipated by Newton's teacher Isaac Barrow (1630-1677).

Example Suppose we wish to find the area inside a circle of radius one centered at the origin. Of course, we have all learned that the answer is π. But why? Indeed, what does it mean to find the area of a disk?

Area is best defined for polygons, regions in the plane with line segments for sides. One can start by defining the area of a 1×1 square to be one unit. The area of any other polygonal figure is then determined by how many squares may be fit into it, with suitable cutting as necessary. For example, it is seen that the area of a rectangle with base of length b and height a should be ab. Since a parallelogram with base of length b and height a may be cut and pasted onto a rectangle of length b and height a (see Problem 1), it follows that the area of such a parallelogram is also ab. As a triangle with height a and a base of length b is one-half of a parallelogram of height a and base length b (see Problem 2), it easily follows that the area of such a triangle is $\frac{1}{2}ab$. The area of any other polygon can be calculated, at least in theory, by decomposing it into a suitable number of triangles. However, a circle does not have straight sides and so may not be handled so easily. Hence we resort to approximations.

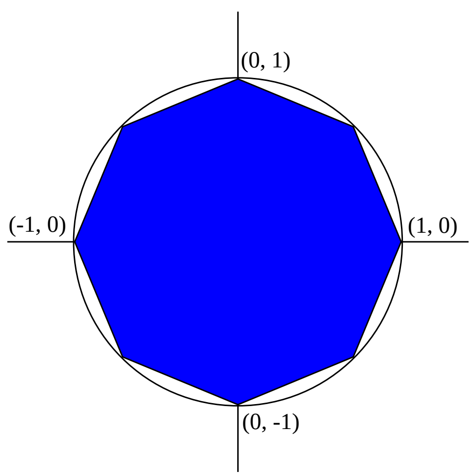


Figure 1.1.1 A regular octagon inscribed in a unit circle

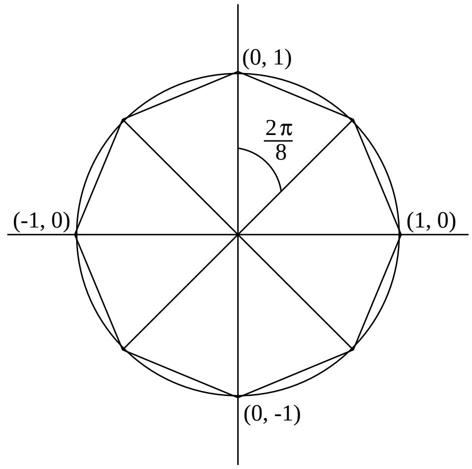


Figure 1.1.2 Decomposition of a regular octagon into eight isosceles triangles

Let P_n be a regular n-sided polygon inscribed in the unit circle centered at the origin and let A_n be the area of P_n. For example, Figure 1.1.1 shows P_8 inscribed in the unit circle. We may decompose P_n into n congruent isosceles triangles by drawing line segments from the center of the circle to the vertices of the polygon, as shown in Figure 1.1.2 for P_8. For each of these triangles, the angle with vertex at the center of the circle has measure $\frac{360}{n}$ degrees, or $\frac{2\pi}{n}$ radians, where π represents the ratio of the circumference of a circle to its diameter. Hence, since the equal sides of each of the triangles are of length one, each triangle has a height of

$$h_n = \cos\left(\frac{\pi}{n}\right)$$

and a base of length

$$b_n = 2\sin\left(\frac{\pi}{n}\right)$$

(see Problem 3). Thus the area of a single triangle is given by

$$\frac{1}{2}b_n h_n = \cos\left(\frac{\pi}{n}\right)\sin\left(\frac{\pi}{n}\right) = \frac{1}{2}\sin\left(\frac{2\pi}{n}\right),$$

where we have used the fact that

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$$

for any angle α. Multiplying by n, we see that the area of P_n is

$$A_n = \frac{n}{2}\sin\left(\frac{2\pi}{n}\right).$$

We now have a sequence of numbers, A_1, A_2, A_3, $\ldots$, each number in the sequence being an approximation to the area of the circle. Moreover, although not entirely obvious, each term in the sequence is a better approximation than its predecessor since the corresponding regular polygon more closely approximates the circle. For example, to five decimal places we have

$$\begin{aligned} A_3 &= 1.29904, \\ A_4 &= 2.00000, \\ A_5 &= 2.37764, \\ A_6 &= 2.59808, \\ A_7 &= 2.73641, \\ A_8 &= 2.82843, \\ A_9 &= 2.89254, \\ A_{10} &= 2.93893, \\ A_{11} &= 2.97352, \end{aligned}$$

and

$$A_{12} = 3.00000.$$

Continuing in this manner, we find $A_{20} = 3.09017$, $A_{50} = 3.13333$, and $A_{100} = 3.13953$. As we would expect, the sequence is increasing and appears to be approaching π. Indeed, if we take a polygon with 1644 sides, we have $A_{1644} = 3.14159$, which is π to five decimal places.

Alternatively, instead of defining π to be the ratio of the circumference of a circle to its diameter, we could define it to be the area of a circle of radius one. That is, we could define π to be the limiting value of the sequence A_n. Symbolically, we express this by writing

$$\pi = \lim_{n \to \infty} A_n.$$

In that case, let B be the area of a circle of radius r and let B_n be the area of a regular n-sided polygon Q_n inscribed in the circle. If we decompose Q_n into n isosceles triangles in the same manner as P_n above, then each triangle in this decomposition is similar to any one of the triangles in the decomposition of P_n. Since the ratios of the lengths of corresponding sides of similar triangles must all be the same, the sides of a triangle in the decomposition of Q_n must be r times the length of the corresponding sides of any triangle in the decomposition of P_n. Hence each of the triangles in the decomposition of Q_n must have a base of length rb_n and a height of rh_n, where h_n is the height and b_n is the length of the base of one of the isosceles triangles in the decomposition of P_n. Thus the area of one of the triangles in the decomposition of Q_n into isosceles triangles will be

$$\frac{1}{2}(rb_n)(rh_n) = \frac{1}{2}r^2 b_n h_n,$$

from which it follows that

$$B_n = \frac{n}{2}r^2 b_n h_n = r^2\left(\frac{n}{2}b_n h_n\right) = r^2 A_n.$$

Since r is a fixed constant, we would then expect that, in the limit as the number of sides grows toward infinity,

$$B = \lim_{n \to \infty} B_n = \lim_{r^2 A_n} = r^2 \lim_{n \to \infty} A_n = \pi r^2.$$

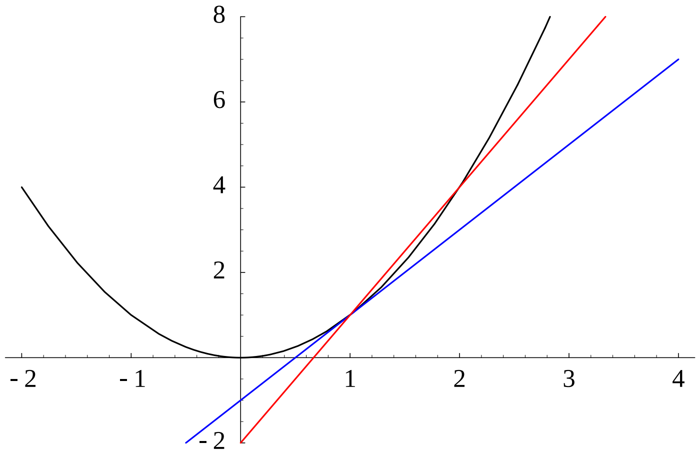


Figure 1.1.3 Parabola $y = x^2$ with tangent line (blue) and a secant line (red)

Hence we arrive at the famous formula for the area of a circle of radius r, in which the constant π has been defined to be the area of a circle of radius one.

Example In this example we wish to find the line tangent to the curve $y = x^2$, a parabola, at the point $(1, 1)$. This problem may not at first seem as useful as that of finding the area of a planar region, but we shall find that the ideas behind the solution have many applications, and are, ultimately, important in the solution of the area problem as well.

First there is the question of exactly what is a tangent line. At the present it will be sufficient to leave the notion at an intuitive level: a tangent line is a line which just touches a given curve at a point, giving a close approximation between curve and line. In Chapter 3, we will see that a line ℓ is tangent to a curve C at a point P on C if ℓ passes through P and, in a sense that we will make precise at that time, gives a better approximation to C for points close to P than any other line.

Now let C be the curve with equation $y = x^2$, let $P = (1, 1)$, and let ℓ be the line tangent to C at P. Since ℓ passes through P, in order to find the equation of ℓ we need only find its slope m. Unfortunately, to find m in the standard way we need to know two points on ℓ, and we know only one, namely P. Hence we will again have to resort to approximations. For example, the line through the points $(1, 1)$ and $(2, 4)$ is not ℓ (it is a secant line, rather than a tangent line), but since it intersects C at P and at another point which is close to P, its slope should approximate m (see Figure 1.1.3). Namely, we have

$$m \approx \frac{4-1}{2-1} = 3.$$

Since $(\frac{3}{2}, \frac{9}{4})$ is on C and is closer to P than $(2, 4)$, a better approximation is given by the slope of the line passing through $(1, 1)$ and $(\frac{3}{2}, \frac{9}{4})$, that is,

$$m \approx \frac{\frac{9}{4} - 1}{\frac{3}{2} - 1} = \frac{\frac{5}{4}}{\frac{1}{2}} = \frac{5}{2}.$$

More generally, let n be a positive integer and let m_n be the slope of the line through the points

$$\left(1+\frac{1}{n}, \left(1+\frac{1}{n}\right)^2\right)$$

and P. For example, we have just seen that $m_1 = 3$ and $m_2 = \frac{5}{2}$. Now, in general,

$$\begin{aligned} m_n &= \frac{\left(1+\frac{1}{n}\right)^2 - 1}{\left(1+\frac{1}{n}\right) - 1} \\ &= \frac{1 + \frac{2}{n} + \frac{1}{n^2} - 1}{\frac{1}{n}} \\ &= n\left(\frac{2}{n} + \frac{1}{n^2}\right) \\ &= 2 + \frac{1}{n} \end{aligned}$$

for $n = 1, 2, 3, \ldots$. Hence

$$\begin{aligned} m_3 &= 2 + \frac{1}{3} = \frac{7}{3}, \\ m_4 &= 2 + \frac{1}{4} = \frac{9}{4}, \\ m_5 &= 2 + \frac{1}{5} = \frac{11}{5}, \end{aligned}$$

and so on. Moreover, as n increases, $\frac{1}{n}$ decreases toward 0, and so we would expect that as n increases, m_n decreases toward 2. At the same time, as n increases m_n more closely approximates m. Thus we should have

$$m = \lim_{n\to\infty} m_n = \lim_{n\to\infty}\left(2 + \frac{1}{n}\right) = 2.$$

That is, the slope of the line tangent to C at P is 2. Then the tangent line ℓ has equation

$$y - 1 = 2(x - 1),$$

or

$$y = 2x - 1.$$

Here we have used the fact that the equation of a line with slope m and passing through the point (a, b) is given by

$$y - b = m(x - a).$$

The rest of this chapter will be concerned with the study of sequences and their limits. The next section will consider the basic definitions and computational techniques, while the remaining sections will discuss some applications. We will return to the problem of finding tangent lines in Chapter 3 and the problem of computing areas in Chapter 4.

Problems

1. Use Figure 1.1.4 to verify that a parallelogram with height a and base of length b has area ab.

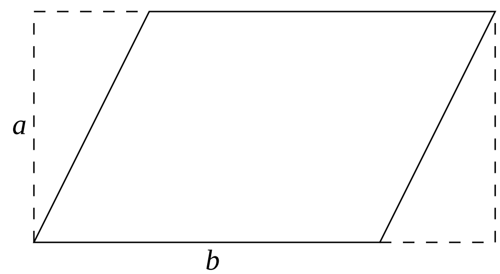


Figure 1.1.4 A parallelogram

2. Explain how any triangle is one-half of a parallelogram, and use this to verify the formula for the area of a triangle.

3. Use Figure 1.1.5 to verify the formulas given for the height and base of one of the isosceles triangles in the decomposition of P_n.

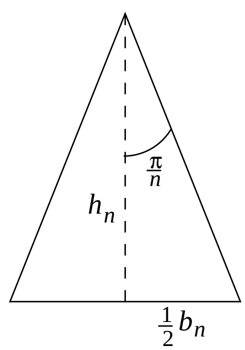


Figure 1.1.5 An isosceles triangle from the decomposition of P_n

4. Try the procedure of the tangent example to find the equation of the line tangent to the following curves at the indicated point.

   (a) $y = 2x^2$ at $(1, 2)$ (b) $y = x^2 + 1$ at $(1, 2)$

   (c) $y = x^3$ at $(1, 1)$ (d) $y = x^2$ at $(2, 4)$

5. For the area example, find the number of sides necessary for the area of the inscribed polygon to approximate π to 6, 7, 8, 9, and 10 digits after the decimal point.

6. For the tangent example, how large would n have to be in order for $|m_n - 2|$ to be less than 0.005?

7. For the tangent example, let p be the smallest positive integer such that $|m_p - 2| < 0.01$.
   (a) What is p?
   (b) What can you say about $|m_n - 2|$ for values of n greater than p?
8. For each of the following sequences $\{a_n\}$, compute a_{10}, a_{20}, a_{100}, a_{500}, and a_{1000}.
   (a) $a_n = n \sin\left(\frac{1}{n}\right)$
   (b) $a_n = \left(1 + \frac{1}{n}\right)^n$
   (c) $a_n = \frac{10^n}{n!}$, where $n! = n(n-1)(n-2)\cdots(2)(1)$
9. As we saw in the area example, there is more than one way to define the number π. For example, we can define it either as the area of a circle of unit radius or as the ratio of the circumference of a circle to its diameter (of course, if the latter approach is taken, one has to show that this ratio is the same for every circle). Suppose we define π as the area of a circle of unit radius. Consider a circle with radius r, diameter d, circumference C, and area A. Then we have seen that $A = \pi r^2$. The following steps show that we also have $\pi = \frac{C}{d}$.
   (a) Let P_n be a regular n-sided polygon inscribed in the circle. Let s be the length of a side of P_n. By dividing P_n into n equal isosceles triangles as we did in the area example, argue that
   $$A \approx \frac{nrs}{2}.$$
   (b) Can you see why as n goes to infinity, ns approaches C?
   (c) Now can you see why
   $$A = \lim_{n\to\infty} \frac{nrs}{2} = \frac{rC}{2}?$$
   (d) Use the result in part (c) to show that
   $$\pi = \frac{C}{d}.$$
10. You may find an interesting discussion of techniques for computing areas and volumes up to the time of Archimedes (287-212 B.C.) in the first two chapters of *The Historical Development of Calculus* by C. H. Edwards (Springer-Verlag New York Inc., 1979). In particular, there is a discussion on pages 31-35 of Archimedes' proof that the two definitions of π mentioned in the area example yield the same number.

Difference Equations to Differential Equations

Section 1.2

Sequences

Recall that a sequence is a list of numbers, such as

$$\begin{aligned}&1, 2, 3, 4, \ldots,\\&2, 4, 6, 8, \ldots,\\&0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots,\\&1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots,\end{aligned}$$

or

$$1, -1, 1, -1, \ldots.$$

As we noted in Section 1.1, listing the first few terms of a sequence does not uniquely specify the remaining terms of the sequence. To fully specify a sequence, we need a formula that describes an arbitrary term in the sequence. For example, the first example above lists the first four terms of the sequence $\{a_n\}$ with

$$a_n = n$$

for $n = 1, 2, 3, \ldots$; the second example lists the first four terms of $\{b_n\}$ with

$$b_n = 2n$$

for $n = 1, 2, 3, \ldots$; the third example lists the first four terms of $\{c_n\}$ with

$$c_n = 1 - \frac{1}{n}$$

for $n = 1, 2, 3, \ldots$; the fourth lists the first four terms of $\{d_n\}$ with

$$d_n = \frac{(-1)^n}{2^n}$$

for $n = 0, 1, 2, 3, \ldots$; and the fifth lists the first four terms of $\{e_n\}$ with

$$e_n = (-1)^n$$

for $n = 0, 1, 2, \ldots$.

As indicated in Section 1.1, we are often interested in the value, if one exists, which a sequence approaches. For example, the sequences $\{a_n\}$ and $\{b_n\}$ increase beyond any possible bound as n increases, and hence they have no limiting value. To visualize what is happening here, you might plot the points of the sequence on the real line. For both of these sequences, the plotted points will march off to the right without any upper limit. Although a limit does not exist in these cases, we usually write

$$\lim_{n\to\infty} a_n = \infty$$

and

$$\lim_{n\to\infty} b_n = \infty$$

to express the fact that the limits do not exist because the terms in the sequence are growing without any positive bound. On the other hand, if we plot the points of the sequence $\{c_n\}$, as in Figure 1.2.1, we see that although they are always increasing (that is, moving toward the right), nevertheless they never increase beyond 1. Moreover, even though no term in the sequence is ever equal to 1, we can see that the points become arbitrarily close to 1. Hence we say that the limit of the sequence is 1 and we write

$$\lim_{n\to\infty} c_n = 1.$$

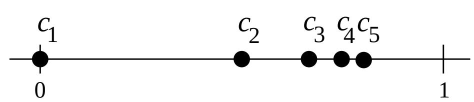


Figure 1.2.1 The first five values of $c_n = 1 - \frac{1}{n}$

Even though they oscillate between positive and negative values, the terms in the sequence $\{d_n\}$ approach closer and closer to 0 as n increases. Since it is possible to make d_n as close as we like to 0 by taking n suitably large, we may write

$$\lim_{n\to\infty} d_n = 0.$$

Finally, for the sequence $\{e_n\}$ there are only two points to plot, alternating between 1 and -1. Since the terms of this sequence oscillate between two numbers, and so do not approach any fixed limiting value, we say that the sequence does not have a limit.

Another approach to visualizing the limiting behavior of a sequence $\{a_n\}$ is to plot the ordered pairs (n, a_n) in the plane for some range of values of n. For example, Figure 1.2.2 shows a plot of the points (n, c_n), $n = 1, 2, 3, \ldots, 50$ for the sequence $\{c_n\}$ given above. Note how the points approach the horizontal line $y = 1$, indicating, as mentioned above, that

$$\lim_{n\to\infty} c_n = 1.$$

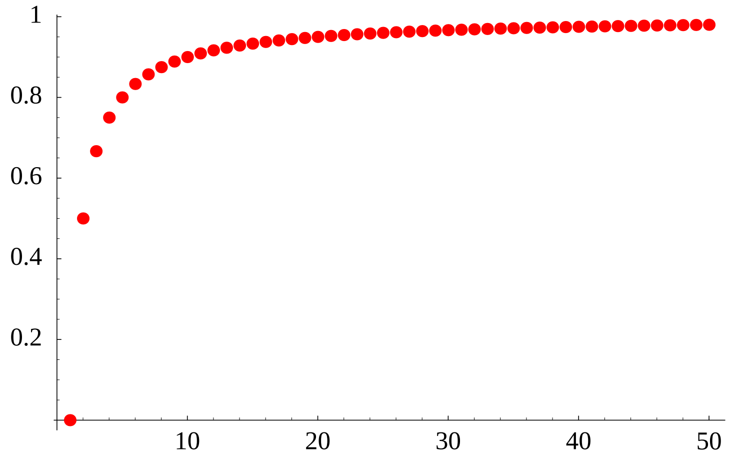


Figure 1.2.2 Plot of $(n, 1 - \frac{1}{n})$ for $n = 1, 2, 3, \ldots, 50$

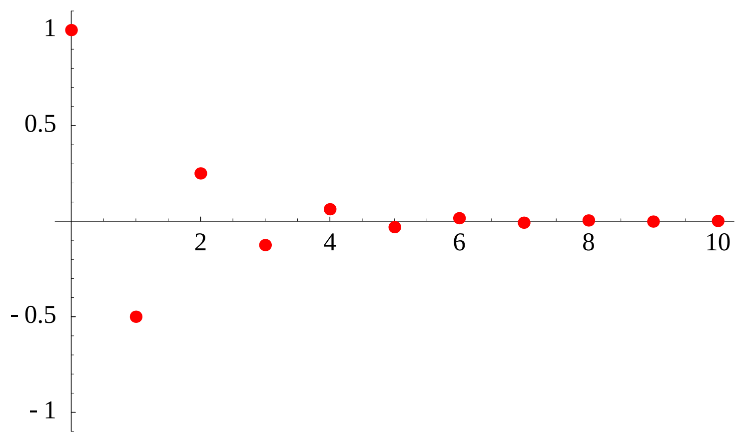


Figure 1.2.3 Plot of $(n, \frac{(-1)^n}{2^n})$ for $n = 0, 1, 2, \ldots, 10$

Similarly, Figure 1.2.3 shows a plot of the points (n, d_n), $n = 0, 1, 2, \ldots, 10$; here the points approach the horizontal axis, $y = 0$, consistent with our claim that

$$\lim_{n \to \infty} d_n = 0.$$

Figure 1.2.4 shows a plot of (n, e_n), $n = 0, 1, 2, \ldots, 20$. The fact that this sequence does not have a limit is manifest in seeing the vertical coordinate of the points oscillate between 1 and -1.

As the concept of a limit is fundamental to the understanding of calculus, it is important that we make the notion more concrete than we have so far. That is, we need to have a formal definition of limit which exactly captures what we have been discussing intuitively. The idea is that we should say L is the limit of a sequence $\{a_n\}$ if for any open interval I containing L, no matter how small, we can find a point in the sequence beyond which all values of the sequence lie in I. Graphically, this means that if we start plotting the points of the sequence, there will come a time when all points from then on will lie

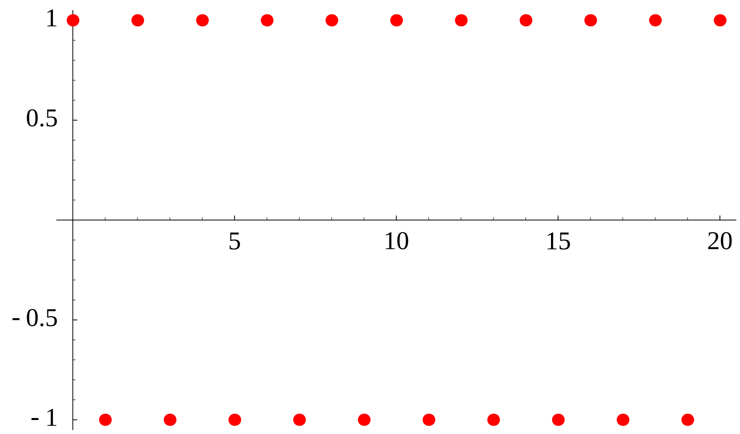


Figure 1.2.4 Plot of $(n, (-1)^n)$ for $n = 0, 1, 2, \ldots, 20$

within the interval I. This idea is formalized in the following definition, where the open interval I is expressed in the form $(L-\epsilon, L+\epsilon)$ and the idea that all values of the sequence beyond a certain point are in this interval is expressed by requiring that $|a_n - L| < \epsilon$, that is, the distance between a_n and L is less than ϵ, for all $n > N$.

Definition We say that the *limit* of the sequence $\{a_n\}$ is L, written

$$\lim_{n \to \infty} a_n = L,$$

if for every $\epsilon > 0$ there exists an integer N such that $|a_n - L| < \epsilon$ whenever $n > N$.

Hence to show that the limit of a sequence is a number L, one must show that for any positive number ϵ, it is possible to find an integer N such that the numbers $a_{N+1}, a_{N+2}, a_{N+3}, \ldots$ are all in the interval $(L - \epsilon, L + \epsilon)$. See Figure 1.2.5.

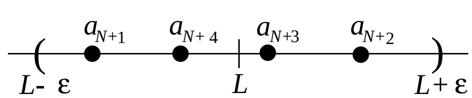


Figure 1.2.5 a_n in $(L - \epsilon, L + \epsilon)$ for $n > N$

Example We will show that

$$\lim_{n \to \infty} \frac{1}{n} = 0.$$

To do so, we must show that for any given $\epsilon > 0$, we can find an integer N such that

$$\left| \frac{1}{n} - 0 \right| < \epsilon$$

whenever $n > N$. Now

$$\left| \frac{1}{n} - 0 \right| = \frac{1}{n},$$

so we need only determine the values of n for which

$$\frac{1}{n} < \epsilon.$$

Since

$$\frac{1}{n} < \epsilon \text{ if and only if } n > \frac{1}{\epsilon},$$

it follows that we may take N to be the largest integer less than or equal to $\frac{1}{\epsilon}$. Then whenever $n > N$, we have

$$n > \frac{1}{\epsilon},$$

from which it follows that

$$\frac{1}{n} < \epsilon.$$

This is exactly what we need in order to conclude, by the definition, that

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

The following definition is useful in situations, such as in the previous example, when we want the largest integer less than or equal to some given value.

Definition For any real number x, we may define the *floor function*, denoted $\lfloor x \rfloor$, by

$$\lfloor x \rfloor = \text{ the largest integer less than or equal to } x, \tag{1.2.1}$$

and the *ceiling function*, denoted $\lceil x \rceil$, by

$$\lceil x \rceil = \text{ the smallest integer greater than or equal to } x. \tag{1.2.2}$$

For example, $\lfloor 5.3 \rfloor = 5$, $\lceil \pi \rceil = 4$, $\lfloor 3 \rfloor = 3$, and $\lceil 3 \rceil = 3$. With this notation, we could define N in the previous example by

$$N = \left\lfloor \frac{1}{\epsilon} \right\rfloor.$$

Example We will show that

$$\lim_{n\to\infty} \frac{1}{2^n} = 0.$$

This time we must show that for any $\epsilon > 0$, we can find an integer N such that

$$\left| \frac{1}{2^n} - 0 \right| < \epsilon$$

whenever $n > N$. Now

$$\left|\frac{1}{2^n} - 0\right| = \frac{1}{2^n} = \left(\frac{1}{2}\right)^n,$$

so we need to determine the values of n for which

$$\left(\frac{1}{2}\right)^n < \epsilon.$$

We need to solve this inequality for n. Since n is in the exponent, we may use logarithms to simplify the inequality. Although we will not provide a careful treatment of logarithms until Chapter 6, we will assume for the moment some acquaintance with logarithms using base 10. Now

$$\left(\frac{1}{2}\right)^n < \epsilon$$

if and only if

$$\log_{10}\left(\frac{1}{2}\right)^n < \log_{10}(\epsilon).$$

Since

$$\log_{10}\left(\frac{1}{2}\right)^n = n\log_{10}\left(\frac{1}{2}\right),$$

we have

$$\left(\frac{1}{2}\right)^n < \epsilon$$

if and only if

$$n\log_{10}\left(\frac{1}{2}\right) < \log_{10}(\epsilon).$$

Now $\log_{10}\left(\frac{1}{2}\right) < 0$, so

$$n\log_{10}\left(\frac{1}{2}\right) < \log_{10}(\epsilon)$$

if and only if

$$n > \frac{\log_{10}(\epsilon)}{\log_{10}\left(\frac{1}{2}\right)}.$$

Thus if we let

$$N = \left\lfloor \frac{\log_{10}(\epsilon)}{\log_{10}\left(\frac{1}{2}\right)} \right\rfloor,$$

then

$$\left|\frac{1}{2^n} - 0\right| < \epsilon$$

whenever $n > N$. For example, if we take $\epsilon = 0.001$, then, to two decimal places,

$$\frac{\log_{10}(\epsilon)}{\log_{10}\left(\frac{1}{2}\right)} = 9.97,$$

and so we would have

$$N = \lfloor 9.97 \rfloor = 9.$$

This N works because, for $n > 9$,

$$\left|\frac{1}{2^n} - 0\right| = \frac{1}{2^n} \leq \frac{1}{2^{10}} = \frac{1}{1024} < 0.001.$$

Problem 12 at the end of this section will ask you to generalize the previous example to show that

$$\lim_{n\to\infty} r^n = 0$$

whenever $|r| < 1$. This is an important fact that we will make use of later.

In this course we will be concerned more with the development of an intuitive understanding of limits and a computational facility with limits than with the formalism of verifying a specific limit using the above definition. That is not to say that the definition is unimportant; rather a good grasp of the concept in the definition is important for a full understanding of much of what we will do in calculus. In fact, mathematicians of the 19th century arrived at the definition we have stated in their attempts to clarify confusions that had developed in mathematics since the time of Newton and Leibniz. However, for the most part these difficulties are beyond the scope of a text such as this one.

We will see that a few basic properties of limits, combined with a few simple limits like the ones in the previous two examples, will enable us to compute easily a large number of limits. To begin considering these properties, consider the case where we already know that

$$\lim_{n\to\infty} a_n = L \tag{1.2.3}$$

and we want to compute

$$\lim_{n\to\infty} ka_n$$

for some constant $k \neq 0$. Now (1.2.3) tells us that for any $\epsilon > 0$, we may find an integer N such that for $n > N$,

$$|a_n - L| < \frac{\epsilon}{|k|}.$$

It follows that for $n > N$,

$$|ka_n - kL| = |k||a_n - L| < |k|\frac{\epsilon}{|k|} = \epsilon.$$

But this is what it means to say that

$$\lim_{n\to\infty} ka_n = kL. \tag{1.2.4}$$

Note that (1.2.4) is obviously true as well when $k = 0$. Hence we have the following proposition.

Proposition If $\{a_n\}$ is a sequence for which

$$\lim_{n\to\infty} a_n = L,$$

then for any constant k we have

$$\lim_{n\to\infty} ka_n = k \lim_{n\to\infty} a_n = kL. \tag{1.2.5}$$

Example Since we have already seen that

$$\lim_{n\to\infty} \frac{1}{n} = 0,$$

it follows that

$$\lim_{n\to\infty} \frac{350}{n} = 350 \lim_{n\to\infty} \frac{1}{n} = (350)(0) = 0.$$

Now suppose we have two sequences $\{a_n\}$ and $\{b_n\}$ with

$$\lim_{n\to\infty} a_n = L \tag{1.2.6}$$

and

$$\lim_{n\to\infty} b_n = M. \tag{1.2.7}$$

Then (1.2.6) and (1.2.7) tell us that for any $\epsilon > 0$, we can find integers N_1 and N_2 such that

$$|a_n - L| < \frac{\epsilon}{2}$$

whenever $n > N_1$ and

$$|b_n - M| < \frac{\epsilon}{2}$$

whenever $n > N_2$. If we let N be the larger of N_1 and N_2, then whenever $n > N$ we will have

$$\begin{aligned} |(a_n + b_n) - (L + M)| &= |(a_n - L) + (b_n - M)| \\ &\leq |a_n - L| + |b_n - M| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned} \tag{1.2.8}$$

Note that in (1.2.8) we have used the fact, known as the *triangle inequality*, that for any real numbers x and y,

$$|x + y| \leq |x| + |y|. \tag{1.2.9}$$

Thus we have shown

$$\lim_{n\to\infty} (a_n + b_n) = L + M. \tag{1.2.10}$$

Hence we have the following proposition.

Proposition If $\{a_n\}$ and $\{b_n\}$ are sequences with

$$\lim_{n\to\infty} a_n = L$$

and

$$\lim_{n\to\infty} b_n = M,$$

then

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = L + M. \tag{1.2.11}$$

Example We have

$$\lim_{n\to\infty}\left(4 + \frac{8}{n}\right) = \lim_{n\to\infty} 4 + \lim_{n\to\infty}\frac{8}{n} = 4 + 8\lim_{n\to\infty}\frac{1}{n} = 4 + (8)(0) = 4.$$

Note that in the last example we used the fact that if k is a constant and $a_n = k$ for all n, then

$$\lim_{n\to\infty} a_n = k.$$

This follows immediately from the definition since

$$|a_n - k| = 0$$

for all values of k, and so any integer N will work for any value of ϵ.

Again suppose we have two sequences $\{a_n\}$ and $\{b_n\}$ with

$$\lim_{n\to\infty} a_n = L$$

and

$$\lim_{n\to\infty} b_n = M.$$

Then we have

$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty}(-b_n) = \lim_{n\to\infty} a_n + (-1)\lim_{n\to\infty} b_n = L - M. \qquad (1.2.12).$$

Proposition If $\{a_n\}$ and $\{b_n\}$ are sequences with

$$\lim_{n\to\infty} a_n = L$$

and

$$\lim_{n\to\infty} b_n = M,$$

then

$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = L - M. \tag{1.2.13}$$

Example We have

$$\lim_{n\to\infty}\left(\frac{3}{n}-\frac{8}{5^n}\right)=3\lim_{n\to\infty}\frac{1}{n}-8\lim_{n\to\infty}\left(\frac{1}{5}\right)^n=(3)(0)-(8)(0)=0.$$

Note that we have used the result that

$$\lim_{n\to\infty} r^n = 0$$

whenever $|r|<0$.

We will state three more properties of limits without justifications. Although the reasoning behind these results is similar to the reasoning of the previous three propositions, they require a little more care and are best left to a more advanced course.

Proposition If $\{a_n\}$ and $\{b_n\}$ are sequences with

$$\lim_{n\to\infty} a_n = L$$

and

$$\lim_{n\to\infty} b_n = M,$$

then

$$\lim_{n\to\infty} a_n b_n = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n) = LM. \tag{1.2.14}$$

Example We have

$$\lim_{n\to\infty}\frac{1}{n^2}=\left(\lim_{n\to\infty}\frac{1}{n}\right)\left(\lim_{n\to\infty}\frac{1}{n}\right)=(0)(0)=0.$$

Proposition If $\{a_n\}$ and $\{b_n\}$ are sequences with

$$\lim_{n\to\infty} a_n = L$$

and

$$\lim_{n\to\infty} b_n = M,$$

then

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}=\frac{L}{M}, \tag{1.2.15}$$

provided $L \neq 0$ and $b_n \neq 0$ for all n.

Example We have

$$\lim_{n\to\infty}\frac{n-3}{2n+4}=\lim_{n\to\infty}\frac{\dfrac{n-3}{n}}{\dfrac{2n+4}{n}}=\lim_{n\to\infty}\frac{1-\dfrac{3}{n}}{2+\dfrac{4}{n}}=\frac{\lim\limits_{n\to\infty}\left(1-\dfrac{3}{n}\right)}{\lim\limits_{n\to\infty}\left(2+\dfrac{4}{n}\right)}=\frac{1}{2}.$$

Note that we can apply the previous proposition only when both numerator and denominator have a limit. Hence, in this example, we first divided the numerator and denominator by n to put the problem in a form to which we could apply the proposition.

Proposition Suppose $\{a_n\}$ is a sequence with

$$\lim_{n\to\infty} a_n = L.$$

Moreover, suppose p is a rational number, a_n^p is defined for all n, and L^p is defined. Then

$$\lim_{n\to\infty} a_n^p = (\lim_{n\to\infty} a_n)^p = L^p. \tag{1.2.16}$$

Example We have

$$\lim_{n\to\infty} \sqrt{4 - \frac{3}{n}} = \sqrt{\lim_{n\to\infty} \left(4 - \frac{3}{n}\right)} = \sqrt{4} = 2.$$

Example For any rational number $p > 0$, we have

$$\lim_{n\to\infty} \frac{1}{n^p} = \left(\lim_{n\to\infty} \frac{1}{n}\right)^p = 0^p = 0.$$

Example We have

$$\lim_{n\to\infty} \left(18 - \frac{5}{n} + \frac{23}{n^5}\right) = \lim_{n\to\infty} 18 - 5 \lim_{n\to\infty} \frac{1}{n} + 23 \lim_{n\to\infty} \frac{1}{n^5} = 18 - (5)(0) + (23)(0) = 18.$$

Example We have

$$\lim_{n\to\infty} \frac{4n^5 + 5n^2 - 6}{3n^5 + 4n - 18} = \lim_{n\to\infty} \frac{4 + \dfrac{5}{n^3} - \dfrac{6}{n^5}}{3 + \dfrac{4}{n^4} - \dfrac{18}{n^5}} = \frac{\lim\limits_{n\to\infty} \left(4 + \dfrac{5}{n^3} - \dfrac{6}{n^5}\right)}{\lim\limits_{n\to\infty} \left(3 + \dfrac{4}{n^4} - \dfrac{18}{n^5}\right)} = \frac{4}{3}.$$

In general, for sequences of the form of the previous example it is useful to divide both numerator and denominator by the highest power of n which occurs in the denominator.

Example As another illustration of the idea in the previous example, we have

$$\lim_{n\to\infty} \frac{3n^2 + 2n - 1}{2n^3 - 16n} = \lim_{n\to\infty} \frac{\dfrac{3}{n} + \dfrac{2}{n^2} - \dfrac{1}{n^3}}{2 - \dfrac{16}{n^2}} = \frac{0}{2} = 0.$$

Definition If $\lim_{n\to\infty} a_n$ exists, we say the sequence $\{a_n\}$ *converges*. If the sequence $\{a_n\}$ does not have a limit, we say the sequence *diverges*.

An important class of divergent sequences are those for which a limit does not exist either because the terms grow without an upper bound or because they decrease without any lower bound, as defined in the following definition.

Definition A sequence $\{a_n\}$ is said to *diverge to infinity* if for any real number M there exists an integer N such that $a_n > M$ whenever $n > N$, in which case we write

$$\lim_{n\to\infty} a_n = \infty.$$

A sequence $\{a_n\}$ is said to *diverge to negative infinity* if for any real number M there exists an integer N such that $a_n < M$ whenever $n > N$, in which case we write

$$\lim_{n\to\infty} a_n = -\infty.$$

Example Clearly

$$\lim_{n\to\infty} n^p = \infty$$

for any value of $p > 0$. For given any M, we need only take

$$N = \left\lfloor \sqrt[p]{|M|} \right\rfloor$$

to guarantee that $a_n > M$ whenever $n > N$.

Example We have

$$\lim_{n\to\infty} 2^n = \infty$$

since, given any M, $2^n > M$ for all n if $M \leq 0$ and $2^n > M$ provided

$$n > \frac{\log_{10}(M)}{\log_{10}(2)}$$

if $M > 0$.

Suppose the sequence $\{a_n\}$ diverges and $k \neq 0$ is a constant. Then the sequence $\{ka_n\}$ must also diverge since if $\{ka_n\}$ converged, then the sequence with nth term

$$\frac{1}{k}(ka_n) = a_n$$

would also converge, contradicting our assumption that $\{a_n\}$ diverges.

Proposition If the sequence $\{a_n\}$ diverges and $k \neq 0$ is a constant, then the sequence $\{ka_n\}$ also diverges.

If the sequence $\{a_n\}$ diverges and the sequence $\{b_n\}$ converges, then the sequence $\{a_n + b_n\}$ also diverges since, if it converged, then the sequence with nth term

$$(a_n + b_n) - b_n = a_n$$

would also converge, contradicting our assumption that $\{a_n\}$ diverges. Similarly, the sequence $\{a_n - b_n\}$ diverges.

Proposition If the sequence $\{a_n\}$ diverges and the sequence $\{b_n\}$ converges, then the sequences $\{a_n + b_n\}$ and $\{a_n - b_n\}$ both diverge.

Suppose the sequence $\{a_n\}$ diverges, the sequence $\{b_n\}$ converges, and

$$\lim_{n\to\infty} b_n \neq 0. \tag{1.2.17}$$

Now (1.2.17) implies that we can find an integer N such that $b_n \neq 0$ for all $n > N$. So if the sequence $\{a_n b_n\}$ converged, then the sequence with, for $n > N$, nth term,

$$\frac{1}{b_n}(a_n b_n) = a_n$$

would also converge, contradicting our assumption that $\{a_n\}$ diverges. Hence $\{a_n b_n\}$ must diverge.

Proposition If the sequence $\{a_n\}$ diverges, the sequence $\{b_n\}$ converges, and

$$\lim_{n\to\infty} b_n \neq 0,$$

then the sequence $\{a_n b_n\}$ diverges

Finally, if the sequence $\{a_n\}$ diverges, the sequence $\{b_n\}$ converges, and $b_n \neq 0$ for all n, then the sequence

$$\left\{\frac{a_n}{b_n}\right\}$$

diverges since, if it converged, the sequence with nth term

$$b_n\left(\frac{a_n}{b_n}\right) = a_n$$

would also converge, contradicting our assumption that $\{a_n\}$ diverges.

Proposition If the sequence $\{a_n\}$ diverges, the sequence $\{b_n\}$ converges, and $b_n \neq 0$ for all n, then the sequence

$$\left\{\frac{a_n}{b_n}\right\}$$

diverges.

Example Consider

$$\lim_{n\to\infty} \frac{4n^3+n-2}{5n^2-7n} = \lim_{n\to\infty} \frac{4n+\frac{1}{n}-\frac{2}{n^2}}{5-\frac{7}{n}}. \tag{1.2.18}$$

Now

$$\lim_{n\to\infty} 4n = \infty$$

and

$$\lim_{n\to\infty} \left(\frac{1}{n}-\frac{2}{n^2}\right) = 0,$$

so

$$\lim_{n\to\infty} \left(4n+\frac{1}{n}-\frac{2}{n^2}\right) = \infty.$$

Moreover,

$$\lim_{n\to\infty} \left(5-\frac{7}{n}\right) = 5.$$

Thus the numerator in (1.2.18) diverges while the denominator converges. Hence the ratio diverges. In fact, it should be clear that

$$\lim_{n\to\infty} \frac{4n^3+n-2}{5n^2-7n} = \lim_{n\to\infty} \frac{4n+\frac{1}{n}-\frac{2}{n^2}}{5-\frac{7}{n}} = \infty.$$

Note that in the previous example it was once again useful to divide numerator and denominator by the highest power of n in the denominator.

Example We have

$$\lim_{n\to\infty} \frac{15-26n^5}{13+n^2} = \lim_{n\to\infty} \frac{\frac{15}{n^2}-26n^3}{\frac{13}{n^2}+1} = -\infty.$$

Example The absolute values of the terms of the sequence $\{(-2)^n\}$ grow without bound, and so the sequence diverges. However, since the terms alternate in sign, the sequence neither diverges to ∞ nor to $-\infty$.

Monotone sequences

It is sometimes possible to determine that a given sequence converges without explicitly computing the limit. One important case involves *monotone sequences.*

Definition We say a sequence $\{a_n\}$ is *monotone increasing* if $a_n \leq a_{n+1}$ for all n. We say a sequence $\{a_n\}$ is *monotone decreasing* if $a_n \leq a_{n+1}$ for all n. We say a sequence is *monotone* if it is either monotone increasing or monotone decreasing.

Now suppose $\{a_n\}$ is a monotone increasing sequence. For such a sequence there either exists a number P such that $a_n \leq P$ for all n or there does not exist such an P. In the latter case, given any real number M, it is then possible to find integer N such that $a_N > M$. Since the sequence is monotone, it follows that $a_n > M$ for all $n > N$, and so the sequence diverges to infinity. On the other hand, if there does exist a number P such that $a_n \leq P$ for all n, then there in fact exists a number B such that $a_n \leq B$ for all n and $B \leq P$ for any number P with the property that $a_n \leq P$ for all n. The existence of B, known as the *least upper bound* of the sequence $\{a_n\}$, is not at all obvious; indeed, the subtle properties of the real numbers that imply the existence of B were not fully understood until the middle part of the 19th century. However, given the existence of B, it is easy to see that given any $\epsilon > 0$, there exists a integer N for which $a_N > B - \epsilon$ (if not, then $B - \epsilon$ would be an upper bound for the sequence smaller than B). Since the sequence is monotone increasing and $a_n < B$ for all n, it follows that

$$|a_n - B| < \epsilon$$

for all $n > N$. That is, we have shown that the sequence converges and

$$\lim_{n\to\infty} a_n = B.$$

Similar results hold for sequences which are monotone decreasing.

Monotone sequence theorem Suppose the sequence $\{a_n\}$ is monotone. If the sequence is monotone increasing and there exists a number P such that $a_n \leq P$ for all n, then the sequence converges. If the sequence is monotone increasing and no such number P exists, then

$$\lim_{n\to\infty} a_n = \infty.$$

If the sequence is monotone decreasing and there exists a number Q such that $a_n \geq Q$ for all n, then the sequence converges. If the sequence is monotone decreasing and no such number Q exists, then

$$\lim_{n\to\infty} a_n = -\infty.$$

Example As we shall see in Sections 1.4 and 1.5, we often work with sequences without having an explicit formula for each term in the sequence. For example, suppose all we know about the sequence $\{a_n\}$ is that $a_1 = 4$ and

$$a_{n+1} = \frac{1}{2}a_n$$

for $n = 1, 2, 3, \ldots$. That is, the first term in the sequence is 4 and then each successive term is one-half of its predecessor. Thus

$$\begin{aligned} a_1 &= 4, \\ a_2 &= 2, \\ a_3 &= 1, \\ a_4 &= \frac{1}{2}, \end{aligned}$$

and so on. Hence $\{a_n\}$ is monotone decreasing. Moreover, every term in the sequence is positive, so $a_n \geq 0$ for all n. Thus, by the Monotone Sequence Theorem, $\{a_n\}$ converges. Moreover, note that

$$a_{n+1} = \frac{1}{2} a_n$$

implies that

$$\lim_{n \to \infty} a_{n+1} = \frac{1}{2} \lim_{n \to \infty} a_n. \tag{1.2.19}$$

If we let

$$L = \lim_{n \to \infty} a_n = \lim_{n \to \infty} a_{n+1},$$

then (1.2.19) becomes

$$L = \frac{1}{2} L.$$

Hence $L = 0$. That is,

$$\lim_{n \to \infty} a_n = 0.$$

Problems

1. For each of the following, find a general expression for the nth term of a sequence which would yield these values as the first four terms.

   (a) $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$ (b) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

   (c) $1, \frac{3}{2}, \frac{5}{3}, \frac{7}{4}, \ldots$ (d) $-\frac{1}{3}, \frac{1}{5}, -\frac{1}{7}, \frac{1}{9}, \ldots$

2. For each of the following, decide whether the given sequence converges or diverges. If the sequence converges, find its limit.

   (a) $a_n = \dfrac{1}{3^n}$, $n = 0, 1, 2, \ldots$ (b) $a_n = \pi^n$, $n = 0, 1, 2, \ldots$

   (c) $b_n = \dfrac{3n - 1}{2n + 6}$, $n = 1, 2, 3, \ldots$ (d) $c_n = \cos(\pi n)$, $n = 0, 1, 2, \ldots$

   (e) $a_n = \dfrac{3n^4 - 6n^3 + 1}{5n^3 + n^2 + 2}$, $n = 1, 2, 3, \ldots$ (f) $b_n = \dfrac{2n^5 - 3n^2 + 23}{7n^5 + 13n^4 - 12}$, $n = 1, 2, 3, \ldots$

(g) $c_n = \dfrac{45 - 16n^2}{13 + 5n + 6n^3}$, $n = 1, 2, 3, \ldots$

(h) $b_n = \dfrac{3n + 1}{\sqrt{4n^2 + 1}}$, $n = 1, 2, 3, \ldots$

(i) $a_n = (-2)^{2n+1}$, $n = 1, 2, 3, \ldots$

(j) $a_n = \dfrac{10 - 16n^3}{1 + n^2}$, $n = 1, 2, 3, \ldots$

(k) $a_n = \sqrt{\dfrac{3n^2 + n - 6}{5n^2 + 16}}$, $n = 1, 2, 3, \ldots$

(l) $b_n = \dfrac{(-1)^n}{5^n}$, $n = 0, 1, 2, \ldots$

3. Explain why
$$-1 \le \frac{\sin(n)}{n} \le 1$$
for $n = 1, 2, 3, \ldots$. What can you conclude about $\displaystyle\lim_{n \to \infty} \frac{\sin(n)}{n}$?

4. Let $a_n = \left(1 + \dfrac{1}{n}\right)^n$, $n = 1, 2, 3, \ldots$.
   (a) Compute a_1, a_2, a_3, a_4, and a_5 using a calculator.
   (b) Compute values of a_n for $n = 1, 2, 3, \ldots, 200$.
   (c) Plot the points (n, a_n) for $n = 1, 2, 3, \ldots, 200$, along with the horizontal line $y = e$.
   (d) Does it seem reasonable that $\displaystyle\lim_{n \to \infty} a_n = e$?
   (e) What is the smallest value of n for which $a_n > e$?
   (f) What is the first value of n for which $|a_n - e| < 0.01$? Recall that $e = 2.71828$ to five decimal places.

5. Let $a_n = n \sin\left(\dfrac{1}{n}\right)$, $n = 1, 2, 3, \ldots$.
   (a) Compute a_1, a_2, a_3, a_4, and a_5 using a calculator.
   (b) Compute values of a_n for $n = 1, 2, 3, \ldots, 200$.
   (c) Plot the points (n, a_n) for $n = 1, 2, 3, \ldots, 200$, along with the horizontal line $y = 1$.
   (d) Does it seem reasonable that $\displaystyle\lim_{n \to \infty} a_n = 1$?
   (e) What is the smallest value of n for which $a_n > 0.999$?
   (f) What is the first value of n for which $|a_n - 1| < 0.0001$?

6. Let $a_n = 1.01^n$ and $b_n = 0.99^n$ for $n = 0, 1, 2, \ldots$. On the same graph, plot the points (n, a_n) and (n, b_n) for $n = 0, 1, 2, \ldots, 200$. How do these two plots compare? Do the sequences converge?

7. Let $a_n = \dfrac{10^n}{n!}$ for $n = 1, 2, 3, \ldots$.
   (a) Plot the points (n, a_n) for $n = 1, 2, 3, \ldots, 100$.
   (b) From the picture in part (a), can you guess $\displaystyle\lim_{n \to \infty} a_n$?
   (c) What is the maximum value of a_n for $n = 1, 2, 3, \ldots, 100$?

(d) Can you see why
$$\lim_{n\to\infty} \frac{k^n}{n!} = 0$$
for any constant k?

8. Consider the sequence $\{a_n\}$ with $a_1 = 10$ and
$$a_{n+1} = \frac{1}{3}a_n$$
for $n = 1, 2, 3, \ldots$. Plot the points (n, a_n) for $n = 1, 2, 3, \ldots 50$. Do you think this sequence has a limit? Can you verify this?

9. Consider the sequence $\{a_n\}$ with $a_1 = 2$ and
$$a_{n+1} = 2a_n$$
for $n = 1, 2, 3, \ldots$. Plot the points (n, a_n) for $n = 1, 2, 3, \ldots, 50$. Can you find the limit of this sequence using the same method you used in part Problem 8? Does this sequence have a limit?

10. Consider the sequence $\{a_n\}$ with $a_1 = 0.9$ and
$$a_{n+1} = 2a_n(1 - a_n)$$
for $n = 1, 2, 3, \ldots$. Plot the points (n, a_n) for $n = 1, 2, 3, \ldots, 100$. Do you think this sequence has a limit? If so, can you find it?

11. In each of the following, for an arbitrary $\epsilon > 0$, find the smallest integer N for which $|a_n - L| < \epsilon$ whenever $n > N$. Verify that your value for N works in the particular case $\epsilon = 0.001$.

(a) $a_n = 1 - \frac{1}{n}$, $L = 1$ (b) $a_n = 0.98^n$, $L = 0$

(c) $a_n = \frac{1}{n^2}$, $L = 0$ (d) $a_n = \frac{3n^3 - 1}{n^3}$, $L = 3$

12. Show that for any $-1 < r < 1$, $\lim_{n\to\infty} r^n = 0$.

13. Find sequences $\{a_n\}$ and $\{b_n\}$ such that $\{a_n\}$ and $\{b_n\}$ both diverge, but $\{a_n + b_n\}$ converges.

14. Find sequences $\{a_n\}$ and $\{b_n\}$ such that $\{a_n\}$ diverges, $\{b_n\}$ converges, and $\{a_n b_n\}$ converges.

Section 1.3

The Sum of a Sequence

This section considers the problem of adding together the terms of a sequence. Of course, this is a problem only if more than a finite number of terms of the sequence are nonzero. In this case, we must decide what it means to add together an infinite number of nonzero numbers. The first example shows how a relatively simple question may lead to such infinite summations.

Example Suppose a game is played in which a fair coin is tossed until the first time a head appears. What is the probability that a head appears for the first time on an even-numbered toss? To solve this problem, we first need to determine the probability of obtaining a head for the first time on any given even numbered toss, and then we need to add all these probabilities together. Let P_n denote the probability that the first head appears on the nth toss, $n = 1, 2, 3, \ldots$. Then, since the coin is assumed to be fair,

$$P_1 = \frac{1}{2}.$$

Now in order to get a head for the first time on the second toss, we must toss a tail on the first toss and then follow that with a head on the second toss. Since one-half of all first tosses will be tails and then one-half of those tosses will be followed by a second toss of heads, we should have

$$P_2 = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}.$$

Similarly, since one-fourth of all sequences of coin tosses will begin with two tails and then half of these sequences will have a head for the third toss, we have

$$P_3 = \left(\frac{1}{4}\right)\left(\frac{1}{2}\right) = \frac{1}{8}.$$

Continuing in this fashion, it should seem reasonable that, for any $n = 1, 2, 3, \ldots$,

$$P_n = \frac{1}{2^n}.$$

Hence we have a sequence of probabilities $\{P_n\}$ for $n = 1, 2, 3, \ldots$, and, in order to find the desired probability, we need to add up the even-numbered terms in this sequence. Namely, the probability that a head appears for the first time on an even toss is given by

$$P_2 + P_4 + P_6 + \cdots = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots. \tag{1.3.1}$$

But this involves adding together an infinite number of nonzero values. Is this possible? Can we perform the operation of addition an infinite number of times? In this case the answer is yes, but we will need a few preliminaries before we can finish this particular example.

We begin with a definition of the sum of a sequence $\{a_n\}$. The idea is to create a new sequence by successively adding together the terms of the original sequence. That is, we define a new sequence $\{s_n\}$ where s_n is the sum of the first n terms of the original sequence. If

$$\lim_{n \to \infty} s_n$$

exists, then this indicates that, as we add together more and more terms of $\{a_n\}$, the resulting sums approach a limiting value. It is then reasonable to call this limiting value the sum of the sequence. For example, if

$$a_n = \frac{1}{2^n}$$

for $n = 1, 2, 3, \ldots$, then we would have

$$\begin{aligned} s_1 &= \frac{1}{2}, \\ s_2 &= \frac{1}{2} + \frac{1}{4} = \frac{3}{4}, \\ s_3 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}, \\ s_4 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}, \end{aligned}$$

and so on. If you plot these points on the real line, you may think of starting at $\frac{1}{2}$, moving $\frac{1}{2}$ the distance to 1 to plot the next point, then $\frac{1}{2}$ the remaining distance to 1 to plot the next point, and so on. After n points, you would be at

$$s_n = 1 - \frac{1}{2^n}. \tag{1.3.2}$$

Clearly,

$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} \left(1 - \frac{1}{2^n}\right) = 1,$$

and it would be reasonable to say that the sequence adds up to 1. That is,

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots = 1. \tag{1.3.3}$$

This idea is formalized in the following definition.

Definition Given a sequence $\{a_n\}$, $n = 1, 2, 3, \ldots$, we define a new sequence $\{s_n\}$ by letting

$$s_n = a_1 + a_2 + \ldots + a_n \tag{1.3.4}$$

for $n = 1, 2, 3, \ldots$. If the sequence $\{s_n\}$ converges, then we call

$$s = \lim_{n \to \infty} s_n$$

the *sum* of the sequence $\{a_n\}$. The sequence $\{s_n\}$ is called an *infinite series* and an individual term s_n of this sequence is called a *partial sum* of the sequence $\{a_n\}$.

Note that we have assumed that the first term in the sequence $\{a_n\}$ in the definition is a_1. The sequence could just as well start with any other integer index, in which case the sequence of partial sums $\{s_n\}$ would start with the same index. For example, if the first term of the sequence is a_0, then the first partial sum is s_0.

Since summations involving an infinite number, or even a large finite number, of terms are cumbersome to write using the standard plus sign of addition, Σ (the capital Greek *sigma*) is used to denote the process of summation. In particular, we would write

$$s_n = \sum_{j=1}^{n} a_j = a_1 + a_2 + \cdots + a_n \tag{1.3.5}$$

and

$$s = \lim_{n \to \infty} s_n = \lim_{n \to \infty} \sum_{j=1}^{n} a_j = \lim_{n \to \infty} (a_1 + a_2 + \cdots + a_n). \tag{1.3.6}$$

Since (1.3.6) is we what mean by an infinite sum, we will in fact write

$$s = \sum_{j=1}^{\infty} a_j = a_1 + a_2 + \cdots + a_n + \cdots. \tag{1.3.7}$$

For example, in this notation, we may restate our earlier results as

$$\sum_{n=1}^{n} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}.$$

and

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots = 1.$$

We should note that, since the sum of a sequence is the limit of another sequence, and not all sequences have limits, there are sequences which do not have sums. For example, the sequence with terms $a_n = 1$ for $n = 1, 2, 3, \ldots$ does not have a sum since

$$s_n = \underbrace{1 + 1 + \cdots + 1}_{n \text{ times}} = n,$$

from which it follows that

$$\lim_{n\to\infty} s_n = \infty.$$

For another example, the sequence $\{(-1)^n\}$, $n = 0, 1, 2, \ldots$, does not have a sum since

$$s_n = \begin{cases} 1, & \text{if } n = 0, 2, 4, \ldots, \\ 0, & \text{if } n = 1, 3, 5, \ldots, \end{cases}$$

a sequence which clearly does not have a limit.

In general it may be difficult to determine the sum of a sequence; in fact, it may be difficult to determine even if the sequence has a sum. We will return to this problem in Chapter 5 when we have more tools at our disposal, as well as more motivation for studying infinite series. For now we will look at an important class of sequences for which the sum is determined with relative ease. These are the sequences for which the terms are in geometric progression; that is, sequences for which successive terms have a common ratio. We call the infinite series which corresponds to such a sequence a *geometric series.*

Geometric series

Suppose $\{a_n\}$ is a sequence with $a_n = cr^{n-1}$, where $c \neq 0$ and r are constants and $n = 1, 2, 3, \ldots$. Then the partial sums are

$$\begin{aligned} s_n &= a_1 + a_2 + a_3 + \cdots + a_n \\ &= c + cr + cr^2 + \cdots + cr^{n-1} \\ &= c(1 + r + r^2 + \cdots + r^{n-1}). \end{aligned}$$

If $r = 1$, $s_n = nc$ and so $\{s_n\}$ does not converge. If $r \neq 1$, it is easy to see, using long division (or the derivation in outlined in Problem 4), that

$$\frac{1 - r^n}{1 - r} = 1 + r + r^2 + \cdots + r^{n-1}. \tag{1.3.8}$$

Hence, if $r \neq 1$,

$$s_n = \frac{c(1 - r^n)}{1 - r}. \tag{1.3.9}$$

From (1.3.9), it is clear that $\{s_n\}$ does not converge if $|r| \geq 1$. But if $-1 < r < 1$, then

$$\lim_{n\to\infty} r^{n-1} = 0,$$

and so the sequence has the sum

$$s = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{c(1 - r^n)}{1 - r} = \frac{c}{1 - r}. \tag{1.3.10}$$

That is, we have now seen that

$$\sum_{n=1}^{\infty} cr^{n-1} = \frac{c}{1 - r} \tag{1.3.11}$$

whenever $-1 < r < 1$.

Example We have, using (1.3.11) with $c = 1$ and $r = \frac{1}{2}$,

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \frac{1}{1 - \frac{1}{2}} = 2.$$

Note that this agrees with our previous result that

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1.$$

Example We have

$$\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n} = \sum_{n=1}^{\infty} \frac{2}{3}\left(\frac{2}{3}\right)^{n-1} = \frac{\frac{2}{3}}{1 - \frac{2}{3}} = 2.$$

Example We have

$$\sum_{n=0}^{\infty} \left(\frac{1}{5^n}\right) = \sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^{n} = \frac{1}{1 - \frac{1}{5}} = \frac{5}{4}.$$

Note that in this example the sum starts with $n = 0$ instead of $n = 1$ as in (1.3.11). However, it is the initial power of r in (1.3.11) that is important, not how we write the index. Hence, the sum in this example could be written equally well as

$$\sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^{n-1},$$

or

$$\sum_{n=2}^{\infty} \left(\frac{1}{5}\right)^{n-2},$$

or

$$\sum_{n=100}^{\infty} \left(\frac{1}{5}\right)^{n-100},$$

as well as many other ways. The key in applying (1.3.11) is that we identify c and r so that the first term in the sum is $cr^0 = c$.

Example We have

$$\sum_{n=2}^{\infty} 4(0.34)^n = \sum_{n=2}^{\infty} 4(0.34)^2(0.34)^{n-2} = \frac{4(0.34)^2}{1 - 0.34} = 0.7006,$$

where we have used (1.3.11) with $c = 4(0.34)^2$ and $r = 0.34$.

We are now in a position to compute the sum in (1.3.1), and hence complete our first example

Example Let P be the probability that, when tossing a fair coin repeatedly, the first head appears on an even toss. Then we have seen that

$$P = P_2 + P_4 + P_6 + \cdots = \sum_{n=1}^{\infty} P_{2n},$$

where

$$P_{2n} = \left(\frac{1}{2}\right)^{2n} = \left(\left(\frac{1}{2}\right)^2\right)^n = \left(\frac{1}{4}\right)^n$$

for $n = 1, 2, 3, \ldots$. Thus

$$P = \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n = \sum_{n=1}^{\infty} \frac{1}{4}\left(\frac{1}{4}\right)^{n-1} = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{1}{3}.$$

Example Economists often talk about the *multiplier effect* of an infusion of money into an economy which results in new spending many times greater than the original amount spent. This is a consequence of the recipients of the money spending a certain percentage of their new money, the recipients of this spending again spending a certain percentage of their gain, and so on. For example, suppose the government spends three million dollars, and suppose that at each stage the recipients spend 90% of the money they receive. Then the first recipients spend $(3)(0.9) = 2.7$ million dollars, the second recipients spend

$$(3)(0.9)(0.9) = (3)(0.9)^2 = 2.43$$

million dollars (that is, 90% of the 2.7 million spent by the first recipients), the third recipients spend

$$(3)(0.9^2)(0.9) = (3)(0.9)^3 = 2.187$$

million dollars, and so on. If we denote the total amount of spending after n transactions by S_n , then, in millions of dollars,

$$\begin{aligned}
S_1 &= 3, \\
S_2 &= 3 + 3(0.9), \\
S_3 &= 3 + 3(0.9) + 3(0.9)^2, \\
S_4 &= 3 + 3(0.9) + 3(0.9)^2 + 3(0.9)^3,
\end{aligned}$$

and, in general,

$$S_n = 3 + 3(0.9) + 3(0.9)^2 + \cdots + 3(0.9)^{n-1}$$

for $n = 1, 2, 3, \ldots$. Although in actuality there will be only a finite number of transactions, we can see that as n increases the total spending will approach the sum

$$S = \sum_{n=1}^{\infty} 3(0.9)^{n-1} = \frac{3}{1-0.9} = 30$$

million dollars. Thus the initial governmental expenditure of three million dollars results in approximately 30 million dollars, 10 times the initial amount, in new spending in the economy. This partially explains why deficit spending by the government in depressed times can be far more beneficial to the economy than the actual amount spent, and why such spending during other times can be highly inflationary.

Example This example involves slightly more complicated probabilistic reasoning, as well as some additional algebraic simplification, before the problem is reduced to the summation of a geometric series. Suppose that a certain female animal has a 10% chance of dying during any given year of her life. Moreover, suppose the animal does not reproduce during her first year of life, but every year after has a 20% chance of successfully reproducing. What is the probability that this animal has offspring before dying?

Let P be the probability that the animal has offspring before dying and let P_n be the probability that the animal successfully reproduces for the first time in its nth year. Then

$$P = \sum_{n=1}^{\infty} P_n.$$

Note that our sum extends to infinity even though in reality it is highly unlikely that any such animal would live even to an age of 100 years. We do this because the model we are using, as with all mathematical models, is an idealization of the real situation. In this case, by assuming that a given animal of this species has a constant 10% chance of dying in any given year, we have implicitly assumed that there is no fixed upper bound to its life-span. Put another way, we have assigned a positive probability to an animal's living for, as an example, 1000 years, although this probability is very small (namely, $0.9^{1000} \approx 1.748 \times 10^{-46}$) and, hence, is not actually ever going to happen.

Since we have assumed that these animals cannot reproduce in their first year of life, we have $P_1 = 0$. To compute P_2, we note that 90% of all such females will live through their first year and that 20% of these will then have offspring successfully. Hence the proportion of females that successfully reproduce for the first time in their second year is

$$P_2 = (0.9)(0.2).$$

To compute P_3 , first we note that the proportion of females living until their third year will be $(0.9)(0.9)$ (that is, 90% of the 90% who lived through their first year). Now 80% of these will not have produced offspring successfully in their second year, so the proportion of females who reach their third year without having reproduced is $(0.9)^2(0.8)$. Finally, 20% of these will have success in reproducing in their third year. Thus

$$P_3 = (0.9)^2(0.8)(0.2).$$

Similar reasoning yields

$$P_4 = (0.9)^3(0.8)^2(0.2)$$

(that is, this represents a female who has lived through three years, did not reproduce in either her second or third year, but did have offspring in her fourth year) and, in general,

$$P_n = (0.9)^{n-1}(0.8)^{n-2}(0.2)$$

for $n = 2, 3, 4, \ldots$. Hence

$$\begin{aligned}
P &= \sum_{n=1}^{\infty} P_n \\
&= \sum_{n=2}^{\infty}(0.9)^{n-1}(0.8)^{n-2}(0.2) \\
&= \sum_{n=2}^{\infty}(0.2)(0.9)(0.9)^{n-2}(0.8)^{n-2} \\
&= \sum_{n=2}^{\infty}(0.18)(0.72)^{n-2} \\
&= \frac{0.18}{1-0.72} \\
&= 0.6429,
\end{aligned}$$

where the answer has been rounded to four decimal places. Thus we conclude that a given female of this species has just over a 64% chance of reproducing during her lifetime.

The harmonic series

It may happen that a sequence $\{a_n\}$ does not have a sum even though

$$\lim_{n\to\infty} a_n = 0.$$

One important example of this behavior is provided by the sequence $\{a_n\}$ with

$$a_n = \frac{1}{n}$$

for $n = 1, 2, 3, \ldots$. The resulting infinite series with nth partial sum given by

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n}. \tag{1.3.12}$$

is called the *harmonic series*. Since

$$s_{n+1} = 1 + \frac{1}{2} + \cdots + \frac{1}{n} + \frac{1}{n+1} = s_n + \frac{1}{n+1} > s_n, \tag{1.3.13}$$

the sequence $\{s_n\}$ is monotone increasing. Hence, by the Monotone Sequence Theorem, $\{s_n\}$ either converges or diverges to infinity. Now

$$\begin{aligned}
s_1 &= 1, \\
s_2 &= 1 + \frac{1}{2}, \\
s_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{1}{2} + \frac{1}{2} = 1 + 2\left(\frac{1}{2}\right), \\
s_8 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \\
&= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + 3\left(\frac{1}{2}\right),
\end{aligned}$$

and

$$s_{16} = s_8 + \sum_{j=9}^{16} \frac{1}{j} > s_8 + \sum_{j=9}^{16} \frac{1}{16} > 1 + 3\left(\frac{1}{2}\right) + \frac{8}{16} = 1 + 4\left(\frac{1}{2}\right).$$

Continuing in this pattern, we can see that, in general,

$$s_{2^m} > 1 + \frac{m}{2} \tag{1.3.14}$$

for any $m = 0, 1, 2, \ldots$. Thus, since $\frac{m}{2}$ may be made arbitrarily large, the sequence $\{s_n\}$ does not have an upper bound. Hence we must have

$$\lim_{n\to\infty} s_n = \infty, \tag{1.3.15}$$

and so the harmonic series does not have a sum.

Although the partial sums of the harmonic series diverge to infinity, they grow very slowly. For example, if $n = 500,000,000$, then s_n is between 20 and 21. That is,

$$20 < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{500,000,000} < 21. \tag{1.3.16}$$

Problems

1. Find the sum of each of the following infinite series which has a sum.

(a) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{n-1}$ (b) $\displaystyle\sum_{n=1}^{\infty} 4(0.21)^{n-1}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{5^n}$ (d) $\displaystyle\sum_{n=0}^{\infty} \frac{2}{7^n}$

(e) $\displaystyle\sum_{n=1}^{\infty} 7\left(\frac{1}{3}\right)^{n}\left(\frac{2}{5}\right)^{n-1}$ (f) $\displaystyle\sum_{n=3}^{\infty} \left(\frac{2}{3}\right)^{n}$

(g) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^{n-1}$

(h) $\displaystyle\sum_{n=1}^{\infty} 0.99999^n$

(i) $\displaystyle\sum_{n=1}^{\infty} 1.00001^n$

(j) $\displaystyle\sum_{n=30}^{\infty} 5\left(\frac{3^n}{7^{n-1}}\right)$

(k) $\displaystyle\sum_{n=100}^{\infty}\left(\frac{91}{89}\right)^{n-1}$

(l) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{\pi}{4}\right)^{n}$

(m) $\displaystyle\sum_{n=1}^{\infty}\sin(\pi n)$

(n) $\displaystyle\sum_{n=1}^{\infty}\cos(\pi n)$

2. Consider the infinite series with nth partial sum

$$s_n = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \cdots + \frac{1}{n!} = \sum_{j=0}^{\infty}\frac{1}{j!}.$$

Note that, by definition, $0! = 1$.

(a) Show that $s_n < 3$ for all values of n. Hint: Note that

$$n! = (1)(2)(3)\cdots(n) \geq (1)(2)(2)\cdots(2) = 2^{n-1}$$

for $n = 1, 2, 3, \ldots$.

(b) Combine (a) with the fact that $s_{n+1} > s_n$ for all n to conclude that

$$\sum_{j=1}^{\infty}\frac{1}{j!}$$

exists and is less than 3.

(c) In fact,

$$\sum_{j=1}^{\infty}\frac{1}{j!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots$$

is a well known irrational number. Add up a sufficient number of terms to enable you to guess the value of the sum. How many terms did it take?

(d) How many terms are necessary to obtain a partial sum that is within 0.000001 of the sum?

3. The sum

$$4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$$

is a well known irrational number.

(a) Add up a sufficient number of terms to enable you to guess the value of the sum. How many terms did it take?

(b) How many terms are necessary to obtain a partial sum that is within 0.01 of the sum?

4. This problem outlines an alternative method for deriving the result of (1.3.8). Suppose $r \neq 1$. For $n = 1, 2, 3, \ldots$, let

$$s_n = 1 + r + r^2 + \cdots + r^{n-1}.$$

Show that $s_n - rs_n = 1 - r^n$ and conclude that

$$s_n = \frac{1 - r^n}{1 - r}.$$

5. Using the model we used for the multiplier effect, find the total amount of new spending resulting from each of the following.
   (a) The government spends 2 billion dollars; each recipient spends 80% of what he or she receives.
   (b) The government spends 250 million dollars; each recipient spends 95% of what he or she receives.
   (c) The government spends A dollars; each recipient spends $100r\%$, $0 < r < 1$, of what he or she receives.

6. Government regulations specify that a bank may not loan 100% of its deposits; the bank must keep a certain percentage of its deposits in reserve. For example, if a bank must keep 15% of its deposits in reserve, then it may loan out \$850 from a \$1000 deposit. Typically, this \$850 will again be deposited in a bank, and that bank may loan out 85% of it. Again, this money will be deposited and 85% of it given out in loans. As this will continue indefinitely, the multiplier effect comes into play and the total amount of money in all the deposits resulting from the initial \$1000 deposit can be computed in the same manner as in our example.
   (a) Compute the total amount of the deposits resulting from the initial \$1000 deposit.
   (b) How would the answer in (a) change if the reserve rate was changed from 15% to 20%?
   (c) How would the answer in (a) change if the reserve rate was changed from 15% to 10%?

7. A ball is dropped from a height of 10 meters. Suppose that every time it strikes the ground, it bounces back to a height which is 75% of the height of the previous bounce. Assuming an infinite number of bounces (again, an idealized mathematical model), how far does the ball travel before it comes to rest? What would happen if it rebounded to only 25% of its initial height?

8. Suppose the animal in the our final example above could not produce offspring for its first 3 years of life. How would this change the probability of a female's reproducing before dying?

9. Suppose the animal in our final example above has only an 80% chance of living through a given year. How does this change the probability of a female's reproducing before dying?

10. Suppose a female animal of the type discussed in the final example above has a $100r\%$, $0 \le r \le 1$, chance of reproducing each year after its first year.

    (a) Find the probability P of a female's reproducing before dying.

    (b) Plot P as a function of r for $0 \le r \le 1$.

    (c) Find the value of r for which $P = 0.5$.

11. How many terms of the harmonic series are needed to obtain a partial sum larger than 5? How many terms are needed to obtain a partial sum larger than 10?

12. Plot the points (n, s_n), where s_n is the nth partial sum of the harmonic series, for $n = 1, 2, 3, \ldots, 1000$. What does this show you about the rate of growth of the partial sums?

13. The first example of this section is a particular case of the more general problem of computing probabilities associated with the waiting time for some event to occur. As another example, suppose that an electronic switch works with probability p and fails with probability $q = 1 - p$. Then, using reasoning analogous to that used in the coin tossing example, the probability that the first failure will occur on the nth use of the switch is $p^{n-1}q$, $n = 1, 2, 3, \ldots$.

    (a) Can you justify this probability?

    (b) The reliability of the switch is given by the function

    $$\begin{aligned} R(n) &= \text{ probabilty that the switch does not fail until after the } n\text{th use} \\ &= \sum_{j=n+1}^{\infty} p^{j-1}q \end{aligned}.$$

    Show that $R(n) = p^n$, $n = 1, 2, 3, \ldots$.

    (c) Find away to show that $R(n) = p^n$ directly without using an infinite series.

Difference Equations to Differential Equations

Section 1.4

Difference Equations

At this point almost all of our sequences have had explicit formulas for their terms. That is, we have looked mainly at sequences for which we could write the nth term as $a_n = f(n)$ for some known function f. For example, if

$$a_n = \frac{n+1}{n^2+3},$$

then it is an easy matter to compute explicitly, say, $a_{10} = \frac{11}{103}$ or $a_{100} = \frac{101}{10003}$. In such cases we are able to compute any given term in the sequence without reference to any other terms in the sequence. However, it is often the case in applications that we do not begin with an explicit formula for the terms of a sequence; rather, we may know only some relationship between the various terms. An equation which expresses a value of a sequence as a function of the other terms in the sequence is called a *difference equation*. In particular, an equation which expresses the value a_n of a sequence $\{a_n\}$ as a function of the term a_{n-1} is called a *first-order difference equation*. If we can find a function f such that $a_n = f(n)$, $n = 1, 2, 3, \ldots$, then we will have solved the difference equation. In this section we will consider a class of difference equations that are solvable in this sense; in the next section we will discuss an example where an explicit solution is not possible.

Example Suppose a certain population of owls is growing at the rate of 2% per year. If we let x_0 represent the size of the initial population of owls and x_n the number of owls n years later, then

$$x_{n+1} = x_n + 0.02x_n = 1.02x_n \tag{1.4.1}$$

for $n = 0, 1, 2, \ldots$. That is, the number of owls in any given year is equal to the number of owls in the previous year plus 2% of the number of owls in the previous year. Equation (1.4.1) is an example of a first-order difference equation; it relates the number of owls in a given year with the number of owls in the previous year. Hence we know the value of a specific x_n once we know the value of x_{n-1}. To get the sequence started we have to know the value of x_0. For example, if initially we have a population of $x_0 = 100$ owls and we want to know what the population will be after 4 years, we may compute

$$\begin{aligned} x_1 &= 1.02x_0 = (1.02)(100) = 102, \\ x_2 &= 1.02x_1 = (1.02)(102) = 104.04, \\ x_3 &= 1.02x_2 = (1.02)(104.04) = 106.1208, \end{aligned}$$

and

$$x_4 = 1.02x_3 = (1.02)(106.1208) = 108.243216.$$

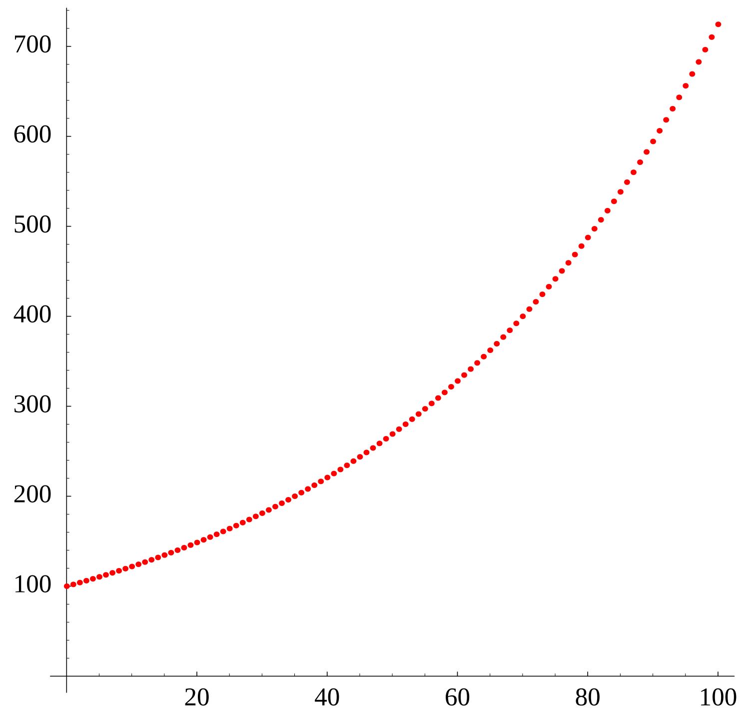


Figure 1.4.1 Plot of (n, x_n), $n = 0, 1, 2, \ldots$, where $x_0 = 100$ and $x_{n+1} = 1.02x_n$

Thus we would expect about 108 owls in the population after 4 years. Note that although it is not possible to have a fractional part of an owl, it is nevertheless important to keep the fractional part in intermediary calculations.

We may work backwards to find x_4 explicitly in terms of x_0:

$$\begin{aligned} x_4 &= 1.02x_3 \\ &= (1.02)(1.02)x_2 \\ &= (1.02)(1.02)(102)x_1 \\ &= (1.02)(1.02)(1.02)(1.02)x_0 \\ &= (1.02)^4 x_0. \end{aligned}$$

This is interesting because it indicates that we can compute x_4 without reference to the values of x_1, x_2, and x_3, provided, of course, that we know the value of x_0. If we do this in general, then we have solved the difference equation $x_{n+1} = 1.02x_n$. Namely, we have, for any $n = 1, 2, 3, \ldots$,

$$x_n = 1.02x_{n-1} = (1.02)^2 x_{n-2} = (1.02)^3 x_{n-3} = \cdots = (1.02)^n x_0. \tag{1.4.2}$$

For example, if $x_0 = 100$ as above, then we can compute

$$x_{20} = (1.02)^{20}(100) \approx 149,$$

or even

$$x_{150} = (1.02)^{150}(100) \approx 1,950,$$

without having to compute any intermediate values.

For a geometric feeling of how the population is changing with time, Figure 1.4.1 shows a plot of the points (n, x_n) for $n = 0, 1, 2, \ldots 100$. Of course, whether or not our model will provide an accurate prediction of the owl population 100 or 200 years into the future is an entirely different question. Frequently, a simple population model like this will be valid only for a short span of time during which the rate of growth of population remains stable.

By replacing 1.02 with an arbitrary constant α in (1.4.2), we arrive at the general result that the solution of the difference equation

$$x_{n+1} = \alpha x_n, \tag{1.4.3}$$

$n = 0, 1, 2, \ldots$, is given by

$$x_n = \alpha^n x_0, \tag{1.4.4}$$

$n = 0, 1, 2, \ldots$. Note that this difference equation, and its solution, are useful whenever we are interested in a sequence of numbers where the $(n+1)$st term is a constant proportion of the nth term. Our first example, where a population was assumed to grow at a constant rate, is a common example of this type of behavior. Another common example is when a quantity decreases at a constant rate over time. This behavior is discussed in the next example in the context of radioactive decay.

Example Radium is a radioactive element which decays at a rate of 1% every 25 years. This means that the amount left at the beginning of any given 25 year period is equal to the amount at the beginning of the previous 25 year period minus 1% of that amount. That is, if x_0 is the initial amount of radium and x_n is the amount of radium still remaining after $25n$ years, then

$$x_{n+1} = x_n - 0.01x_n = 0.99x_n \tag{1.4.5}$$

for $n = 0, 1, 2, \ldots$. Since this is a difference equation of the form of (1.4.3) with $\alpha = 0.99$, we know that the solution is of the form (1.4.4). Namely,

$$x_n = (0.99)^n x_0$$

for $n = 0, 1, 2, \ldots$. For example, the amount left after 100 years is given by

$$x_4 = (0.99)^4 x_0 = 0.9606x_0,$$

where we have rounded the answer to four decimal places. That is, approximately 96% of the initial amount of radium will be left after 100 years. A plot of the amount of radium left versus number of years, assuming an initial amount of 500 grams, is given in Figure 1.4.2.

The half-life of a radioactive element is the number of years required for one-half of an initial amount to decay. Suppose that, for this example, N is the smallest integer for which x_N is less than one-half of the initial amount of radium. This would mean that

$$\frac{1}{2}x_0 \geq (0.99)^N x_0,$$

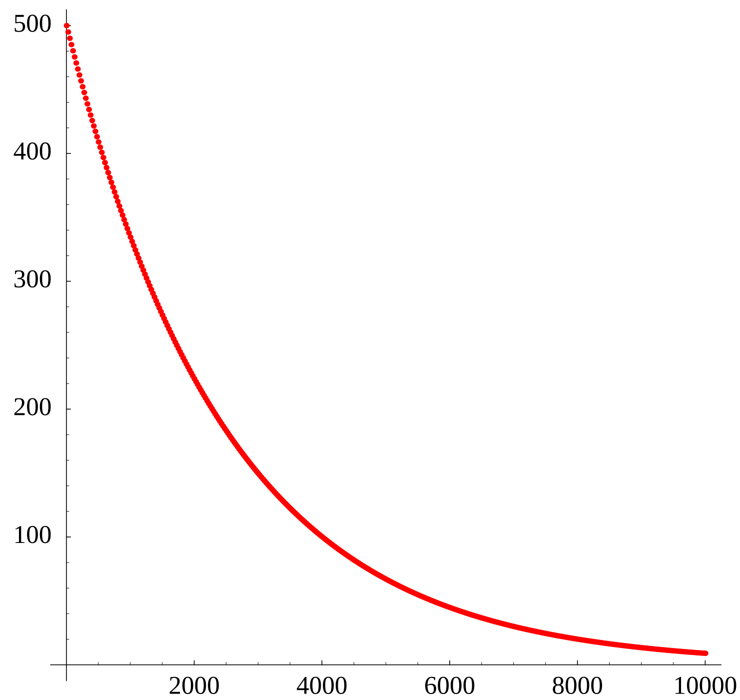


Figure 1.4.2 Plot of amount of radium versus number of years

which implies that

$$\frac{1}{2} \geq (0.99)^N.$$

Taking logarithms, we have

$$\log_{10}\left(\frac{1}{2}\right) \geq \log_{10}\left((0.99)^N\right),$$

which implies that

$$\log_{10}\left(\frac{1}{2}\right) \geq N \log_{10}(0.99).$$

Solving for N, and remembering that $\log_{10}(0.99) < 0$, we have

$$N \geq \frac{\log_{10}\left(\frac{1}{2}\right)}{\log_{10}(0.99)} = 68.98,$$

rounding to two decimal places. Hence, since N must be an integer, we have $N = 69$. Recalling that we are working with 25 year units of time, this shows that the half-life of radium is approximately $(25)(69) = 1725$ years. For example, this means that if we started with an initial amount of 100 grams of radium, after 1725 years we would still have 50 grams left. It would then take an additional 1725 years until the remaining amount would be reduced to 25 grams.

Although we have stated the results of the preceding example in discrete time units, namely, units of 25 years each, later we will see that the results hold for continuous time as well. In other words, although the difference equation (1.4.5) has been set up for nonnegative integer values of n, the solution (1.4.6) is valid for arbitrary nonnegative values of n. We will hold off discussion of these ideas until we consider differential equations, the continuous time versions of difference equations, in Chapter 6.

It is interesting to compare the plots in Figures 1.4.1 and 1.4.2. The first is an example of *exponential growth*, whereas the second is an example of *exponential decay*. In the first, the steepness of the graph increases with time; in the second, the graph flattens out over time. The difference equation (1.4.3) will always lead to the first behavior when $\alpha > 1$ and to the second when $0 < \alpha < 1$.

First-order linear difference equations

Given constants α and β, a difference equation of the form

$$x_{n+1} = \alpha x_n + \beta, \tag{1.4.6}$$

$n = 0, 1, 2, \ldots$, is called a *first-order linear difference equation*. Note that the difference equation (1.4.3) is of this form with $\beta = 0$. A procedure analogous to the method we used to solve (1.4.3) will enable us to solve this equation as well. Namely,

$$\begin{aligned}
x_n &= \alpha x_{n-1} + \beta \\
&= \alpha(\alpha x_{n-2} + \beta) + \beta \\
&= \alpha^2 x_{n-2} + \beta(\alpha + 1) \\
&= \alpha^2(\alpha x_{n-3} + \beta) + \beta(\alpha + 1) \\
&= \alpha^3 x_{n-3} + \beta(\alpha^2 + \alpha + 1) \\
&\ \ \vdots \\
&= \alpha^n x_0 + \beta(\alpha^{n-1} + \alpha^{n-2} + \cdots + \alpha^2 + \alpha + 1).
\end{aligned}$$

Note that if $\alpha = 1$, this gives us

$$x_n = x_0 + n\beta, \tag{1.4.7}$$

$n = 0, 1, 2, \ldots$, as the solution of the difference equation $x_{n+1} = x_n + \beta$. If $\alpha \neq 1$, we know from Section 1.3 that

$$\alpha^{n-1} + \alpha^{n-2} + \cdots + \alpha^2 + \alpha + 1 = \frac{1 - \alpha^n}{1 - \alpha}.$$

Hence

$$x_n = \alpha^n x_0 + \beta\left(\frac{1 - \alpha^n}{1 - \alpha}\right), \tag{1.4.8}$$

$n = 0, 1, 2, \ldots$, is the solution of the first-order linear difference equation $x_{n+1} = \alpha x_n + \beta$ when $\alpha \neq 1$.

We have seen examples of first-order linear equations in the population growth and radioactive decay examples above. Another interesting example arises in modeling the change in temperature of an object placed in an environment held at some constant temperature, such as a cup of tea cooling to room temperature or a glass of lemonade warming to room temperature. If T_0 represents the initial temperature of the object, S the constant temperature of the surrounding environment, and T_n the temperature of the object after n units of time, then the change in temperature over one unit of time is given by

$$T_{n+1} - T_n = k(T_n - S), \tag{1.4.9}$$

$n = 0, 1, 2, \ldots$, where k is a constant which depends upon the object. This difference equation is known as *Newton's law of cooling.* The equation says that the change in temperature over a fixed unit of time is proportional to the difference between the temperature of the object and the temperature of the surrounding environment. That is, large temperature differences result in a faster rate of cooling (or warming) than do small temperature differences. If S is known and enough information is given to determine k, then this equation may be rewritten in the form of a first order-linear difference equation and, hence, solved explicitly. The next example shows how this may be done.

Example Suppose a cup of tea, initially at a temperature of 180°F, is placed in a room which is held at a constant temperature of 80°F. Moreover, suppose that after one minute the tea has cooled to 175°F. What will the temperature be after 20 minutes?

If we let T_n be the temperature of the tea after n minutes and we let S be the temperature of the room, then we have $T_0 = 180$, $T_1 = 175$, and $S = 80$. Newton's law of cooling states that

$$T_{n+1} - T_n = k(T_n - 80), \tag{1.4.10}$$

$n = 0, 1, 2, \ldots$, where k is a constant which we will have to determine. To do so, we make use of the information given about the change in the temperature of the tea during the first minute. Namely, applying (1.4.10) with $n = 0$, we must have

$$T_1 - T_0 = k(T_0 - 80).$$

That is,

$$175 - 180 = k(180 - 80).$$

Hence

$$-5 = 100k,$$

and so

$$k = -\frac{5}{100} = -0.05.$$

Thus (1.4.10) becomes

$$T_{n+1} - T_n = -0.05(T_n - 80) = -0.05T_n + 4.$$

Hence

$$T_{n+1} = T_n - 0.05T_n + 4 = 0.95T_n + 4 \tag{1.4.11}$$

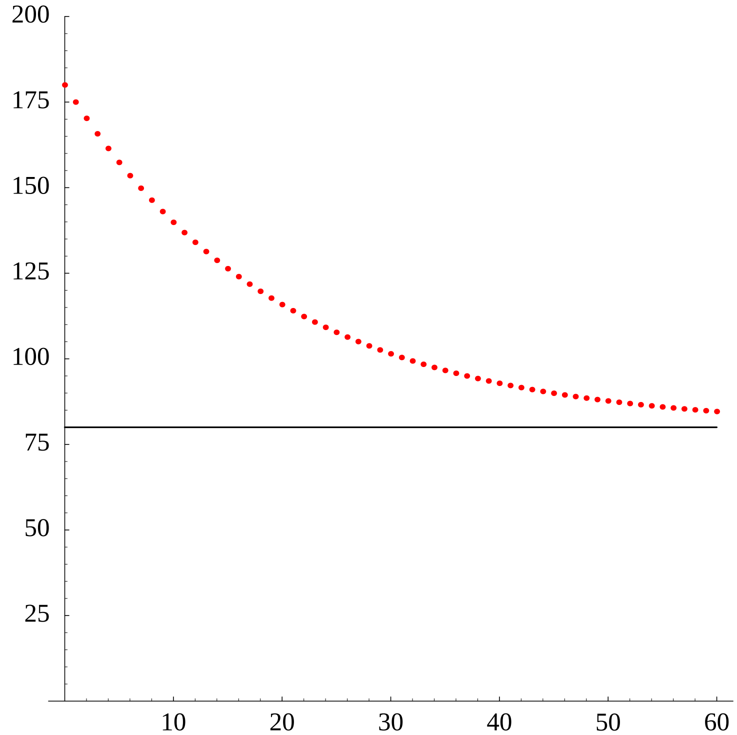


Figure 1.4.3 Tea temperature decreases asymptotically toward room temperature

for $n = 0, 1, 2, \ldots$. Now (1.4.11) is in the standard form of a first-order linear difference equation, so from (1.4.8) we know that the solution is

$$\begin{aligned} T_n &= (0.95)^n(180) + 4\left(\frac{1-(0.95)^n}{1-0.95}\right) \\ &= 180(0.95)^n + 80(1-(0.95)^n) \\ &= 80 + 100(0.95)^n \end{aligned}$$

for $n = 0, 1, 2, \ldots$. In particular,

$$T_{20} = 80 + 100(0.95)^{20} = 115.85,$$

where we have rounded the answer to two decimal places. Hence after 20 minutes the tea has cooled to just under 116°F. Also, since

$$\lim_{n\to\infty}(0.95)^n = 0,$$

we see that

$$\lim_{n\to\infty} T_n = \lim_{n\to\infty}(80 + 100(0.95)^n) = 80. \tag{1.4.12}$$

That is, as we would expect, the temperature of the tea will approach an equilibrium temperature of 80°F, the room temperature. In Figure 1.4.3 we have plotted temperature T_n versus time n for $n = 0, 1, 2, \ldots, 60$, along with the horizontal line $T = 80$. As indicated by (1.4.12), we can see that T_n decreases asymptotically toward 80°F as n increases.

Problems

1. Compute the next five terms of each of the following sequences from the given information.

   (a) $x_0 = 10$, $x_{n+1} = x_n + 4$

   (b) $y_0 = -1$, $y_{n+1} = \dfrac{1}{y_n}$

   (c) $x_0 = 40$, $x_{n+1} = 2x_n - 20$

   (d) $z_0 = 2$, $z_{n+1} = z_n^2 - z_n$

   (e) $x_0 = 2$, $x_1 = 3$, $x_{n+2} = x_{n+1} + x_n$

   (f) $x_0 = 15$, $x_n = \dfrac{1}{3}x_{n-1} + 2$

2. Solve the following difference equations with the given initial condition. Use your solution to find x_{10}.

   (a) $x_{n+1} = 2x_n$, $x_0 = 5$

   (b) $x_{n+1} = \dfrac{3}{4}x_n$, $x_0 = 100$

   (c) $x_{n+1} = 1.8x_n + 10$, $x_0 = 20$

   (d) $4x_{n+1} - 2x_n = 12$, $x_0 = 6$

   (e) $x_{n+1} - x_n = 3x_n + 4$, $x_0 = 2$

   (f) $5x_{n+1} - 3x_n = 2x_{n+1} - x_n$, $x_0 = 100$

3. A population of weasels is growing at rate of 3% per year. Let w_n be the number of weasels n years from now and suppose that there are currently 350 weasels.

   (a) Write a difference equation which describes how the population changes from year to year.

   (b) Solve the difference equation of part (a). If the population growth continues at the rate of 3%, how many weasels will there be 15 years from now?

   (c) Plot w_n versus n for $n = 0, 1, 2, \ldots, 100$.

   (d) How many years will it take for the population to double?

   (e) Find $\lim_{n\to\infty} w_n$. What does this say about the long-term size of the population? Will this really happen?

4. If the rate of growth of the weasel population in Problem 3 was 5% instead of 3%, how many years would it take for the population to double?

5. Suppose that the weasel population of Problem 3 would grow at a rate of 3% a year if left to itself, but poachers kill 6 weasels every year for their fur.

   (a) Write a difference equation which describes how the population changes from year to year.

   (b) Solve the difference equation of part (a). How many weasels will there be in 15 years?

   (c) Find $\lim_{n\to\infty} w_n$. What does this say about the long-term size of the population?

   (d) Will the population eventually double? If so, how long will this take?

   (e) Plot w_n versus n for $n = 0, 1, 2, \ldots, 100$.

6. Suppose that the weasel population of Problem 3 would grow at a rate of 3% a year if left to itself, but poachers kill 15 weasels every year for their fur.
   (a) Write a difference equation which describes how the population changes from year to year.
   (b) Solve the difference equation of part (a). How many weasels will there be in 15 years?
   (c) Find $\lim_{n\to\infty} w_n$. What does this say about the long-term size of the population?
   (d) Will the population eventually double? If so, how long will this take
   (e) Will the population eventually die out? If so, how long will this take?
   (f) Plot w_n versus n for $n = 0, 1, 2, \ldots, 100$.
7. A radioactive element is known to decay at the rate of 2% every 20 years.
   (a) If initially you had 165 grams of this element, how much would you have in 60 years?
   (b) What is the half-life of this element?
   (c) Suppose that the bones of a certain animal maintain a constant level of this element while the animal is living, but the element begins to decay as soon as the animal dies. If a bone of this animal is found and is determined to have only 10% of its original level of this element, how old is the bone?
8. Repeat Problem 7 if the element decays at the rate of 3% every 10 years.
9. A cup of coffee has an initial temperature of 165°F, but cools to 155°F in one minute when placed in a room with a temperature of 70°F. Let T_n be the temperature of the coffee after n minutes.
   (a) Write a difference equation, in standard first order linear form, which describes the change in temperature of the coffee from minute to minute.
   (b) Solve the difference equation from part (a).
   (c) Find the temperature of the coffee after 25 minutes.
   (d) Find $\lim_{n\to\infty} T_n$.
   (e) Plot T_n versus n for $n = 0, 1, 2, \ldots 120$.
   (f) Does the temperature ever reach 70°F?
10. A glass of lemonade, initially at a temperature of 42°F, is placed in a room with a temperature of 78°F. If the lemonade warms to 45°F in 30 seconds, what will its temperature be in 10 minutes?
11. An iron ingot, heated to a temperature of 300°C, is placed in a liquid bath held at a constant temperature of 90°C. If the ingot cools to 250°C in two minutes, what will its temperature be in 20 minutes?
12. A glass of ginger ale is left in a room. Initially, the ginger ale has a temperature of 45°F, but after one minute the temperature has increased to 50°F and after two minutes it has increased to 54°F. What is the temperature of the room?

13. In his book *Liber Abaci* (*Book of the Abacus*), Leonardo of Pisa, also know as Fibonacci, posed the following question: How many pairs of rabbits will be produced in a year, beginning with a single pair, if in every month each pair bears a new pair which becomes productive from the second month on? (See *A History of Mathematics* by Carl B. Boyer, Princeton University Press, 1985, page 281).

    (a) Let f_n be the number of pairs of rabbits in the nth month. Explain why $f_1 = 1$ and $f_2 = 1$.

    (b) Explain why $f_{n+2} = f_{n+1} + f_n$ for $n = 1, 2, 3, \ldots$.

    (c) Compute f_n for $n = 3, 4, 5, 6, 7, 8$ by hand.

    (d) Compute f_n for $n = 1, 2, 3, \ldots, 100$.

    (e) What is $\lim_{n\to\infty} f_n$?

    (f) Compute
    $$r_n = \frac{f_n}{f_{n+1}}$$
    for $n = 1, 2, 3, \ldots, 100$. Do you think $\lim_{n\to\infty} r_n$ exists? If so, what is a good approximation for this limit to five decimal places?

    (g) Show that
    $$r_{n+1} = \frac{1}{1 + r_n}.$$

    (h) Using (g) and assuming that $\lim_{n\to\infty} r_n$ exists, show that
    $$\lim_{n\to\infty} r_n = \frac{\sqrt{5} - 1}{2},$$
    the *golden section ratio.*

14. Given $x_0 = 0$ and $x_{10} = 20$, show that $x_n = 2n$ satisfies the difference equation
$$x_n = \frac{x_{n-1} + x_{n+1}}{2}$$
for $n = 1, 2, 3, \ldots, 9$. This difference equation is a discrete model for the equilibrium heat distribution along a a straight piece of wire running from 0 to 10 with the temperature at 0 held at 0° and the temperature at 10 held at 20°.

15. How would the solution to Problem 14 change if we changed the boundary conditions to $x_0 = 10$ and $x_{10} = 50$?

16. An approximate solution of a two-dimensional version of the model in Problem 14 may be found using a spreadsheet. For example, you might set cells A1-A20 and H1-H20 equal to 10 and cells B1-G1 and B20-G20 equal to 0. This would represent a flat rectangular piece of metal with the temperature along the vertical sides held fixed at 10° and the temperature along the horizontal sides held fixed at 0°. Now set the value of every cell inside the rectangle to be equal to the average of the values of its four

neighboring cells. For example, you would put the formula (A2+C2+B1+B3)/4 in cell B2 and then copy this cell to all the cells in the block from B2 to G19. Now have the spreadsheet repeatedly compute the values of the cells until they stabilize (that is, until they no longer change values when you recompute). If you format the cell values so that they are all integers, this should not take too long. What you have now is the equilibrium heat distribution for the metal plate. Now try different boundary conditions to obtain different equilibrium heat distributions.

Difference Equations to **D**ifferential Equations

Section 1.5

Nonlinear Difference Equations

In Section 1.4 we discussed the difference equation

$$x_{n+1} = \alpha x_n, \tag{1.5.1}$$

$n = 0, 1, 2, \ldots$, as a model for either growth or decay and we saw that its solution is given by

$$x_n = \alpha^n x_0,$$

$n = 0, 1, 2, \ldots$. Now

$$\lim_{n\to\infty} \alpha^n = \begin{cases} 0, & 0 < \alpha < 1, \\ 1, & \alpha = 1, \\ \infty, & \alpha > 1, \end{cases} \tag{1.5.2}$$

from which it follows that if $\{x_n\}$ is a solution of (1.5.1) with $x_0 > 0$, then

$$\lim_{n\to\infty} x_n = x_0 \lim_{n\to\infty} \alpha^n = \begin{cases} 0, & 0 < \alpha < 1, \\ x_0, & \alpha = 1, \\ \infty, & \alpha > 1. \end{cases} \tag{1.5.3}$$

These limiting values are consistent with our radioactive decay example since, in that case, $0 < \alpha < 1$ and we would expect the amount of a radioactive element to decline toward 0 over time. The case $0 < \alpha < 1$ also may make sense for a population model if the population is declining and heading toward extinction. However, the unbounded growth indefinitely into the future implied by the case $\alpha > 1$ is very unlikely for a population model: eventually ecological or even sociological problems come to the forefront, such as when the population begins to overreach the resources available to it, and the rate of growth of the population changes. Even for bacteria growing in a Petri dish, diminishing food and space eventually cause a change in the rate of growth. Hence the equation

$$x_{n+1} = \alpha x_n, \tag{1.5.4}$$

for $n = 0, 1, 2, \ldots$ and $\alpha > 1$, called the *uninhibited*, or *natural*, *growth model*, although often accurate as a model of population growth over short periods of time, is usually too simplistic for predictions over long time spans.

The inhibited growth model

Suppose we wish to model the growth of a certain population which, without ecological constraints, would grow at a rate of $100\beta\%$ per unit of time. That is, if x_n represents the size of the population after n units of time and there are no constraints on the size of the population, then

$$x_{n+1} - x_n = \beta x_n \tag{1.5.5}$$

for $n = 0, 1, 2, \ldots$. However, suppose that, because of the limitation of resources, the population will begin to decline if it ever has more than M individuals. We call M the *carrying capacity* of the available resources, the maximum population which is sustainable over time. Then it would be reasonable to modify our model by forcing the amount of increase over a unit of time to decrease as the size of the population approaches M and to become negative if the size of the population ever exceeds M. One way to accomplish this is to multiply the term βx_n in (1.5.5) by

$$\frac{M - x_n}{M},$$

a ratio which is close to 1 when x_n is small, close to 0 when x_n is close to M, and negative when x_n exceeds M. This leads us to the difference equation

$$x_{n+1} - x_n = \beta x_n \left(\frac{M - x_n}{M} \right),$$

$n = 0, 1, 2, \ldots$, or, equivalently,

$$x_{n+1} = x_n + \frac{\beta}{M} x_n (M - x_n), \tag{1.5.6}$$

$n = 0, 1, 2, \ldots$, which we call the *inhibited growth model*, also known as the *discrete logistic equation*. This is an example of a nonlinear difference equation because if we multiply out the right-hand side of the equation we have a quadratic term, namely, $\frac{\beta}{M} x_n^2$. Such equations are, in general, far more difficult to solve than the linear difference equations we considered in Section 1.4; in fact, many nonlinear difference equations are not solvable in terms of the elementary functions of calculus. Hence we will not consider any methods for solving such equations, relying instead on computing specific solutions by iterating the equation using a calculator or, preferably, a computer.

Example Suppose a population of owls, currently numbering 100, has a natural growth rate of 4%, but, because of the limited resources of their natural habitat, can sustain a population of no more than 500. If we let x_n represent the size of the population n years from now, then, using the inhibited growth model, we should have

$$x_{n+1} = x_n + \frac{0.04}{500} x_n (500 - x_n) = x_n + 0.00008 x_n (500 - x_n)$$

for $n = 0, 1, 2, \ldots$. Using this equation we are able to compute, for example, the predicted size of the population for the next 10 years:

Year	0	1	2	3	4	5	6	7	8	9	10
Population	100.0	103.2	106.5	109.8	113.3	116.8	120.3	124.0	127.8	131.5	135.4

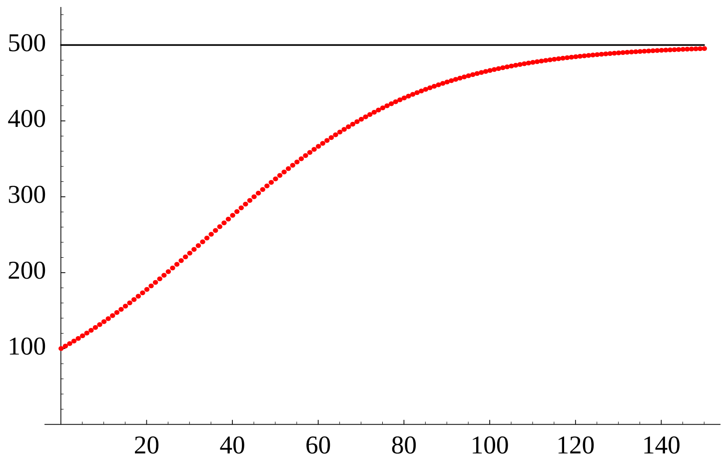


Figure 1.5.1 Inhibited population growth with $\beta = 0.04$

Here, and in subsequent tables, we have rounded our results to the first decimal place. It is interesting to compare these results to the corresponding results for the uninhibited growth model. If we let y_n be the predicted population n years from now using the uninhibited growth model, then we would have

$$y_{n+1} = y_n + 0.04y_n = 1.04y_n,$$

$n = 0, 1, 2, \ldots$, which has the exact solution

$$y_n = 100(1.04)^n$$

for $n = 0, 1, 2, \ldots$. From this model we obtain the following predicted population sizes:

Year	0	1	2	3	4	5	6	7	8	9	10
Population	100.0	104.0	108.2	112.5	117.0	121.7	126.5	131.6	136.9	142.3	148.0

As we would expect, the population is growing more slowly under the inhibited population growth model than under the uninhibited model. Moreover, this difference will become more pronounced over time. For example, after 150 years we would have $x_{150} = 495.4$ and $y_{150} = 35,892$, illustrating how the inhibited growth model is constrained by the carrying capacity of 500 while the uninhibited growth model will have unbounded growth. Figures 1.5.1 and 1.5.2 provide a graphical comparison of the two models for $n = 0, 1, 2, \ldots, 150$. Note that it appears that

$$\lim_{n\to\infty} x_n = 500,$$

while

$$\lim_{n\to\infty} y_n = \infty.$$

With the inhibited growth model, if $0 < \beta < 1$ and $x_n < M$, then

$$\beta\frac{x_n}{M} < 1,$$

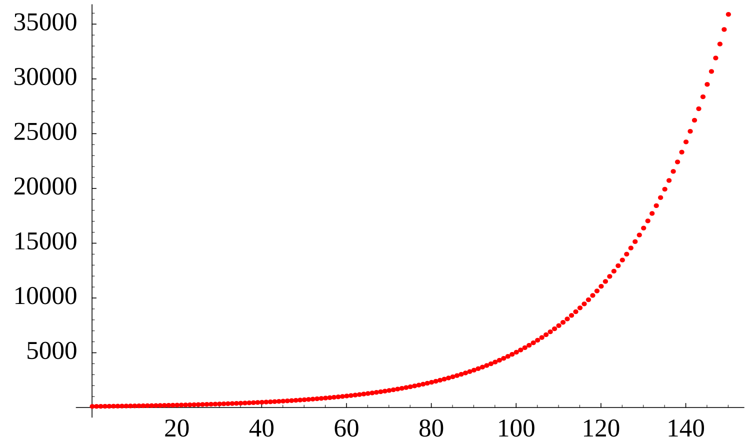


Figure 1.5.2 Uninhibited population growth with $\beta = 0.04$

so

$$x_{n+1} = x_n + \beta\frac{x_n}{M}(M - x_n) < x_n + (M - x_n) = M \tag{1.5.7}$$

for $n = 0, 1, 2, \ldots$. Thus if $0 < \beta < 1$, and we start with $x_0 < M$, then $x_n < M$ for all n. Moreover, since

$$\beta\frac{x_n}{M} > 0,$$

we have

$$x_{n+1} = x_n + \beta\frac{x_n}{M}(M - x_n) > x_n$$

for all n. Hence the sequence $\{x_n\}$ is monotone increasing and bounded, and so must have a limit. In Problem 8 you will be asked to verify that this limit is in fact M, as appeared to be the case in the previous example.

If $\beta > 1$, it may be the case that there are values of n for which $x_n > M$, in which case

$$\beta\frac{x_n}{M}(M - x_n) < 0$$

and, as a consequence, $x_{n+1} < x_n$.

Example Suppose $x_0 = 100$ and $M = 500$ as in the previous example, but now let $\beta = 1.5$. That is,

$$x_{n+1} = x_n + \frac{1.5}{500}x_n(M - x_n) = x_n + 0.003x_n(500 - x_n)$$

for $n = 0, 1, 2, \ldots$. This equation generates the following values:

Year	0	1	2	3	4	5	6	7	8	9	10
Population	100.0	220.0	404.8	520.4	448.5	505.3	497.2	501.4	499.3	500.3	499.8

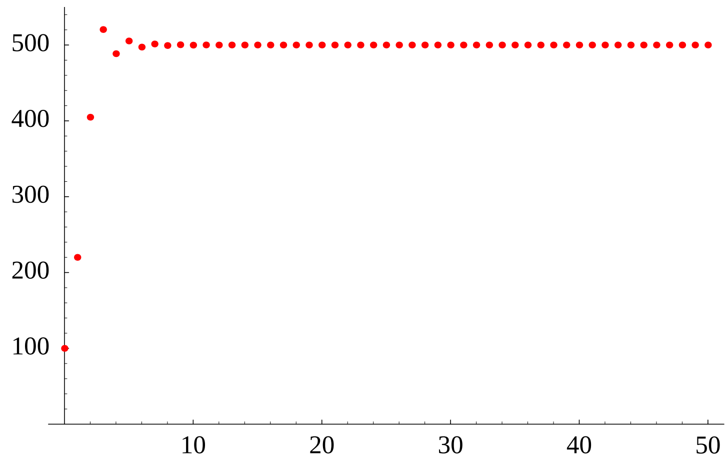


Figure 1.5.3 Inhibited population growth with $\beta = 1.5$

Note how the values increase rapidly (as we should expect with such a large value for β) to above the carrying capacity of 500, but then oscillate about 500, with the oscillations diminishing in size. In fact it may be shown that it is also true in this case that

$$\lim_{n \to \infty} x_n = 500.$$

See Figure 1.5.3.

It is possible to show that, for the inhibited growth model of (1.5.6),

$$\lim_{n \to \infty} x_n = M$$

whenever $0 < \beta \leq 2$. However, there are other possible behaviors when $\beta > 2$.

Example Suppose $x_0 = 100$ and $M = 500$, as in the previous examples, but now let $\beta = 2.3$. That is,

$$x_{n+1} = x_n + \frac{2.3}{500} x_n (M - x_n) = x_n + 0.0046 x_n (500 - x_n)$$

for $n = 0, 1, 2, \ldots$. This equation generates the following values:

Year	0	1	2	3	4	5	6	7	8	9	10
Population	100.0	284.0	566.2	393.8	586.2	353.8	591.7	342.0	590.6	344.5	590.9

Year	11	12	13	14	15	16	17	18	19	20
Population	343.8	590.8	344.0	590.9	343.9	590.8	343.9	590.8	343.9	590.8

Notice that instead of approaching a single limiting value, the population is settling down to an oscillation between 344 and 591. We say that the sequence $\{x_n\}$ is approaching a *limiting cycle of period 2*, as shown in Figure 1.5.4.

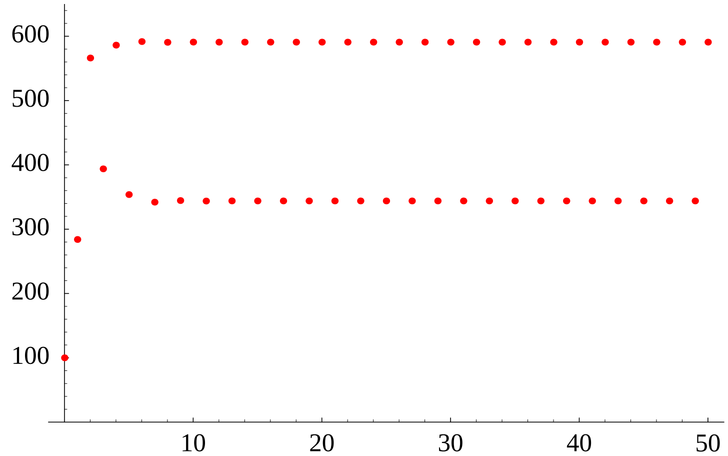


Figure 1.5.4 Inhibited population growth with $\beta = 2.3$

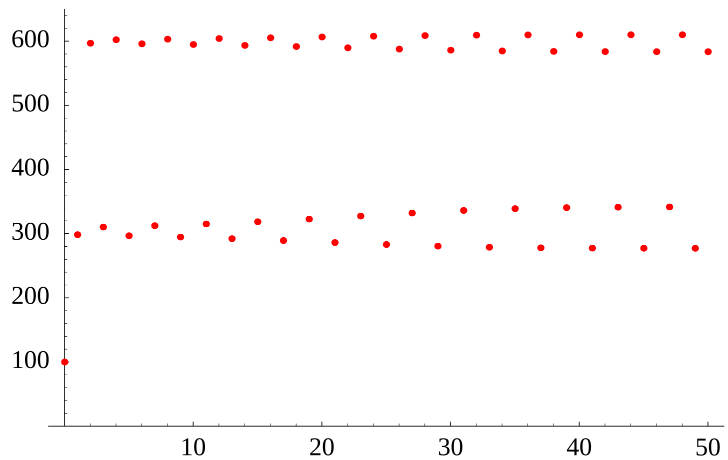


Figure 1.5.5 Inhibited population growth with $\beta = 2.48$

It is possible to obtain limiting cycles of longer periods by increasing β. For example, Figure 1.5.5 shows the effect of letting $\beta = 2.48$. Note that $\{x_n\}$ appears to be approaching a limiting cycle of period 4.

With appropriate choices for β and x_0, it is in fact possible for the inhibited growth model to exhibit limiting cycles of any given period. This is related to the fact that it is possible for this model to behave chaotically. Intuitively, a sequence is *chaotic* if it displays erratic behavior which, although in theory completely determined by a difference equation such as (1.5.6), is in practice unpredictable because small changes in the initial value x_0 yield strikingly different sequences. For example, Figures 1.5.6 and 1.5.7 illustrate the differing behavior of the inhibited growth model with $\beta = 2.95$, first for an initial population of 100 and then for an initial population of 101.

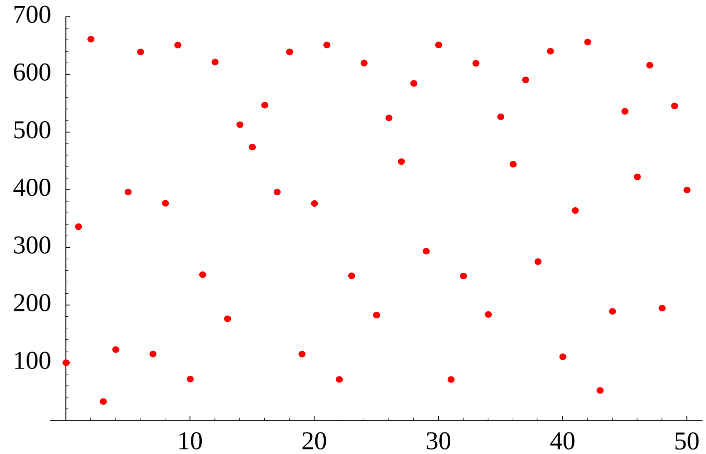


Figure 1.5.6 Inhibited population growth with $\beta = 2.95$ and $x_0 = 100$

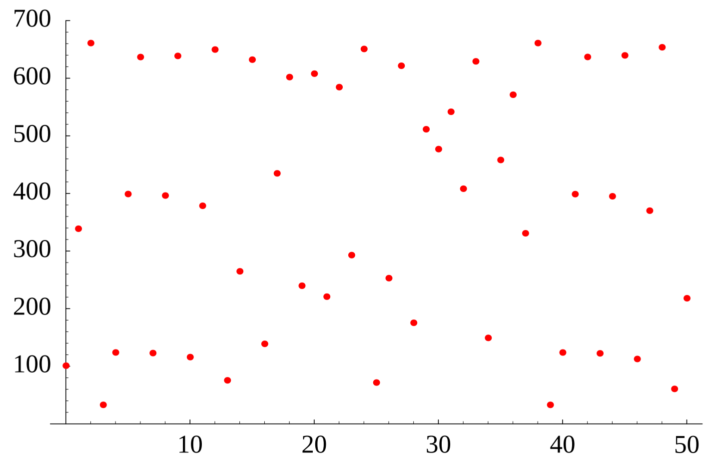


Figure 1.5.7 Inhibited population growth with $\beta = 2.95$ and $x_0 = 101$

Problems

1. A population of weasels has a natural growth rate of 3% per year. Let w_n be the number of weasels n years from now and suppose there are currently 300 weasels.
   (a) Suppose the carrying capacity of the weasel's habitat is 1000. Using an inhibited growth model, write a difference equation which describes how the population changes from year to year.
   (b) Using the difference equation from part (a), compute w_n for $n = 1, 2, \ldots, 150$.
   (c) How many years will it take for the population to double? To triple?
   (d) Plot w_n versus n for $n = 0, 1, 2, \ldots, 150$. From the plot, guess $\lim_{n\to\infty} w_n$.
   (e) Compare your answers with those to Problem 3 in Section 1.4.
2. Suppose a population of northern pike in a lake in Montana has a natural growth rate of 4.5% per year, but the lake can support no more than 10,000 pike. Let p_n be the

number of pike n years from now and suppose $p_0 = 1000$.

(a) Use the inhibited growth model to write a difference equation which describes how the population changes from year to year.

(b) Using the difference equation from part (a), compute p_n for $n = 1, 2, 3, \ldots 50$.

(c) How many years will it take for the population to double? To triple?

(d) Plot p_n versus n for $n = 0, 1, 2, \ldots 200$. From the plot, guess $\lim_{n\to\infty} p_n$.

(e) How many years will it take for the population to reach 9500?

3. Do Problem 2 assuming an uninhibited growth model and no restrictions on the number of pike that the lake can support.

4. Suppose r_n represents the number of snowshoe rabbits in a certain National Forest in Alaska after n years with an initial value of $r_0 = 5000$. Moreover, suppose the forest can support no more than 10,000 rabbits and $\{r_n\}$satisfies the inhibited growth model

$$r_{n+1} = r_n + \frac{\beta}{10,000} r_n (10,000 - r_n)$$

for $n = 0, 1, 2, \ldots$. For each of the following values for β, plot r_n versus n for $n = 0, 1, 2, \ldots, 100$ and comment on the behavior of the sequence, in particular noting any limiting values or limiting cycles

(a) $\beta = 0.5$
(b) $\beta = 1.5$
(c) $\beta = 2.4$
(d) $\beta = 2.5$
(e) $\beta = 2.56$
(f) $\beta = 2.9$

5. Using an initial value of $x_0 = 0.5$, let $\{x_n\}$ be the sequence which satisfies the difference equation

$$x_{n+1} = \mu x_n (1 - x_n),$$

$n = 0, 1, 2, \ldots$. Plot x_n versus n for the following values of μ and comment on the behavior of the sequence, in particular noting any limiting values or limiting cycles.

(a) $\mu = 0.9$
(b) $\mu = 1.0$
(c) $\mu = 1.5$
(d) $\mu = 2.0$
(e) $\mu = 2.5$
(f) $\mu = 3.0$
(g) $\mu = 3.1$
(h) $\mu = 3.5$
(i) $\mu = 3.57$
(j) $\mu = 1 + \sqrt{8}$
(k) $\mu = 3.99$
(l) $\mu = 4.0$

6. Repeat Problem 5 starting with an initial value of $x_0 = 0.6$.

7. If f is any function defined for real numbers, then the difference equation

$$x_{n+1} = f(x_n),$$

$n = 0, 1, 2, \ldots$, is called a *discrete dynamical system*. For any given initial condition x_0, the sequence $\{x_n\}$ which satisfies this equation is called an orbit of f. Note that an orbit of f is simply the sequence of points

$$x_0, f(x_0), f(f(x_0)), f(f(f(x_0))), \ldots .$$

For example, the difference equation in Problem 5 is an example of a discrete dynamical system with $f(x) = \mu x(1 - x)$. For each of the following, compute 50 terms of the given orbit and discuss its behavior.

(a) $x_0 = 10,\ f(x) = 2x$

(b) $x_0 = 100,\ f(x) = 0.8x$

(c) $x_0 = 2\ f(x) = \cos(x)$

(d) $x_0 = 2,\ f(x) = \sin(x)$

(e) $x_0 = 5,\ f(x) = \dfrac{1}{2}\left(x + \dfrac{2}{x}\right)$

(f) $x_0 = 1,\ f(x) = \dfrac{2x^2}{3x^2 - 5}$

(g) $x_0 = 0,\ f(x) = x^2 + 1.0$

(h) $x_0 = 0,\ f(x) = x^2 - 0.5$

(i) $x_0 = 0,\ f(x) = x^2 - 0.8$

(j) $x_0 = 0,\ f(x) = x^2 - 1.0$

(k) $x_0 = 0,\ f(x) = x^2 - 1.9$

(l) $x_0 = 0,\ f(x) = x^2 - 2.0$

8. Assuming that the sequence $\{x_n\}$ satisfying the inhibited growth model equation

$$x_{n+1} = x_n + \frac{\beta}{M} x_n (M - x_n)$$

has a limit, show that $\lim_{n\to\infty} x_n = M$.

Difference Equations to Differential Equations

Section 2.1

Functions And Their Graphs

Since functions are the basic building blocks out of which mathematicians construct models of the physical world, it is essential that any student of mathematics have a firm grasp of the concept. In particular, one must be careful to distinguish between a given function and a notational or graphical representation for it. A function is a type of relationship, a mental concept that cannot be seen or touched. Although pictures and symbolic representations of a function are extremely important in understanding its behavior, the student must always keep in mind the distinction between the function itself and its representations.

Modern methods for giving a formal definition of a function, developed in the latter part of the 19th century, are based on set-theoretic ideas. We will not go into the details necessary to make such a precise definition, but rather aim at an intuitive understanding of the basic concept. For us, a function is a special type of relationship between two quantities. We often think of this relationship to be one of dependence. That is, if the value of one quantity, say y, is determined by the value of another quantity, say x, then we say that y is a function of x. For example, if x represents the height from which a certain rock is dropped and y represents the velocity with which the rock strikes the ground, then the value of y will depend on the value of x and we say that velocity y is a function of height x. Note here that if y is the terminal velocity of the object, then there are many different values of x which yield the same value of y, namely, any value of x which gives the object sufficient time to reach its terminal velocity before striking the ground. On the other hand, for a given value of x, there is only one related value of y. It is this latter property that makes the relationship between height and impact velocity a function. For any quantities represented by y and x, in order to say that y is a function of x we require that every value of x be related to exactly one value of y. Such a relationship often arises through some physical dependency, a cause creating a deterministic effect, but the definition does not require such a link between the quantities in question. A number of examples should help clarify this concept.

Example Sequences are example of functions. That is, if $\{x_n\}$ is a sequence with $n = 1, 2, 3, \ldots$, then every value of n determines exactly one value x_n. For example, the area of a regular polygon inscribed in a unit circle is a function of the number of sides. Also, a difference equation, such as

$$x_{n+1} = 1.02x_n,$$

$n = 0, 1, 2, \ldots$, makes x_n a function of n. For example, the size of a certain population of owls will be a function of the number of years from some starting date.

Example The area of a circle is a function of the radius of the circle.

Example The distance of the earth from the sun is a function of the time of year.

Example The temperature at a certain fixed point in space is a function of time.

In mathematical terminology, if y is a function of x, then we call x the *independent variable* and y the *dependent variable*. Also, the *domain* of this function is the set of permissible values for x and the *range* is the set of all values of y which correspond to some value of x.

Example Recall that the nth term of the sequence which gives the area of a regular n-sided polygon inscribed in a unit circle is

$$A_n = \frac{n}{2}\sin\left(\frac{2\pi}{n}\right),$$

$n = 3, 4, 5, \ldots$ The domain of this function is the set of integers $\{3, 4, 5, \ldots\}$. The range can be specified only by saying that it is the set of numbers

$$\left\{\frac{n}{2}\sin\left(\frac{2\pi}{n}\right) \,\middle|\, n = 3, 4, 5, \ldots\right\}.$$

Before proceeding further, we should recall the notation for intervals of real numbers. Given any real numbers a and b, we have

$$(a, b) = \{x \mid a < x < b\}, \tag{2.1.1}$$

$$(a, b] = \{x \mid a < x \leq b\}, \tag{2.1.2}$$

$$[a, b) = \{x \mid a \leq x < b\}, \tag{2.1.3}$$

$$[a, b] = \{x \mid a \leq x \leq b\}, \tag{2.1.4}$$

$$(a, \infty) = \{x \mid x > a\}, \tag{2.1.5}$$

$$[a, \infty) = \{x \mid x \geq a\}, \tag{2.1.6}$$

$$(-\infty, b) = \{x \mid x < b\}, \tag{2.1.7}$$

and

$$(-\infty, b] = \{x \mid x \leq b\}. \tag{2.1.8}$$

Moreover, we call intervals of the form (2.1.1), (2.1.5), and (2.1.7) *open intervals* and intervals of the form (2.1.4), (2.1.6), and (2.1.8) *closed intervals*.

Example If we let d specify the distance from the sun to the earth and t specify the time of year, then the function that relates d and t has domain

$$\{t \mid 0 \leq t \leq 8760\} = [0, 8760],$$

where t is specified in hours, and range

$$\{d \mid 91.4 \leq d \leq 94.6\} = [91.4, 94.6],$$

where d is specified in millions of miles.

Before learning much about a specific function, a mathematician must represent the function in some concrete form. This can be done in many ways. For example, we might construct a table of values for the function. Such a table might have two rows, one for values of the independent variable and one for the corresponding values of the dependent variable. For example, if T is the temperature, in degrees Fahrenheit, at the Kalispell airport weather station at time t, measured in hours past midnight, on August 3, 1999, then our table might look like the following:

Time (t)	0	1	2	3	4	5	6	7	8	9
Temperature (T)	68	66	64	62	61	59	60	64	68	70
Time (t)	10	11	12	13	14	15	16	17	18	19
Temperature (T)	74	76	78	80	84	84	82	81	79	77
Time (t)	20	21	22	23	24					
Temperature (T)	74	70	69	67	66					

In other words, this table provides a complete listing of the values of the dependent variable T which correspond to each value of the independent variable t.

Of course, if the domain of the function contains a large number of points, it might not be practical to represent the function using a table. Indeed, most of the functions which we will consider in this course have an infinite number of points in their domain, rendering complete representations using tables impossible. Moreover, even with a limited amount of data, it is hard to understand much about the underlying function by looking at a table. One alternative to a tabular representation of a function is a graphical representation. If y is a function of x, the graph of this function is the set of all points in the Cartesian plane with coordinates (x, y). If the domain of the function has only a finite number of points, as in the preceding example, then its graph is just a set of points in the plane, as we see in Figure 2.1.1. However, if we were able to plot this function for all values of t between 0 and 24, then its graph would become a curve passing through the points given by the table. With the given data, we could approximate this curve by plotting the given points and then connecting successive points by straight lines, as in Figure 2.1.2. In either form, the graph gives a good pictorial representation of the function. From this picture, we can easily identify such things as the high and low temperatures for the day, as well as the time at which they occurred, or the time of day when the temperature was changing most rapidly.

It would be hard to overestimate the importance of graphs in studying functions; we will in fact spend much time in this course considering graphs. However, the most concise, and at the same time most complete, representation for a function is a formula which expresses the values of the dependent variable in terms of values of the independent variable. For a given function it may not be possible to find such a formula. For example, the function which gives the temperature at the Kalispell airport for any given time during the day of August 3, 1999, is not expressible by a formula; the only way we can compute values for this function is to record the temperatures as they occur. On the other hand, for a circle of radius r and area A, the formula $A = \pi r^2$ gives us an explicit means for computing values of the dependent variable A for any given value of the independent variable r. The

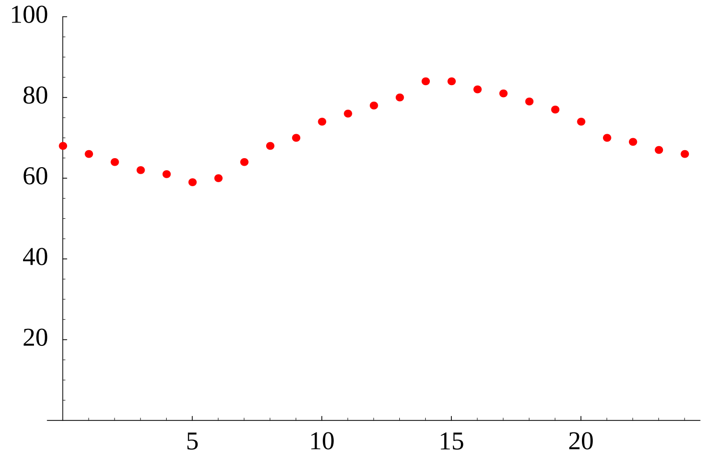


Figure 2.1.1 Plot of temperature data for Kalispell

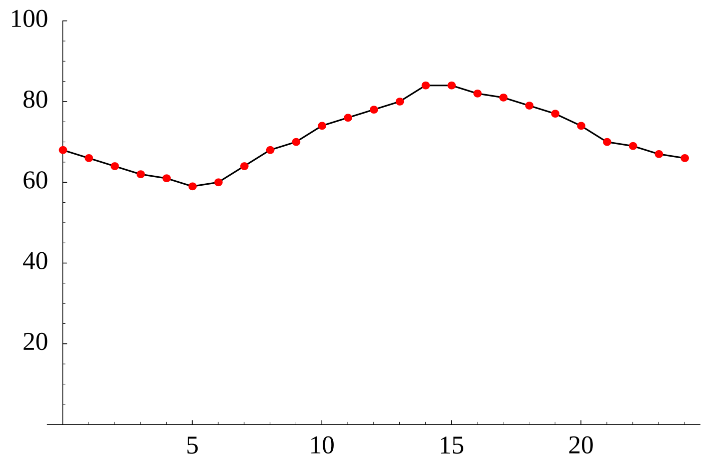


Figure 2.1.2 Plot of temperature date for Kalispell with lines connecting data points

existence of a formula for a function enables us to perform mathematical computations which, at best, could only be approximated otherwise. At the same time, it is important to remember that a function is an abstract object; it is not itself a formula or a number or a graph, but a relationship which exists between quantities specified by numbers. We need to keep this in mind, even as we proceed to work more and more with functions through their representations using formulas and graphs.

Example If V represents the volume and r the radius of a sphere, then V is a function of r and the formula

$$V = \frac{4}{3}\pi r^3$$

expresses this relationship. Note that the domain of this function is the open interval $(0, \infty)$, even though negative values of r can be substituted into the formula without any problems. This emphasizes that the function is determined by the underlying relationship between V and r. Here we also have the range equal to $(0, \infty)$.

Example Suppose the quantity y is related to the quantity x by the formula

$$y = \frac{1}{\sqrt{1-x^2}}.$$

Since, by convention, the square root notation refers to the positive square root of a given number, this relationship makes y a function of x. If we are given no further information about this function, then we should ascribe to it the largest possible domain and range. In this case, the domain is the interval $(-1, 1)$ (that is, values of x for which $1 - x^2$ is positive) and the range is $[1, \infty)$ (that is, the possible results from dividing 1 by numbers in the interval $(0, 1]$).

At this point, we have used notation for the dependent and independent variables of a function, but not for the function itself. As with variables, it is common to use letters to designate functions. For example, we frequently use f to denote a function, in which case f stands for the function itself, a relationship, while expressions like $f(x)$, $f(2)$, and $f(s)$ denote particular values of the function. That is $f(x)$, $f(2)$, and $f(s)$ represent values of the dependent variable which correspond to the values x, 2, and s, respectively, of the independent variable.

Example The expression

$$f(x) = \frac{1}{x^2}$$

tells us that f represents a function which associates the value

$$\frac{1}{x^2}$$

to a given value x of the independent variable. Hence, for example,

$$f(2) = \frac{1}{4},$$

$$f(-1) = 1,$$

$$f(s) = \frac{1}{s^2},$$

and

$$f(z+1) = \frac{1}{(z+1)^2}.$$

Note that the domain of f is $\{x \mid x \neq 0\}$. That is, f is defined for every real number except 0. The range of f is $(0, \infty)$.

Example Suppose S is the function which gives the temperature at the Kalispell airport on August 3, 1990. If we measure time in of hours since midnight, then we know, for example, that $S(2) = 64$ and $S(19) = 77$. However, if we let t represent the independent variable for this function, namely, the number of hours since midnight, then we do not have a general formula to express $S(t)$. For example, we cannot compute $S(7.5)$ or $S(3.2)$, let alone even consider what $S(\pi)$ might be.

It often happens that the output from one function is used as input for another function. For example, suppose a pebble is dropped in a pond and the resulting circular wave has a radius of $20t$ centimeters after t seconds. Then if r is the radius of the wave and A is the area inside the wave, we have $r = 20t$ and $A = \pi r^2$. But the area inside the wave is also a function of time, which may be expressed as

$$A = \pi(20t)^2 = 400\pi t^2.$$

That is, the area of the circle is a function of the radius, which in turn is a function of time. The function that we arrive at, namely, A as a function of t, is called the *composition* of the two original functions. In the notation which uses letters to denote functions, we have the following definition.

Definition If f and g are two functions, then the composition of f and g is the function $f \circ g$ whose value at x is given by

$$f \circ g(x) = f(g(x)). \tag{2.1.9}$$

Example If $f(x) = \sqrt{x}$ and $g(x) = x^2 + 1$, then

$$f \circ g(x) = f(g(x)) = f(x^2 + 1) = \sqrt{x^2 + 1}$$

and

$$g \circ f(x) = g(f(x)) = g(\sqrt{x}) = x + 1.$$

Note that $f \circ g$ and $g \circ f$, as in this example, are not usually the same function.

Classes of functions

The simplest type of functions are those which involve only multiplication and addition. In particular, functions of the form

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \tag{2.1.10}$$

where $a_0, a_1, \ldots, a_n$ are constants and n is a nonnegative integer, are called *polynomials*. If $a_n \neq 0$, the *degree* of the polynomial is n. For example,

$$q(x) = 3x^2 - 13x + 3,$$

$$f(t) = 21t^{34} + 18t^2 - \pi,$$

and

$$g(s) = \frac{1}{2}s^3 + s + 12 - 4s^5$$

are polynomials, of degrees 2, 34, and 5, respectively, whereas

$$h(x) = \frac{3}{x}$$

is not a polynomial. In a sense polynomials are the building blocks for a large family of important functions in calculus. In this regard, one of the major goals of this text is to show how polynomials may be used to approximate more complicated functions.

Functions which may be written in the form of a polynomial divided by a polynomial are called *rational functions*. For example,

$$f(x) = \frac{3x^2 - 4x + 1}{x^4 + 1}$$

and

$$g(s) = \frac{12}{s} + \frac{1}{s^2 - 3s + 1}$$

are both rational functions, the latter because it may be rewritten as a polynomial divided by a polynomial if all the terms are put over the common denominator $s(s^2 - 3s + 1)$.

The function $f(x) = \sqrt{x}$ is neither a polynomial nor a rational function because x is raised to a power which is not an integer. Functions which permit addition, multiplication, division, and rational numbers for powers are called *algebraic functions*. Thus, for example,

$$g(t) = t^{\frac{3}{2}} + 2t^2 - 3$$

is neither a polynomial nor a rational function, but is an algebraic function. Similarly,

$$h(s) = \sqrt{s^2 + 3s + 2}$$

is an algebraic function. We should note that every polynomial is also a rational function and every rational function is also an algebraic function.

Functions which are not algebraic are called *transcendental*. The trigonometric functions are examples of transcendental functions. We shall discuss them in detail in the next section.

Graphs of functions

We are now in a position to say more about the graphs of functions. With the notation we have now, the graph of a function f is the set of all points $(x, f(x))$ in the plane, where x is in the domain of f. For example, you should recall from previous work that the graph of $y = x^2$ is a parabola opening about the x-axis with its vertex at $(0, 0)$. Also, you should recall the shapes of the graphs of such functions as $y = x^2$, $y = x^3$, $y = x^4$, and $y = x^5$. See Figures 2.1.3, and 2.1.4.

Moreover, given a function f and a constant c, you may recall that the graph of $y = f(x) + c$ is the graph of f shifted c units vertically (upward if $c > 0$ and downward if $c < 0$), the graph of $y = f(x - c)$ is the graph of f shifted c units horizontally (to the right

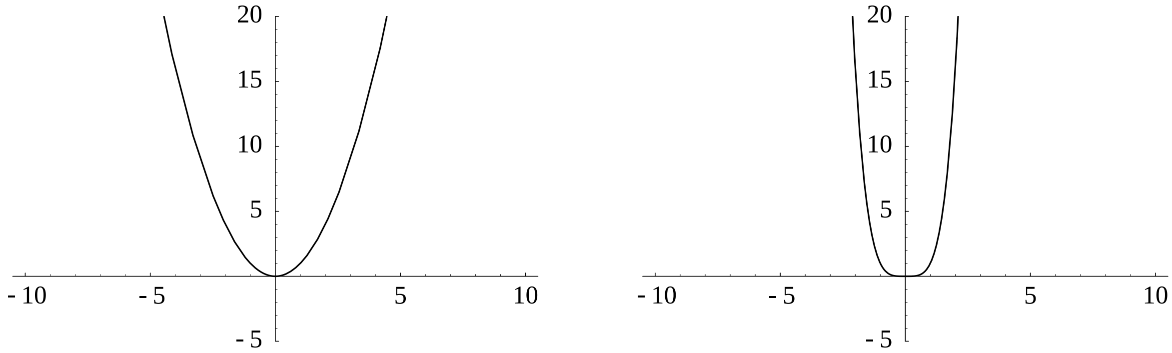


Figure 2.1.3 Graphs of $y = x^2$ and $y = x^4$

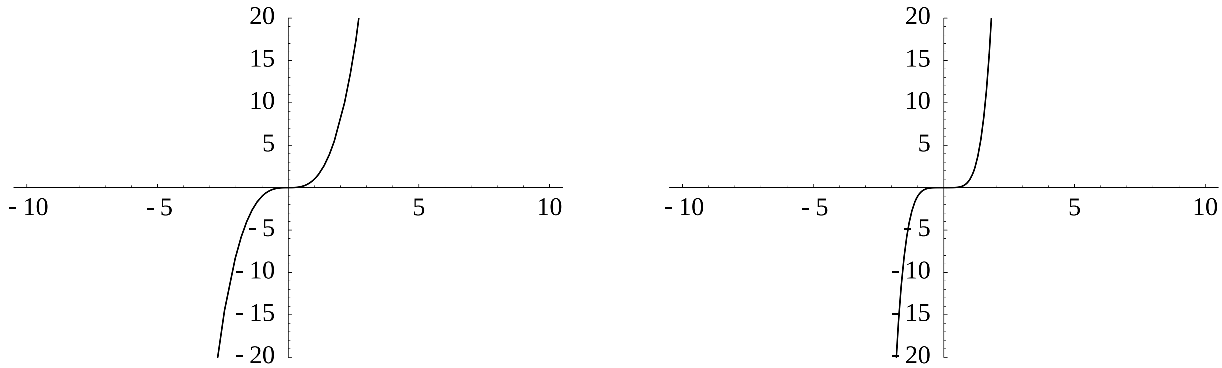


Figure 2.1.4 Graphs of $y = x^3$ and $y = x^5$

if $c > 0$ and to the left if $c < 0$), and the graph of $y = -f(x)$ is the graph of f reflected about the x-axis. Hence, for example, the graph of

$$y = x^2 - 3$$

is a parabola, opening upward about the x-axis, with its vertex at $(0, -3)$; the graph of

$$y = (x + 2)^2 - 3$$

is a parabola, opening upward about the line $x = -2$, with vertex at $(-2, -3)$; and the graph of

$$y = -(x + 2)^2 + 3$$

is a parabola, opening downward about the line $x = -2$, with vertex at $(-2, 3)$. See Figure 2.1.5.

Drawing graphs of functions whose basic shapes are not already known to us can be a difficult problem. If the domain of a function f is finite, then drawing its graph is only a matter of plotting some points in the plane. However, most of the functions we will encounter in this course will have domains containing an infinite number of points; drawing the graphs of such functions requires much more than plotting a few points. The

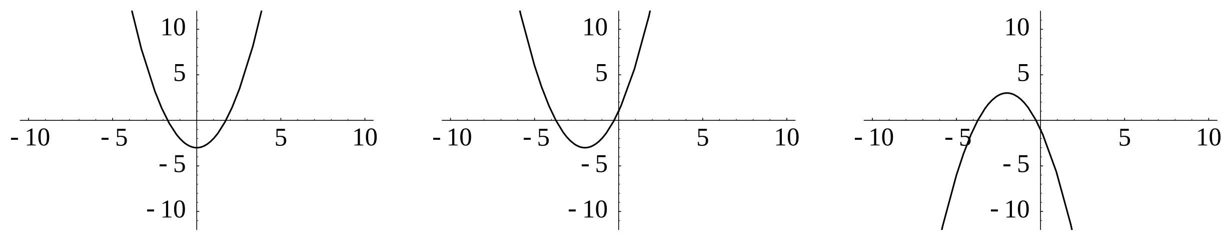


Figure 2.1.5 Graphs of $y = x^2 - 3$, $y = (x+2)^2 - 3$, and $y = -(x+2)^2 + 3$

problem is that no matter how many points we plot, we still do not know how the function is behaving at the other points. For example, if we want to graph a function f on the interval $[0, 1]$, we might first plot the points

$$(0, f(0)), (0.1, f(0.1)), (0.2, f(0.2)), \ldots, (1.0, f(1.0)).$$

Next, to guess at the behavior of the function between the plotted points, we might join successive points by straight lines. Of course, this will only give us an approximation to the true curve, the accuracy of which will depend on the actual behavior of the curve between the plotted points, something about which we frequently have very little information. This is similar to the problem we had with plotting the graph of a temperature function earlier. However, here we can get help if we have a formula for f; for in that case we can try plotting more points, say

$$(0, f(0)), (0.05, f(0.05)), (0.10, f(0.10)), \ldots, (1.00, f(1.00)),$$

or

$$(0, f(0)), (0.01, f(0.01)), (0.02, f(0.02)), \ldots, (1.00, f(1.00)).$$

If the graph of f is a reasonably smooth curve, we will be able to approximate it as well as we like by plotting a sufficient number of points. This raises two questions: How do we know that we have plotted a sufficient number of points? And, given that a sufficient number of points will most likely be a large number, how do we actually plot them all? Of course, the latter question is answered by using a computer. In fact, this approach to graphing a function is unreasonable without access to a computer, or at least a calculator. Computers also provide help in answering the first question. We start by plotting a reasonable number of points, say 100 or so. If we have reason to doubt the accuracy of the resulting graph, perhaps because the curve is not as smooth as we expected it to be, we can double the number of points and plot it again. Because any computer, and in fact many calculators, do this type of work rapidly, it is reasonable to plot the same function several times until we are comfortable with the picture. In Section 3.9 we will learn how to use some of the techniques of calculus to better understand the geometry of the graph of a function. This will help us identify whether or not the output from a computer is an accurate depiction of the graph.

Example Figure 2.1.6 compares the results of plotting the points

$$(0, f(0)), (0.1, f(0.1)), (0.2, f(0.2)), \ldots, (1.0, f(1.0))$$

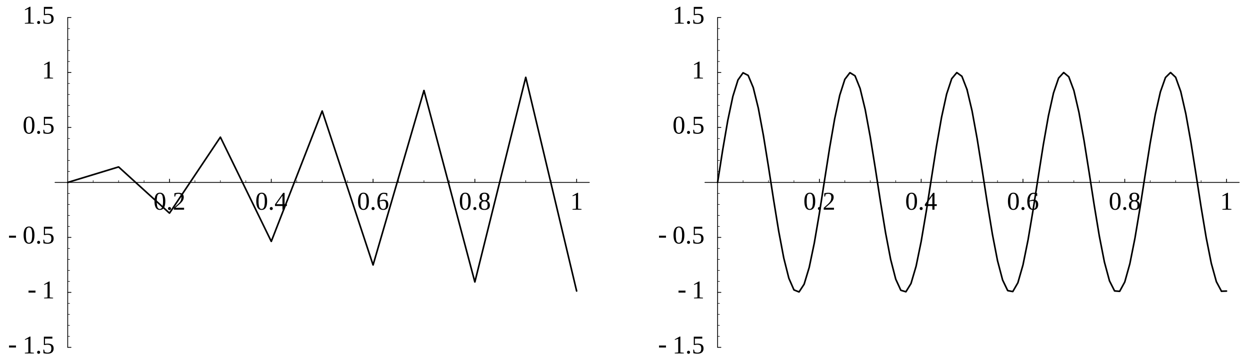


Figure 2.1.6 Plot of $y = \sin(30x)$, using first 11 points and then 101 points

with plotting the points

$$(0, f(0)), (0.01, f(0.01)), (0.02, f(0.02)), \ldots, (1.00, f(1.00))$$

(joining successive points with straight lines) for the function $f(x) = \sin(30x)$. Clearly, the second plot a dramatic improvement over the first.

When using a computer software package or a calculator to graph a function, there are a couple of issues you should keep in mind. First, as we have just noted, plotting a sufficient number of points is crucial to obtaining a good approximation of the graph. Some programs will ask you to specify the number of points to plot, others will plot a predetermined number of points, and still others will determine the number of points to plot based on an estimate of the number of points necessary to provide an accurate picture for the particular function. In the latter two cases it should still be possible to override the program's decision and specify your own choice for the number of points to plot. If plotting more points significantly changes the look of the graph, then you should be wary of the original plot and consider plotting even more points. Second, the computer will plot the function in a rectangle, called a window. The horizontal scale for this window will be the interval over which you want to graph the function. The vertical scale may be chosen by you or by the computer program. If possible, it is usually best first to let the program choose the vertical scale for the window and then to adjust it as necessary to provide a good picture of the graph. If the vertical scale is too small, you may miss part of the graph; if the vertical scale is too large, the interesting features of the graph may be too small to be visible. For example, Figure 2.1.7 shows the graph of $y = \sin(x)$ on the interval $[-4, 4]$, first with the vertical window scale being the interval $[-0.5, -0.5]$ and next with the vertical scale changed to $[-20, 20]$. Certainly, a vertical window scale on the order of $[-1.5, 1.5]$, as shown in Figure 2.1.8, is a more appropriate choice for this graph.

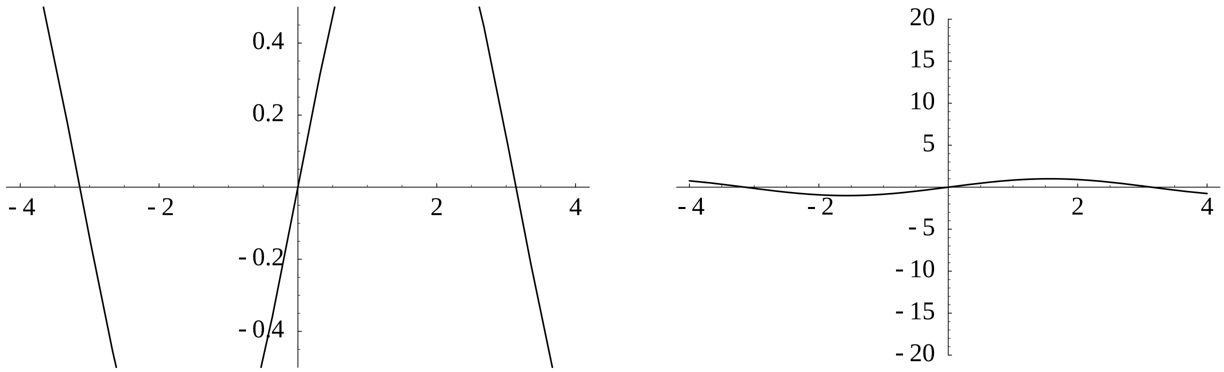


Figure 2.1.7 Graph of $y = \sin(x)$, first with vertical window $[-0.5, 0.5]$ and then $[-20, 20]$

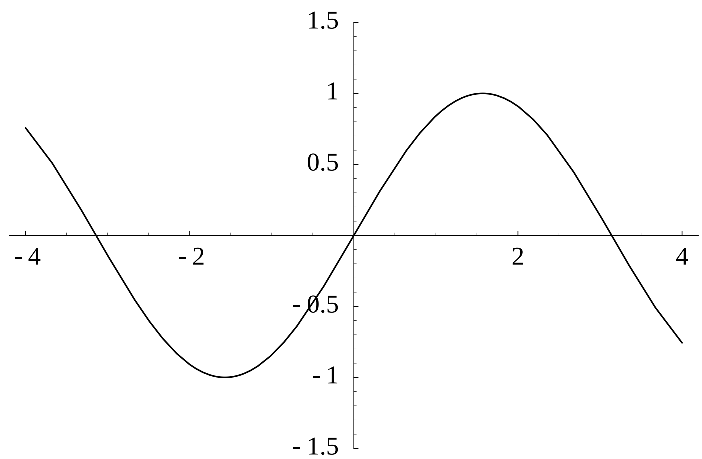


Figure 2.1.8 Graph of $y = \sin(x)$ with vertical window $[-1.5, 1.5]$

Problems

1. In each of the following, x and y denote certain variable quantities. Discuss whether or not y is a function of x. If y is a function of x, can you write a formula that describes the relationship? Also, find the domain and range of each function.

   (a) $x =$ speed of a train; $y =$ distance the train travels in two hours

   (b) $x =$ height above sea level; $y =$ atmospheric pressure

   (c) $x =$ time of the year; $y =$ distance from the earth to the moon

   (d) $x =$ temperature at the Great Falls airport; $y =$ time of the day

   (e) $x =$ length of the side of a square; $y =$ area of the square

   (f) $x =$ area of a circle; $y =$ circumference of the circle

   (g) $x =$ weight of a letter; $y =$ first class postage for the letter

   (h) $x =$ OPEC price for a barrel of oil; $y =$ Dow Jones Industrial Average

2. A projectile was shot vertically into the air. The height h of the projectile was measured at 20 different times t. The following table gives the results, where t is in seconds and h is in meters.

Time (t)	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50
Height (h)	0.00	11.6	22.1	31.2	39.2	46.0	51.5	55.7	58.8	60.6	61.3
Time (t)	2.75	3.00	3.25	3.50	3.75	4.00	4.25	4.50	4.75	5.00	
Height (h)	60.6	58.8	55.7	51.5	46.0	39.2	31.2	22.1	11.6	0.00	

   (a) Graph this data.

   (b) Graph this data with lines connecting successive points. Do you think this is a reasonable approximation to the graph of h as a function of t?

3. Identify the domain of each of the following functions.

   (a) $f(x) = x^2 - 6x$
   (b) $g(x) = \sqrt{x^2 - 9}$
   (c) $f(t) = \sqrt{t^2 + t - 6}$
   (d) $h(x) = \dfrac{3}{x^2 + 6x + 8}$
   (e) $f(s) = \dfrac{41}{s^2 - 9}$
   (f) $y(t) = \dfrac{3}{\sqrt{3 - t^2}}$
   (g) $z(s) = \dfrac{1}{\sqrt{s^2 + 6s + 1}}$
   (h) $f(x) = \dfrac{4}{x(x^2 + 1)}$

4. For each of the following pairs of functions, find $f \circ g(x)$, $g \circ f(x)$, $f \circ g(3)$, and $f \circ g(3)$, when they are defined.

   (a) $f(x) = 4x + 12$, $g(x) = 5x - 2$
   (b) $f(x) = x^2 - 12$, $g(x) = \sqrt{x}$
   (c) $f(x) = 6x - x^2$, $g(x) = \dfrac{1}{x - 9}$

5. (a) If the graph of f is a straight line with slope 3 and the graph of g is a straight line with slope 4, show that the graph of $f \circ g$ is a straight line with slope 12.

   (b) If the graph of f is a straight line with slope m and the graph of g is a straight line with slope n, show that the graph of $f \circ g$ is a straight line with slope mn.

6. Graph each of the following functions on the given interval.

   (a) $f(x) = x^2 + 6x + 1$ on $[-10, 5]$
   (b) $f(x) = x^5 + x^4 + x^3 + x^2 + x + 1$ on $[-5, 5]$
   (c) $f(x) = x^5 + 8x^4 + x^3 + x^2 + x + 1$ on $[-5, 5]$
   (d) $f(x) = x^5 + 8x^4 + x^3 + x^2 + x + 1$ on $[-10, 10]$
   (e) $g(t) = \dfrac{1}{1 + t^2}$ on $[-10, 10]$
   (f) $f(t) = \dfrac{t}{1 + t^2}$ on $[-10, 10]$

(g) $g(t) = \dfrac{t^2}{1+t^2}$ on $[-10, 10]$

(h) $g(x) = \dfrac{x^2-1}{x^4+1}$ on $[-5, 5]$

7. Another problem in graphing using a computer or a calculator arises when the function in question is not defined at some of the points in the interval of interest. For example, try graphing
$$f(x) = \frac{1}{x}$$
on the interval $[-5, 5]$. There are several problems which may arise. First, if the computer program tries to evaluate f at $x = 0$, you may get an error message. In this case you may have to change the number of points being plotted so that $x = 0$ is missed. Second, if the program evaluates f for values of x very close to 0, the output from the function will be very large. The result might be that the vertical scale of your graphing window is much too large. Hence, you may wish to change the scale of the vertical axis. Another problem that may occur arises because the graph of f has two pieces; that is, the part of the graph to the left of the y-axis is not connected to the part of the graph to the right of the y-axis. If your graphing program simply connects points as it moves from left to right, it will connect points on opposite sides of the y-axis which should not be connected. This may be hard to avoid with some software, but you should be aware of the problem and, consequently, interpret your results with care.

8. In light of the remarks in Problem 7, try graphing the following functions.

   (a) $h(s) = \dfrac{1}{1-s^2}$ on $[-4, 4]$

   (b) $g(x) = \dfrac{x^3}{1-x^2}$ on $[-4, 4]$

9. Recall that $\lfloor x \rfloor$ denotes the largest integer less than or equal to x and $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. Let $f(x) = \lfloor x \rfloor$ and $g(x) = \lceil x \rceil$.

   (a) What is the domain of f? What is the range of f?

   (b) What is the domain of g? What is the range of g?

   (c) Graph f and g on the interval $[-5, 5]$.

   (d) Graph $h(x) = \lfloor x^2 \rfloor$ on the interval $[-2, 2]$.

10. We say a function f is *periodic* if there exists a constant T such that $f(t + T) = f(t)$ for every value of t in the domain of f.

    (a) Is it possible for a polynomial to be periodic?

    (b) Are any of the functions in Problem 1 periodic?

    (c) Suppose x represents the number of days since January 1, 1950, and y represents the amount of rainfall in Spokane on day x. Do you think y is a periodic function of x? If not, might it in some way be close to periodic?

Difference Equations to Differential Equations

Section 2.2

Trigonometric Functions

Many processes in nature are cyclic. A pendulum oscillates back and forth, repeating its motion over and over; a weight hanging at the end of a spring bobs up and down; the Earth repeats its orbit about the Sun every 365 days; a population of arctic wolves has periods of growth followed by periods of decrease, following the fluctuations in the population of their prey; the monthly rainfall at an agricultural research station varies cyclically over the years and over the decades. To model such natural behavior, a mathematician needs functions which repeat their values over intervals of fixed length. These functions are the periodic functions. Precisely, a function f is *periodic* if there is a fixed constant T such that $f(t+T) = f(t)$ for every value of t in the domain of f. The smallest such positive T for which this property holds is called the *period* of f.

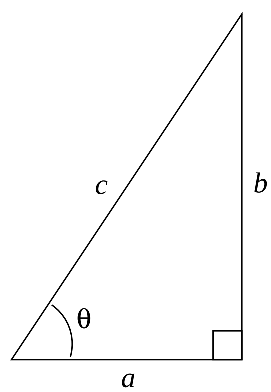


Figure 2.2.1 A right triangle

The class of periodic functions that we will consider in this section are the trigonometric functions. Although these functions were originally invented to work with problems of measurement, their importance in modern mathematics stems more from their periodic behavior. We will begin with a definition in terms of measuring the sides of a right triangle. Consider a right triangle with legs of lengths a and b and hypotenuse of length c. Moreover, suppose, as in Figure 2.2.1, the angle opposite the leg of length b has measure θ. Then we define the *sine* of θ, which we write as $\sin(\theta)$, by

$$\sin(\theta) = \frac{b}{c} \tag{2.2.1}$$

and the *cosine* of θ, which we write as $\cos(\theta)$, by

$$\cos(\theta) = \frac{a}{c}. \tag{2.2.2}$$

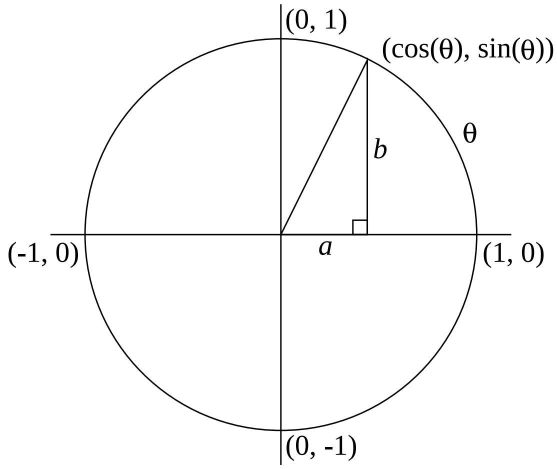


Figure 2.2.2 A right triangle with a vertex on the unit circle

The properties of similar triangles, known even by the ancient Egyptians and Babylonians, show that these ratios depend only on the value of θ, not on the size of the particular right triangle being measured. Hence if we know the value of θ and the length of just one side of the triangle, and we have access to a table of values for the sine and cosine functions, then it is possible to compute the lengths of the other two sides of the triangle. The ancient Greek mathematicians exploited these facts in order to compute distances which are inaccessible to direct measurement, such as the distance from the earth to the moon and from the earth to the sun.

Since the values of the sine and cosine functions do not depend on the size of any particular right triangle, for the purpose of definitions we may restrict our attention to right triangles with hypotenuses of length one. In particular, if we have a right triangle with legs of lengths a and b and hypotenuse of length 1 (so that $a^2 + b^2 = 1$), then we may draw it in the Cartesian plane with one leg running from $(0, 0)$ to $(a, 0)$ and the other from $(a, 0)$ to (a, b). If θ is the measure of the angle opposite the side of length b, then we have

$$\sin(\theta) = b \tag{2.2.3}$$

and

$$\cos(\theta) = a. \tag{2.2.4}$$

In that case, the vertex (a, b) lies on the unit circle $x^2 + y^2 = 1$. In particular, $(\cos(\theta), \sin(\theta)$ is a point on the unit circle centered at the origin. This also gives us a method for measuring angles. We will say that the measure of the angle opposite the side of length b is θ *radians* if the length of the arc of the unit circle from $(a, 0)$ to (a, b) is θ. See Figure 2.2.2.

So far our definitions of sine and cosine include only angles that are between 0 and $\frac{\pi}{2}$ radians. However, the considerations of the previous paragraph show us how to generalize our definitions. Let t be any real number and let C be the unit circle centered at $(0, 0)$. If $t \geq 0$, let (a, b) be the point reached by traversing C a distance of t units in the counterclockwise direction starting from $(1, 0)$. If $t < 0$, let (a, b) be the point reached by traversing C a distance of $|t|$ units in the clockwise direction starting from $(1, 0)$. Note that if $t \geq 2\pi$ or $t \leq -2\pi$, then we will have to travel around C one or more times. We now define the sine and cosine of t by

$$\sin(t) = b \tag{2.2.5}$$

and

$$\cos(t) = a. \tag{2.2.6}$$

In this way we have sine and cosine defined as functions on the entire real line. That is, both sine and cosine now have domain $(-\infty, \infty)$.

Our final definitions of the sine and cosine functions have several immediate consequences. Most importantly, since the circumference of the unit circle is 2π, both functions are periodic with period 2π. Hence

$$\sin(t + 2\pi) = \sin(t) \tag{2.2.7}$$

and

$$\cos(t + 2\pi) = \cos(t) \tag{2.2.8}$$

for any value of t. Also, since $(\cos(t), \sin(t))$ is a point on the unit circle, we have

$$\sin^2(t) + \cos^2(t) = 1 \tag{2.2.9}$$

for all values of t. Recall that, in this notation,

$$\sin^2(t) = (\sin(t))^2$$

and

$$\cos^2(t) = (\cos(t))^2.$$

We will consider other interesting and useful identities involving sine and cosine in the problems at the end of this section and later on as the need for them arises.

Although numerical approximations of $\sin(t)$ and $\cos(t)$ are easily available from a calculator for any value of t, it is useful to know some exact numerical values for these functions. First of all, directly from the definition we have

$$\sin(0) = 0 \qquad \sin\left(\frac{\pi}{2}\right) = 1 \qquad \sin(\pi) = 0 \qquad \sin\left(\frac{3\pi}{2}\right) = -1 \qquad \sin(2\pi) = 0$$

and

$$\cos(0) = 1 \qquad \cos\left(\frac{\pi}{2}\right) = 0 \qquad \cos(\pi) = -1 \qquad \cos\left(\frac{3\pi}{2}\right) = 0 \qquad \cos(2\pi) = 1.$$

Second, with a little more work, it can be shown that

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \qquad \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \qquad \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

and

$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \qquad \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \qquad \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}.$$

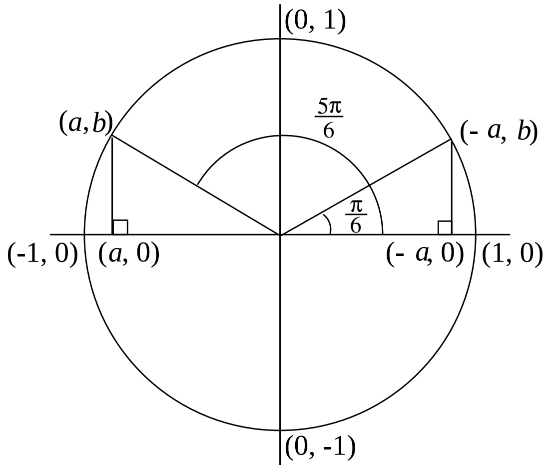


Figure 2.2.3 Finding $\sin\left(\frac{5\pi}{6}\right)$ and $\cos\left(\frac{5\pi}{6}\right)$

Moreover, combining these values with basic knowledge of the geometry of the unit circle, it is possible to find exact numerical values for the values of $\sin(t)$ and $\cos(t)$ for $t = \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{5\pi}{4}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{4}$, and $\frac{11\pi}{6}$.

Example Let (a, b) be the point on the unit circle corresponding to the angle $\frac{5\pi}{6}$. Since (a, b) is a distance $\frac{\pi}{6}$ along the unit circle before $(-1, 0)$, the point $(-a, b)$ is a distance $\frac{\pi}{6}$ along the unit circle after $(1, 0)$. Hence

$$-a = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

and

$$b = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}.$$

Thus

$$\cos\left(\frac{5\pi}{6}\right) = a = -\frac{\sqrt{3}}{2}$$

and

$$\sin\left(\frac{5\pi}{6}\right) = b = \frac{1}{2}.$$

This is all best seen using a picture such as Figure 2.2.3. Note that the triangle with vertices at $(0, 0)$, (a, b), and $(a, 0)$ is congruent to the triangle with vertices at $(0, 0)$, $(-a, b)$, and $(-a, 0)$.

Of course, because both sine and cosine have period 2π, it is also easy to find exact values for $\sin(t)$ and $\cos(t)$ if t differs from one of the above values by a multiple of 2π.

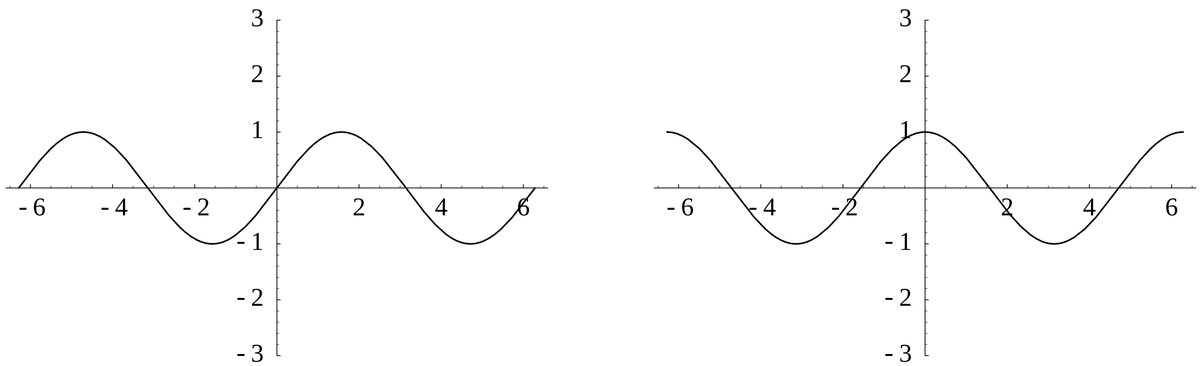


Figure 2.2.4 Graphs of $y = \sin(t)$ and $y = \cos(t)$

Graphs of sine and cosine

The graphs of $y = \sin(t)$ and $y = \cos(t)$ are shown in Figures 2.2.4. Since both functions have period 2π, the graphs will continue this behavior as t goes to $-\infty$ or ∞, completing one oscillation over every interval of length 2π.

The only difference between the graph of $y = a\sin(t)$, where $a > 0$, and the graph of $y = \sin(t)$ is that the former oscillates between $-a$ and a instead of between -1 to 1. In general, for any constant $a \neq 0$, the graph of $y = a\sin(t)$ oscillates between $-|a|$ and $|a|$. We call $|a|$ the *amplitude* of the function $y = a\sin(t)$. Of course, if $a < 0$, then the graph of $y = a\sin(t)$ is the graph of $y = |a|\sin(t)$ reflected about the t-axis.

Example The graph of $y = 2\sin(t)$ is shown in Figure 2.2.5.

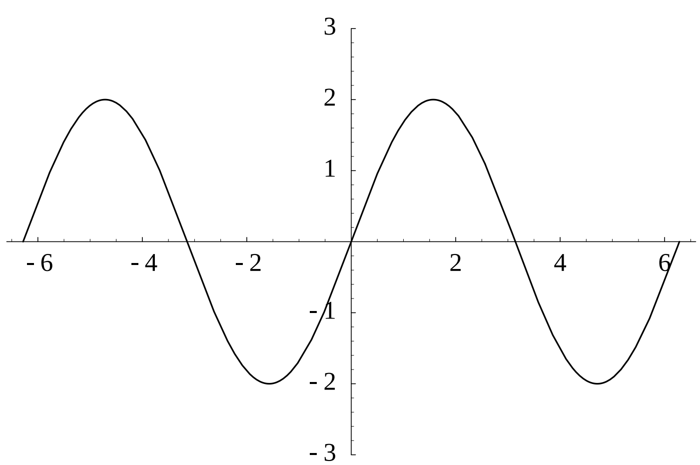


Figure 2.2.5 Graph of $y = 2\sin(t)$

Now consider the graph of the function $y = \sin(bt)$. Since the sine function has period 2π, this function goes through one complete oscillation as t goes from 0 to $\frac{2\pi}{|b|}$. That is, $y = \sin(bt)$ has period $\frac{2\pi}{|b|}$. Hence, if $b > 0$ the only difference between the graphs of $y = \sin(t)$ and $y = \sin(bt)$ is the length of the period of oscillation. If $b < 0$, we may use

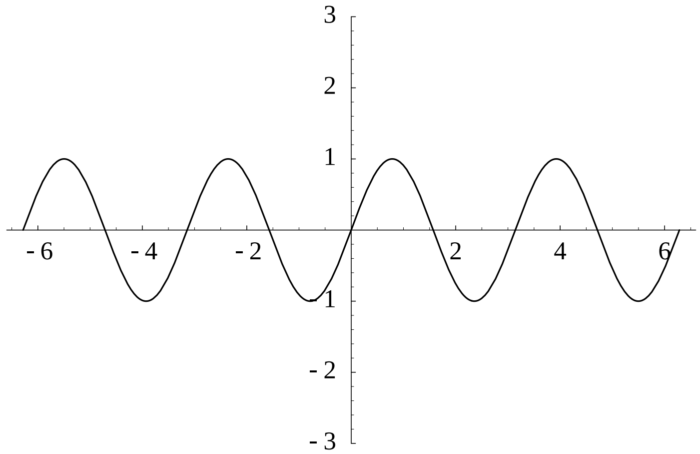


Figure 2.2.6 Graph of $y = \sin(2t)$

the fact that

$$\sin(bt) = \sin(-|b|t) = -\sin(|b|t),$$

which follows from Problem 10.

Example The graph of $y = \sin(2t)$ is shown in Figure 2.2.6.

Finally, consider the graph of $y = \sin(t - c)$. As mentioned in Section 2.1, the effect of the c is to shift the graph $y = \sin(t)$ horizontally by $|c|$ units, to the right if $c > 0$ and to the left if $c < 0$. We call c the *phase angle*.

Example The graph of $y = \sin(t - \pi)$ is shown in Figure 2.2.7.

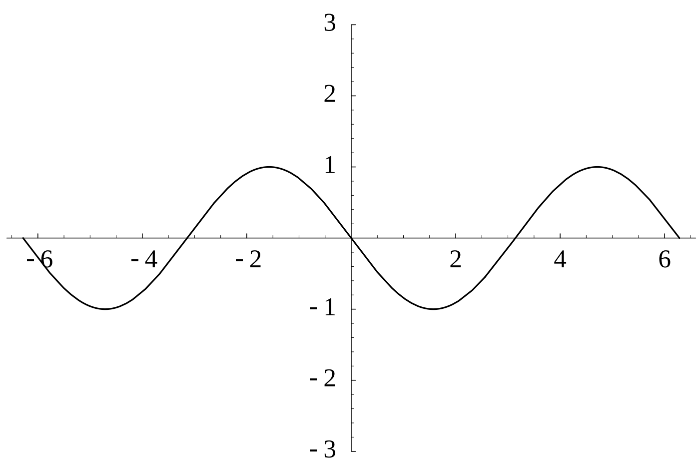


Figure 2.2.7 Graph of $y = \sin(t - \pi)$

Summarizing the previous comments, the function $y = a\sin(b(t - c))$ has amplitude $|a|$, period $\frac{2\pi}{|b|}$, and phase angle c.

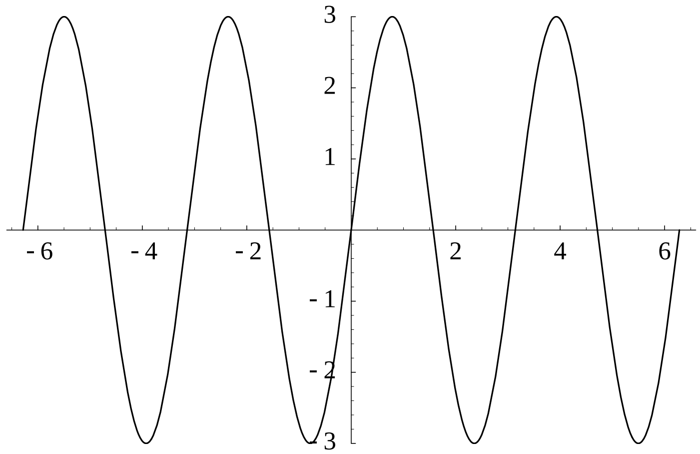


Figure 2.2.8 Graph of $y = -3\sin(2t - \pi)$

Example Consider the function $f(t) = -3\sin(2t - \pi)$. If we write $f(t)$ in the form

$$f(t) = -3\sin\left(2\left(t - \frac{\pi}{2}\right)\right),$$

then we see that f has amplitude 3, period π, and phase angle $\frac{\pi}{2}$. The graph of f is shown in Figure 2.2.8.

Similar remarks hold for the graph of the function $y = a\cos(b(t-c))$, the only difference being that, since $\cos(-t) = \cos(t)$ for all t (see Problem 10), we have

$$\cos(bt) = \cos(|b|t)$$

even when $b < 0$.

Related functions

The other four trigonometric functions are defined in terms of the sine and cosine functions. The *tangent* function is defined by

$$\tan(t) = \frac{\sin(t)}{\cos(t)}. \tag{2.2.10}$$

Note that $\tan(t)$ is the slope of the line from $(0,0)$ to $(\cos(t), \sin(t))$. The graph of $y = \tan(t)$ has vertical asymptotes at every value of t for which $\cos(t) = 0$, as can be seen in Figure 2.2.9.

The *cotangent* function is the reciprocal of the tangent function; namely,

$$\cot(t) = \frac{1}{\tan(t)} = \frac{\cos(t)}{\sin(t)}. \tag{2.2.11}$$

Finally, the *secant* and *cosecant* functions are the reciprocals of the cosine and sine functions, respectively. Hence

$$\sec(t) = \frac{1}{\cos(t)} \tag{2.2.12}$$

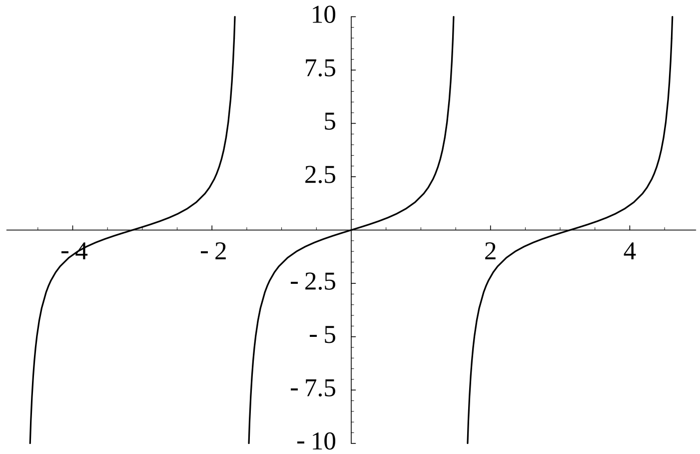


Figure 2.2.9 Graph of $y = \tan(t)$

and

$$\csc(t) = \frac{1}{\sin(t)}. \tag{2.2.13}$$

As with the tangent function, the graph of $y = \sec(t)$ has vertical asymptotes at all points t where $\cos(t) = 0$, as seen in Figure 2.2.10.

Clearly both the secant and cosecant functions have period 2π. However, the tangent and cotangent functions both have period π. We will leave that fact, along with the graphs of the cotangent and cosecant functions, to the problems at the end of this section.

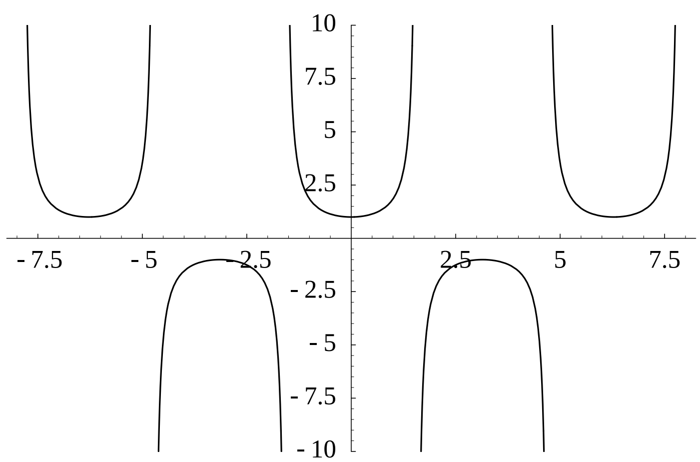


Figure 2.2.10 Graph of $y = \sec(t)$

Periodic motion

As mentioned earlier, many natural phenomena change in a periodic fashion. For example, suppose we have a pendulum and for a given time t we let $x(t)$ represent the angle between the current position of the pendulum and its rest position, taking x to be positive if the

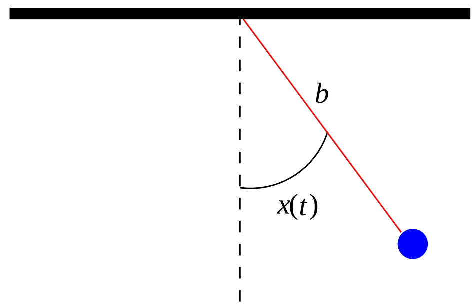


Figure 2.2.11 A pendulum

pendulum is to the right of its rest position and negative otherwise (see Figure 2.2.11). If initially the pendulum is held at a small angle $\alpha > 0$ and then released, that is, $x(0) = \alpha$, then, if we ignore friction, it can be shown that

$$x(t) = \alpha \cos\left(\sqrt{\frac{g}{b}}t\right) \tag{2.2.14}$$

where g is the acceleration due to gravity (32 feet per second per second or 9 .8 meters per second per second) and b is the length of the pendulum. Actually, this is an approximation which holds very well for small values of α. Note that the period of x, namely,

$$\frac{2\pi}{\sqrt{\frac{g}{b}}} = 2\pi\sqrt{\frac{b}{g}},$$

does not depend upon the amplitude α. This is an important fact, supposedly first noticed by Galileo, which is crucial in the operation of pendulum clocks. We will consider this problem more closely in Chapter 8, where we will derive (2.2.14) and see exactly how the approximation enters the picture.

Periodic motions need not always be as simple as the motion of a pendulum. Consider, for example, the motion of a molecule of air as a sound wave passes. The action of the sound wave causes a particular molecule of air to oscillate back and forth about some equilibrium position. If we let $x(t)$ represent the position of the air molecule at time t, with $x = 0$ corresponding to the equilibrium position and x considered to be positive in one direction from the equilibrium position and negative in the other, then for many sounds x will be a periodic function of t. In general, this will be true for musical sounds, but not true for sounds we would normally classify as noise. Moreover, even if x is a periodic function, it need not be simply a sine or cosine function. The graph of x for a musical sound, although periodic, may be very complicated. However, many simple sounds, such as the sound of a tuning fork, are represented by sine curves. For example, if x is the displacement of an air molecule for a tuning fork which vibrates at 440 cycles per second with a maximum displacement from equilibrium of 0.002 centimeters, then

$$x(t) = 0.002\sin(880\pi t).$$

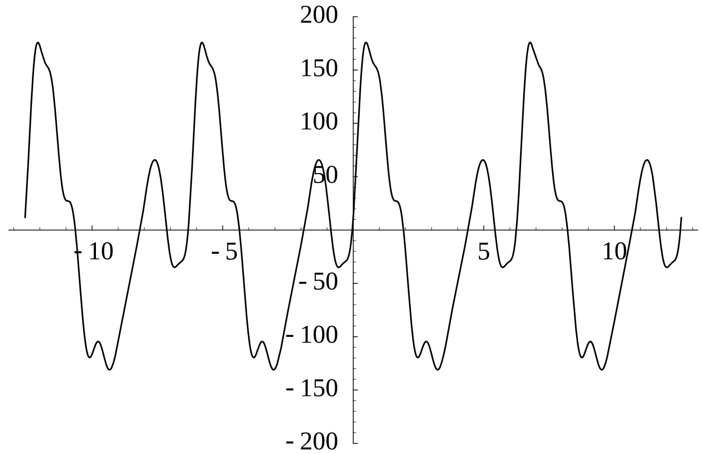


Figure 2.2.12 Graph of the air displacement due to the organ note C_3

Notice that this function has period $\frac{2\pi}{880\pi} = \frac{1}{440}$, and hence has a frequency of 440 cycles per second.

In the early part of the 19th century, Joseph Fourier (1768-1830) showed that the story does not end here. Fourier demonstrated that any "nice" periodic curve (for example, one which is connected) can be approximated as closely as desired by a sum of sine and cosine functions. In particular, this means that for any musical sound the function x may be approximated well by a sum of sine and cosine functions. For example, in his book *The Science of Musical Sounds* (Macmillan, New York, 1926), Dayton Miller shows that, with an appropriate choice of units,

$$\begin{aligned} x(t) = {} & 22.4\sin(t) + 94.1\cos(t) + 49.8\sin(2t) - 43.6\cos(2t) + 33.7\sin(3t) \\ & - 14.2\cos(3t) + 19.0\sin(4t) - 1.9\cos(4t) + 8.9\sin(5t) - 5.22\cos(5t) \\ & - 8.18\sin(6t) - 1.77\cos(6t) + 6.40\sin(7t) - 0.54\cos(7t) + 3.11\sin(8t) \\ & - 8.34\cos(8t) - 1.28\sin(9t) - 4.10\cos(9t) - 0.71\sin(10t) - 2.17\cos(10t) \end{aligned}$$

gives a very good approximation to the displacement curve of a sound wave generated by the tone C_3 of an organ pipe. From the graph of x, shown in Figure 2.2.12, we can see its complexity as well as its periodicity. Notice that the terms in this expression for $x(t)$ are written in pairs with frequencies which are always integer multiples of the frequency of the first pair. This is a general fact which is part of Fourier's theory; if we added more terms to obtain more accuracy, the next terms would be of the form $a\sin(11t) + b\cos(11t)$ for some constants a and b. Notice also that the amplitudes of the sine and cosine curves tend to decrease as the frequencies are increasing. As a consequence, the higher frequencies have less impact on the total curve. Put another way, Fourier's theorem says that every musical sound is the sum of simple tones which could be generated by tuning forks. Hence in theory, although certainly not in practice, the instruments of any orchestra could all be replaced by tuning forks. On a more practical level, Fourier's analysis of periodic functions has been fundamental for the development of such modern conveniences as radios, televisions, stereos, and compact disc players. Unfortunately, this is a story which will have to be told elsewhere.

Problems

1. Find the exact values of $\sin(t)$, $\cos(t)$, $\tan(t)$, and $\sec(t)$ for the following values of t.

   (a) $\frac{4\pi}{3}$ (b) $\frac{7\pi}{6}$

   (c) $\frac{2\pi}{3}$ (d) $-\frac{\pi}{4}$

   (e) $-\frac{2\pi}{3}$ (f) $\frac{21\pi}{4}$

   (g) $\frac{11\pi}{6}$ (h) $-\frac{11\pi}{6}$

2. Sketch the graph of each of the following functions over an interval that contains at least one period of the function both to the right and to the left of the vertical axis. Also, identify the amplitude, period, and phase angle of each curve.

   (a) $y = \sin(3t)$ (b) $y = 3\cos(2t)$

   (c) $y = \cos(t - \pi)$ (d) $x = \sin(2t) + 1$

   (e) $x = 4\sin(\pi t)$ (f) $y = -2\cos(2t - \pi)$

   (g) $x = 5\sin(2t + \pi)$ (h) $y = -3\sin(2\pi t)$

3. Starting with the identity $\sin^2(x) + \cos^2(x) = 1$, explain why

$$1 + \tan^2(x) = \sec^2(x).$$

4. The addition formulas for sine and cosine are

$$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

   and

$$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y).$$

   Use these to derive the double-angle formulas:

   (a) $\sin(2x) = 2\sin(x)\cos(x)$ (b) $\cos(2x) = \cos^2(x) - \sin^2(x)$

5. Use the double-angle formulas of Problem 4 to derive the half-angle formulas:

   (a) $\cos^2(x) = \dfrac{1 + \cos(2x)}{2}$ (b) $\sin^2(x) = \dfrac{1 - \cos(2x)}{2}$

6. Use the addition formulas of Problem 4 to derive the shift formulas:

   (a) $\sin\left(x - \frac{\pi}{2}\right) = -\cos(x)$ (b) $\cos\left(x - \frac{\pi}{2}\right) = \sin(x)$

   (c) $\sin\left(x + \frac{\pi}{2}\right) = \cos(x)$ (d) $\cos\left(x + \frac{\pi}{2}\right) = -\sin(x)$

7. Can you picture the identities of Problem 6 in terms of the definitions of sine and cosine using the unit circle? What do these identities say about the relationship between the graphs of sine and cosine?

8. Using the addition formulas of Problem 4, show that the tangent and cotangent functions have period π. That is, show that

$$\tan(t+\pi) = \tan(t)$$

and

$$\cot(t+\pi) = \cot(t)$$

for all values of t.

9. Graph each of the following functions.

   (a) $y = \tan(2t)$
   (b) $y = \cot(t)$
   (c) $y = \tan\left(\frac{t}{2}\right)$
   (d) $y = \csc(t)$
   (e) $x = \sec(2t)$
   (f) $y = \tan(4t) + 3$

10. Using the definitions of sine and cosine, convince yourself that

$$\sin(-x) = -\sin(x)$$

and

$$\cos(-x) = \cos(x)$$

for all values of x. Now sketch the graphs of $y = \sin(-3x)$ and $y = \cos(\pi - x)$.

11. According to Dayton Miller in *The Science of Musical Sounds*, the function

$$x(t) = 151\sin(t) - 67\cos(t) + 24\sin(2t) + 55\cos(2t) + 27\sin(3t) + 5\cos(3t)$$

gives a good approximation to the shape of the displacement curve for the tone B_4 played on the E string of a violin.

   (a) Graph each of the individual terms of x on the interval $[-15, 15]$. Use a common scale for the vertical axis.
   (b) Graph x on $[-15, 15]$.
   (c) Graph x and its individual terms (a total of 7 graphs) together on the interval $[-15, 15]$.

12. Suppose we define a function f by saying that it is periodic with period 1 and that $f(x) = 1 - 2x$ for $0 \le x < 1$.

   (a) Sketch the graph of f over the interval $[-3, 3]$.
   (b) Let

$$g_n(x) = 2\left(\frac{1}{\pi}\sin(2\pi x) + \frac{1}{2\pi}\sin(4\pi x) + \frac{1}{3\pi}\sin(6\pi x) + \cdots + \frac{1}{n\pi}\sin(2n\pi x)\right)$$

for $n = 1, 2, 3, \ldots$. For example,

$$g_1(x) = \frac{2}{\pi}\sin(2\pi x),$$

$$g_2(x) = \frac{2}{\pi}\sin(2\pi x) + \frac{1}{\pi}\sin(4\pi x),$$

and

$$g_3(x) = \frac{2}{\pi}\sin(2\pi x) + \frac{1}{\pi}\sin(4\pi x) + \frac{2}{3\pi}\sin(6\pi x).$$

What is the period of g_n? Graph g_1, g_2, g_3, g_4, g_5, and g_{10} over the interval $[-3, 3]$.

(c) What do you think happens to g_n as n gets large?

13. Graph $f(x) = \lfloor \sin(x) \rfloor$ on the interval $[-\pi, \pi]$.

14. For an interesting account of sound waves, Fourier's theorem, and related ideas in electromagnetism, read Chapters 19 ("The Sine of G Major") and 20 ("Mastery of the Ether Waves") in Morris Kline's *Mathematics in Western Culture* (Oxford University Press, 1953).

Difference Equations to Differential Equations

Section 2.3

Limits And The Notion Of Continuity

Of particular interest in mathematics and its applications to the physical world are functional relationships in which the dependent variable changes continuously with changes in the independent variable. Intuitively, changing continuously means that small changes in the independent variable do not produce abrupt changes in the dependent variable. For example, a small change in the radius of a circle does not produce an abrupt change in the area of the circle; we would say that the area of the circle changes continuously with the radius of the circle. Similarly, a small change in the height from which some object is dropped will result in a related small change in the object's terminal velocity; hence terminal velocity is a continuous function of height. On the other hand, when an electrical switch is closed, there is an abrupt change in the current flowing through the circuit; the current flow through the circuit is not a continuous function of time. The purpose of this section is to introduce the terminology and concepts that will give us a proper mathematical basis for discussing continuity in the next section.

To begin our study of continuity, we will first look at two examples of functions which are not continuous. In this way we will discover what properties to exclude when forming our definition of a continuous function.

Example Consider the function H defined by

$$H(t) = \begin{cases} 0, & \text{if } t < 0, \\ 1, & \text{if } t \geq 0. \end{cases} \tag{2.3.1}$$

This function, known as the *Heaviside function*, might be used in connection with modeling the current passing through a switch which is open until time $t = 0$ and then closed. The graph of this function consists of two horizontal half-lines with a vertical gap of unit length at the origin, as shown in Figure 2.3.1. Since this function has a break in its graph at 0, its output changes abruptly as t passes from negative values to positive values. In fact, if $t < 0$, $H(t) = 0$ no matter how close t is to 0, whereas if $t > 0$, $H(t) = 1$ no matter how close t is to 0. Hence, near 0, small changes in t may result in sudden changes in $H(t)$. We say that H has a discontinuity at $t = 0$.

In this section we will develop the language and notation necessary to describe this situation mathematically. In particular, note that for any sequence $\{t_n\}$ with $t_n < 0$ for all n and $\lim_{n\to\infty} t_n = 0$, we have

$$\lim_{n\to\infty} H(t_n) = 0$$

since $H(t_n) = 0$ for all n. We say that the limit of $H(t)$ as t approaches 0 from the left is 0, which we denote by

$$\lim_{t\to 0^-} H(t) = 0.$$

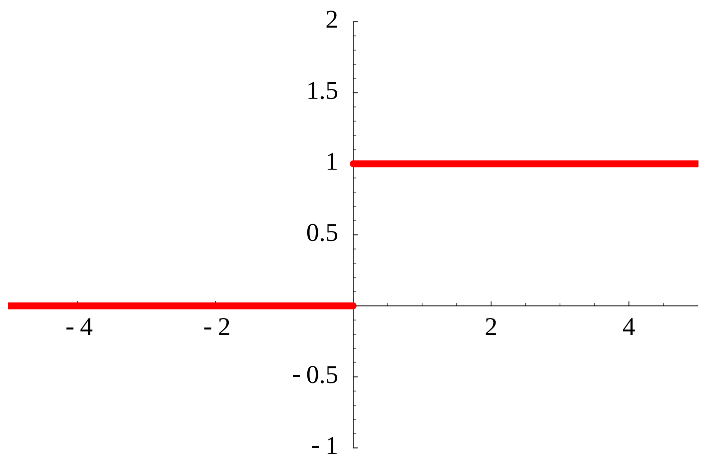


Figure 2.3.1 Graph of the Heaviside function

However, for any sequence $\{t_n\}$ with $t_n > 0$ for all n and $\lim_{n\to\infty} t_n = 0$, we have

$$\lim_{n\to\infty} H(t_n) = 1$$

since $H(t_n) = 1$ for all n. We say that the limit of $H(t)$ as t approaches 0 from the right is 1, which we denote by

$$\lim_{t\to 0^+} H(t) = 1.$$

Since these two limiting values do not agree, we say that $H(t)$ does not have a limit as t approaches 0. Hence, in this case, the discontinuity of H at 0 is characterized by the absence of a limiting value for $H(t)$ at 0. In the next section we will make the existence of a limiting value one of the criteria for a function to be continuous at a point.

Example Now consider the function

$$g(x) = \begin{cases} -x, & \text{if } x < 0, \\ 1, & \text{if } x = 0, \\ x, & \text{if } x > 0. \end{cases}$$

As with the previous example, this function does not change continuously as x passes from negative to positive values. However, the discontinuity arises in a different manner. Note that if $\{x_n\}$ is a sequence with $x_n < 0$ for all n and $\lim_{n\to\infty} x_n = 0$, then

$$\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty}(-x_n) = -\lim_{n\to\infty} x_n = 0.$$

Thus

$$\lim_{x\to 0^-} g(x) = 0.$$

Similarly, if $\{x_n\}$ is a sequence with $x_n > 0$ for all n and $\lim_{n\to\infty} x_n = 0$, then

$$\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} x_n = 0.$$

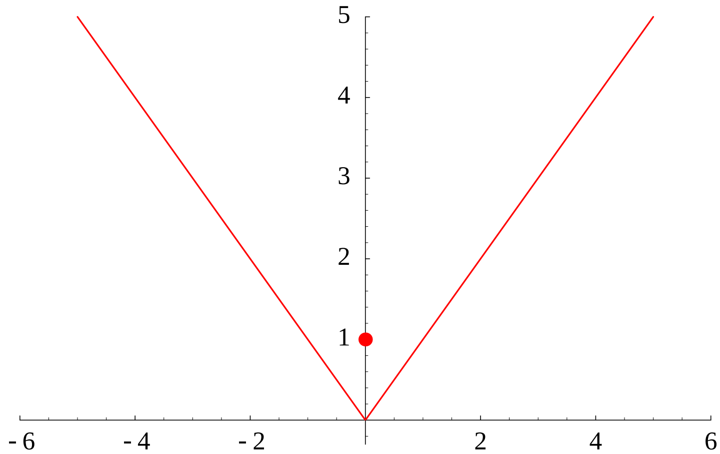


Figure 2.3.2 Graph of $y = g(x)$

Thus

$$\lim_{x \to 0^+} g(x) = 0.$$

Hence in this case $g(x)$ does have a limiting value as x approaches 0 and we can write

$$\lim_{x \to 0} g(x) = 0.$$

However, there is still an sudden change in the value of the function at 0 because $g(0) = 1$, not 0. Graphically, this shows up as a hole in the graph of g at the origin, as shown in Figure 2.3.2. Thus the abrupt change in values of $g(x)$ results not from the lack of a limiting value as x approaches 0, but rather from the fact that

$$g(0) = 1 \neq 0 = \lim_{x \to 0} g(x).$$

This illustrates another type of behavior that we will have to exclude in our definition of continuity.

These examples illustrate two ways in which a function may fail to be continuous. The definition which we will discuss in Section 2.4 essentially says that a function is continuous if it does not have either of the problems that we have seen with H and g. However, before pursuing this question further, we must first introduce the notion of a limit for a function defined on an interval of real numbers. We have already seen the pattern for this definition in the previous examples. Namely, in order to define, for some function f, the limit of $f(x)$ as x approaches some number c, we consider sequences $\{x_n\}$ that converge to c and ask if the sequence $\{f(x_n)\}$ has a limit. Hence we reduce our new question to the old problem of limits of sequences that we considered back in Section 1.2. However, we must be careful about two points. First, there will always be more than one sequence $\{x_n\}$ which converges to a given point c. As we saw in the examples, in order to understand the behavior of a function near c, we must take into account how the function behaves on all possible sequences that converge to c. Second, we want the limit to describe what is happening to the function for values of x close to c, but not equal to c. Thus we must restrict the sequences $\{x_n\}$ to those for which $x_n \neq c$ for all values of n. With these ideas in mind, we now have the following definition.

Definition Let I be an open interval and let c be a point in I. Let J be the set consisting of all points of I except the point c; that is, $J = \{x \mid x \text{ is in } I, x \neq c\}$. Suppose J is in the domain of the function f. We say the *limit* of $f(x)$ as x approaches c is L, denoted

$$\lim_{x \to c} f(x) = L, \tag{2.3.2}$$

if for every sequence $\{x_n\}$ of points in J we have

$$\lim_{n \to \infty} f(x_n) = L \tag{2.3.3}$$

whenever

$$\lim_{n \to \infty} x_n = c. \tag{2.3.4}$$

In other words, to determine the value of $\lim\limits_{x \to c} f(x)$, we ask for the limit of the sequence $\{f(x_n)\}$, where $\{x_n\}$ is any sequence in J which is approaching c. If $\{f(x_n)\}$ approaches L for all such sequences , then L is the limit of $f(x)$ as x approaches c.

We define one-sided limits in a similar fashion. Namely, if J is an open interval of the form (c, b) in the domain of f, then we say the *limit of $f(x)$ as x approaches c from the right* is L, denoted

$$\lim_{x \to c^+} f(x) = L, \tag{2.3.5}$$

if for every sequence $\{x_n\}$ of points in J we have

$$\lim_{n \to \infty} f(x_n) = L \tag{2.3.6}$$

whenever

$$\lim_{n \to \infty} x_n = c. \tag{2.3.7}$$

Similarly, if J is an open interval of the form (a, c) in the domain of f, then we say the *limit of $f(x)$ as x approaches c from the left* is L, denoted

$$\lim_{x \to c^-} f(x) = L, \tag{2.3.8}$$

if for every sequence $\{x_n\}$ of points in J we have

$$\lim_{n \to \infty} f(x_n) = L \tag{2.3.9}$$

whenever

$$\lim_{n \to \infty} x_n = c. \tag{2.3.10}$$

Note that the existence of a one-sided limit only requires that the limiting value of $\{f(x_n)\}$ be the same for all sequences $\{x_n\}$ which approach c from the same side, whereas the existence of a limit requires that the limiting value of be the same for all sequences $\{x_n\}$ which approach c. In particular, this means that if

$$\lim_{x \to c} f(x) = L$$

then we must have both

$$\lim_{x \to c^+} f(x) = L$$

and

$$\lim_{x \to c^-} f(x) = L.$$

Not surprisingly, this also works in the other direction; in general, we have

$$\lim_{x \to c} f(x) = L \text{ if and only if both } \lim_{x \to c+} f(x) = L \text{ and } \lim_{x \to c^-} f(x) = L. \tag{2.3.11}$$

Since the above definitions are all in terms of limits of sequences, we may use all the properties of limits of sequences developed in Section 1.2 when discussing the limit of a function defined on an interval of real numbers.

Example Consider the constant function $f(x) = 2$ for all x. To compute, for example, $\lim\limits_{x \to 3} f(x)$, we need to compute $\lim\limits_{n \to \infty} f(x_n)$ for an arbitrary sequence $\{x_n\}$ with $\lim\limits_{n \to \infty} x_n = 3$. For such a sequence, we have

$$\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} 2 = 2.$$

Hence

$$\lim_{x \to 3} f(x) = 2.$$

In fact, it should be easy to see that for any value of c

$$\lim_{x \to c} f(x) = \lim_{x \to c} 2 = 2,$$

and, more generally, for any constant k,

$$\lim_{x \to c} k = k.$$

Example Suppose $f(x) = x$. To find $\lim\limits_{x \to 5} f(x)$, first let $\{x_n\}$ be any sequence with $\lim\limits_{n \to \infty} x_n = 5$. Then

$$\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} x_n = 5,$$

so

$$\lim_{x \to 5} f(x) = 5.$$

In fact, we could replace 5 by an arbitrary c in this computation and obtain the general result that

$$\lim_{x \to c} x = c. \tag{2.3.12}$$

Example To find $\lim_{x \to 6} x^2$, let $\{x_n\}$ be any sequence with $\lim_{n \to \infty} x_n = 6$. Then

$$\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} x_n^2 = \left(\lim_{n \to \infty} x_n\right)^2 = 6^2 = 36.$$

Thus

$$\lim_{x \to 6} x^2 = 36.$$

Again we can generalize this statement by replacing 6 by an arbitrary c, in which case we have

$$\lim_{x \to c} x^2 = c^2.$$

Moreover, we may replace the power 2 by any rational number p for which x^p and c^p are defined and have

$$\lim_{x \to c} x^p = c^p. \tag{2.3.13}$$

Example Let $f(x) = 4x^3 - 6x^2 + x - 7$. To find $\lim_{x \to 2} f(x)$, let $\{x_n\}$ be any sequence with $\lim_{n \to \infty} x_n = 2$. Then

$$\begin{aligned}
\lim_{n \to \infty} f(x_n) &= \lim_{n \to \infty} (4x_n^3 - 6x_n^2 + x_n - 7) \\
&= 4\left(\lim_{n \to \infty} x_n\right)^3 - 6\left(\lim_{n \to \infty} x_n\right)^2 + \lim_{n \to \infty} x_n - 7 \\
&= (4)(2^3) - (6)(2^2) + 2 - 7 = 3.
\end{aligned}$$

Hence

$$\lim_{x \to 2} f(x) = 3,$$

which is just $f(3)$.

Example Now let f be an arbitrary polynomial, say,

$$f(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

for some constants $a_0, a_1, a_2, \ldots, a_m$. If $\{x_n\}$ is any sequence with $\lim_{n \to \infty} x_n = c$, then

$$\begin{aligned}
\lim_{n \to \infty} f(x_n) &= \lim_{n \to \infty} (a_m x_n^m + a_{m-1}x_n^{m-1} + \cdots + a_2 x_n^2 + a_1 x_n + a_0) \\
&= a_m \left(\lim_{n \to \infty} x_n\right)^m + a_{m-1}\left(\lim_{n \to \infty} x_n\right)^{m-1} + \ldots + a_2\left(\lim_{n \to \infty} x_n\right)^2 \\
&\qquad + a_1\left(\lim_{n \to \infty} x_n\right) + a_0 \\
&= a_m c^n + a_{m-1}c^{m-1} + \cdots + a_2 c^2 + a_1 c + a_0 \\
&= f(c).
\end{aligned}$$

Hence, for any polynomial f and any real number c,

$$\lim_{x \to c} f(x) = f(c).$$

The previous example is important enough to state as a proposition.

Proposition If f is a polynomial and c is any real number, then

$$\lim_{x \to c} f(x) = f(c). \tag{2.3.14}$$

If we combine this result with our result about the limits of quotients in Section 1.2, we have the following proposition.

Proposition If f and g are both polynomials and c is any real number for which $g(c) \neq 0$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}. \tag{2.3.15}$$

In short, if h is a rational function and h is defined at c, then the value of the limit of $h(x)$ as x approaches c is simply the value of h at c. That is,

$$\lim_{x \to c} h(x) = h(c) \tag{2.3.16}$$

for any rational function h which is defined at c.

Example Using our result about polynomials, we have

$$\lim_{x \to 2}(3x^4 - 6x + 12) = (3)(2^4) - (6)(2) + 12 = 48.$$

Example Using our result about rational functions, we have

$$\lim_{x \to 3} \frac{3x + 4}{2x^2 + 2x - 1} = \frac{(3)(3) + 4}{(2)(3^2) + (2)(3) - 1} = \frac{13}{23}.$$

Example Now consider

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}.$$

Note that our result about the limits of rational functions does not apply here since the denominator is 0 at $x = 2$. However, since the numerator is also 0 at $x = 2$, the numerator and the denominator must have a common factor of $x - 2$. Canceling this common factor will simplify the problem and enable us to evaluate the limit. That is,

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x-2)(x+2)}{x - 2} = \lim_{x \to 2}(x + 2) = 4.$$

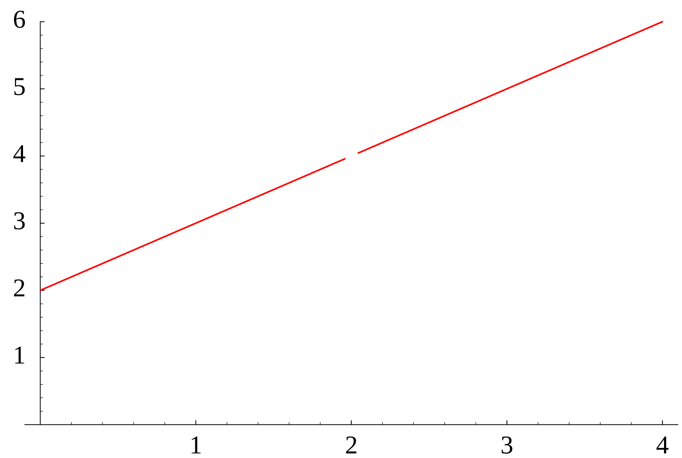


Figure 2.3.3 Graph of $f(x) = \frac{x^2-4}{x-2}$

Although technical, it is worth noting that the functions

$$f(x) = \frac{x^2 - 4}{x - 2}$$

and

$$g(x) = x + 2$$

are different functions. In particular, f is not defined at $x = 2$, whereas g is. However, for every point $x \neq 2$, $f(x) = g(x)$. As a result,

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} g(x),$$

since the limits depend only on the values of f and g for points close to, but not equal to, 2. See the graph of f in Figure 2.3.3.

Example As another example of the technique used in the previous example, we have

$$\begin{aligned} \lim_{t \to -1} \frac{t^2 - 1}{t^3 + 1} &= \lim_{t \to -1} \frac{(t+1)(t-1)}{(t+1)(t^2 - t + 1)} \\ &= \lim_{t \to -1} \frac{t - 1}{t^2 - t + 1} \\ &= \frac{-1 - 1}{1 + 1 + 1} = -\frac{2}{3}. \end{aligned}$$

In the last two examples, we have used the algebraic fact that if c is a root of a polynomial $f(x)$, then $x - c$ is a factor of $f(x)$. In particular, this means that if both the numerator and the denominator of a rational function are 0 at $x = c$, then they have a common factor of $x - c$. However, if the numerator is not 0 at c, but the limit of the denominator is 0, then the limit will not exist. For example, if

$$f(x) = \frac{1}{x^2},$$

then $\lim_{x \to 0} f(x)$ does not exist, since dividing 1 by x_n^2, where $\{x_n\}$ is a sequence with $\lim_{n \to \infty} x_n = 0$, will always result in a sequence of positive numbers which are growing without bound. Borrowing from the notation we developed in Section 1.2, we may write

$$\lim_{x \to 0} \frac{1}{x^2} = \infty.$$

As before, we must be careful to remember that this notation means that, although the function does not have a limit as x approaches 0, the value of the function grows without any bound as x approaches 0. Similarly, since

$$\frac{1}{x} > 0$$

when $x > 0$, we have

$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

and, since

$$\frac{1}{x} < 0$$

when $x < 0$,

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty.$$

However, since

$$f(x) = \frac{1}{x}$$

behaves differently as x approaches 0 from the right than it does when x approaches 0 from the left, all we can say about the limit of $f(x)$ as x approaches 0 is that it does not exist.

Graphically, for a given function f,

$$\lim_{x \to c^-} f(x) = \infty$$

or

$$\lim_{x \to c^-} f(x) = -\infty$$

tells us that the graph of f will approach the vertical line $x = c$ asymptotically as x approaches c from the left. The graph will go off along $x = c$ in the positive direction in the first case and in the negative direction in the second case. Similar remarks hold for x approaching c from the right when

$$\lim_{x \to c^+} f(x) = \infty$$

or

$$\lim_{x \to c^+} f(x) = -\infty.$$

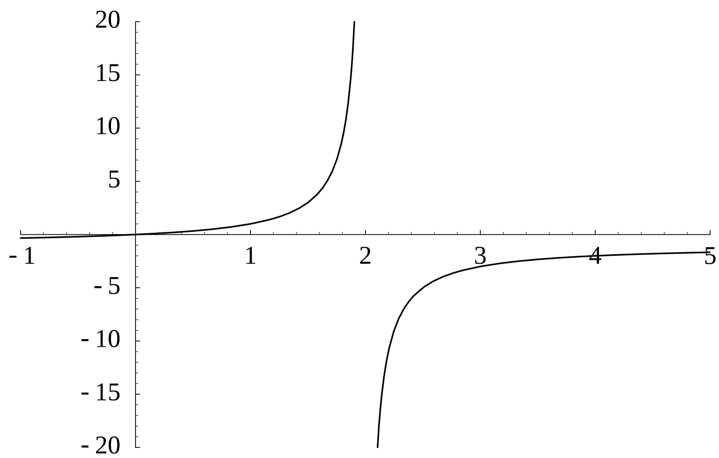


Figure 2.3.4 Graph of $y = \frac{x}{2-x}$

Example Since $2 - x > 0$ when $x < 2$ and $2 - x > 0$ when $x > 2$, it follows that

$$\lim_{x \to 2^-} \frac{x}{2-x} = \infty$$

and

$$\lim_{x \to 2^+} \frac{x}{2-x} = -\infty.$$

It follows that the line $x = 2$ is a vertical asymptote for the graph of

$$y = \frac{x}{2-x},$$

with the curve going off in the positive direction from the left and in the negative direction from the right. See Figure 2.3.4

The next examples illustrate the use of one-sided limits, using (2.3.8), in determining the existence of certain limits.

Example Suppose

$$g(z) = \begin{cases} z^2 + 1, & \text{if } z \leq 1, \\ 3z + 4, & z > 1. \end{cases}$$

Then, since $g(z) = z^2 + 1$ when $z < 1$,

$$\lim_{z \to 1^-} g(z) = \lim_{z \to 1^-} (z^2 + 1) = 2$$

and, since $g(z) = 3z + 4$ when $z > 1$,

$$\lim_{z \to 1^+} g(z) = \lim_{z \to 1^+} (3z + 4) = 7.$$

Since these limits are not the same, we know from (2.3.11) that $g(z)$ does not have a limiting value as z approaches 1. Graphically, we see this as a break in the graph of g at

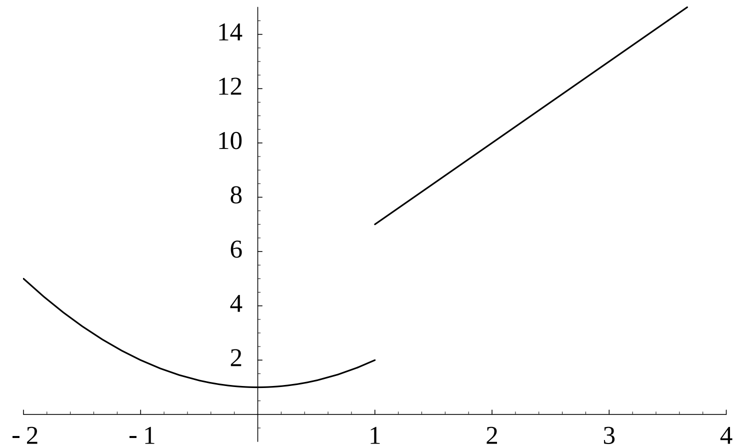


Figure 2.3.5 Graph of $y = g(z)$

$z = 1$, as shown in Figure 2.3.5. Note, however, that for any $c \neq 1$, $\lim_{z \to c} g(z) = g(c)$. For example

$$\lim_{z \to -4} g(z) = \lim_{z \to -4} (z^2 + 1) = 17.$$

Example Now consider

$$h(t) = \begin{cases} 2t + 3, & \text{if } t \leq 2, \\ 2t^2 - 1, & \text{if } t > 2. \end{cases}$$

Then

$$\lim_{t \to 2^-} h(t) = \lim_{t \to 2^-} (2t + 3) = 7$$

and

$$\lim_{t \to 2^+} h(t) = \lim_{t \to 2^+} (2t^2 - 1) = 7.$$

In this case both one-sided limits are equal to 7, so we have, using (2.3.11),

$$\lim_{t \to 2} h(t) = 7.$$

Graphically, the graph of h does not have a break at $t = 2$, even though the formula for computing $h(t)$ changes at this point. See Figure 2.3.6.

We may also use limits to inquire into the behavior of the values of a function f as x increases, or decreases, without bound. This leads to the following definition.

Definition Suppose f is a function defined on an interval J of the form (a, ∞). We say that the *limit of* $f(x)$ *as* x *approaches* ∞ is L, denoted

$$\lim_{x \to \infty} f(x) = L, \tag{2.3.17}$$

if for every sequence $\{x_n\}$ in J we have

$$\lim_{n \to \infty} f(x_n) = L \tag{2.3.18}$$

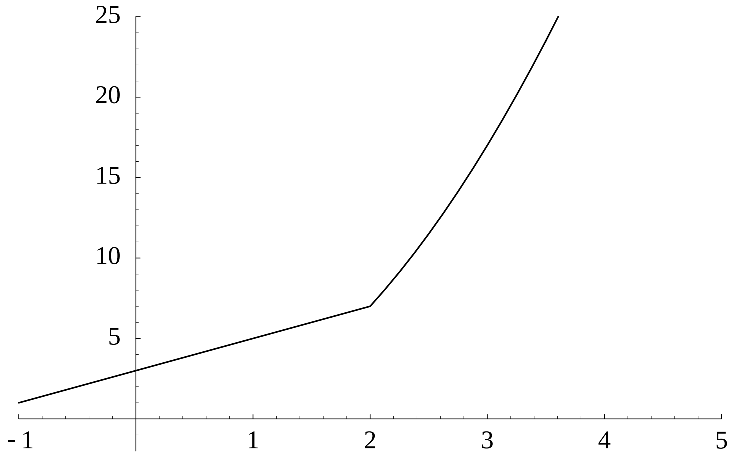


Figure 2.3.6 Graph of $y = h(t)$

whenever

$$\lim_{n\to\infty} x_n = \infty. \tag{2.3.19}$$

Similarly, suppose f is a function defined on an interval J of the form $(-\infty, b)$. We say that the *limit of* $f(x)$ *as* x *approaches* $-\infty$ is L, denoted

$$\lim_{x\to-\infty} f(x) = L, \tag{2.3.20}$$

if for every sequence $\{x_n\}$ in J we have

$$\lim_{n\to\infty} f(x_n) = L \tag{2.3.21}$$

whenever

$$\lim_{n\to\infty} x_n = -\infty. \tag{2.3.22}$$

Example Suppose $\{x_n\}$ is a sequence and $\lim\limits_{n\to\infty} x_n = \infty$. Given $\epsilon > 0$, there must exist an integer N such that

$$x_n > \frac{1}{\epsilon}$$

whenever $n > N$. Hence

$$\frac{1}{x_n} < \epsilon$$

whenever $n > N$. That is,

$$\lim_{n\to\infty} \frac{1}{x_n} = 0.$$

Since this true for any such sequence $\{x_n\}$, it follows that

$$\lim_{n\to\infty} \frac{1}{x} = 0.$$

In a similar fashion, we may show that

$$\lim_{n\to-\infty}\frac{1}{x}=0.$$

With these two basic limits, it is possible to compute limits of these types for any rational function using the same techniques we used in Section 1.2. Namely, given a rational function, dividing numerator and denominator by the highest power appearing in the denominator simplifies the expression to a form where the limit may be evaluated easily.

Example $\displaystyle\lim_{x\to\infty}\frac{3x^2+4x-6}{2x^2-6x+2}=\lim_{x\to\infty}\frac{3+\frac{4}{x}-\frac{6}{x^2}}{2-\frac{6}{x}+\frac{2}{x^2}}=\frac{3}{2}.$

Example $\displaystyle\lim_{x\to\infty}\frac{3x^2-6x}{4x^3+2}=\lim_{x\to\infty}\frac{\frac{3}{x}-\frac{6}{x^2}}{4+\frac{2}{x^3}}=\frac{0}{4}=0.$

Example We have

$$\lim_{x\to-\infty}\frac{4x^3-3}{2x^2+6}=\lim_{x\to-\infty}\frac{4x-\frac{3}{x^2}}{2+\frac{6}{x^2}}=-\infty,$$

since the denominator is approaching 2 while the numerator decreases without bound as x goes to $-\infty$. Note that, as usual, although the limit does not exist, we make use of this notation to indicate the manner in which the limit fails to exist.

Graphically, $\lim_{x\to\infty}f(x)=L$ tells us that the graph of $y=f(x)$ approaches the horizontal line $y=L$ asymptotically as x increases without bound. Similarly, $\lim_{x\to-\infty}f(x)=L$ tells us that the graph of $y=f(x)$ approaches the horizontal line $y=L$ asymptotically as x decreases without bound.

Example Since

$$\lim_{x\to\infty}\frac{x}{x^2+1}=\lim_{x\to\infty}\frac{\frac{1}{x}}{1+\frac{1}{x^2}}=0$$

and

$$\lim_{x\to-\infty}\frac{x}{x^2+1}=\lim_{x\to-\infty}\frac{\frac{1}{x}}{1+\frac{1}{x^2}}=0,$$

we know that the line $y=0$, that is, the x-axis, is a horizontal asymptote for the graph of

$$y=\frac{x}{x^2+1}.$$

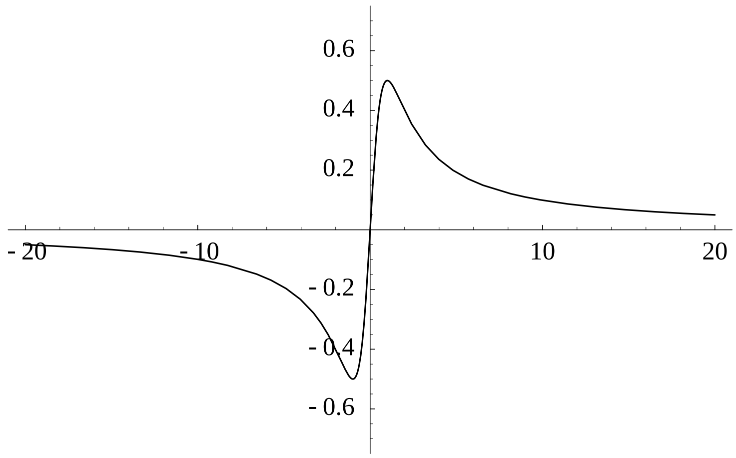


Figure 2.3.7 Graph of $y = \frac{x}{x^2+1}$

Moreover, since

$$\frac{x}{x^2+1} > 0$$

for $x > 0$ and

$$\frac{x}{x^2+1} < 0$$

for $x < 0$, we know that the approach to the x-axis is from above as x increases and from below as x decreases. See Figure 2.3.7.

The following proposition summarizes the basic properties of limits. These are essentially restatements of the properties of limits of sequences that we discussed in Section 1.2. The fact that they hold here follows from the way we have defined limits in this section in terms of limits of sequences. Moreover, the properties listed in this proposition also hold for one-sided limits.

Proposition Suppose $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, where L and M are real numbers and c is a real number, ∞, or $-\infty$. Then

$$\lim_{x \to c} kf(x) = kL \text{ for any constant } k, \tag{2.3.23}$$

$$\lim_{x \to c} (f(x) + g(x)) = L + M, \tag{2.3.24}$$

$$\lim_{x \to c} (f(x) - g(x)) = L - M, \tag{2.3.25}$$

$$\lim_{x \to c} (f(x)g(x)) = LM, \tag{2.3.26}$$

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \tag{2.3.27}$$

and, provided p is a rational number for which $(f(x))^p$ and L^p are defined,

$$\lim_{x \to c} (f(x))^p = L^P. \tag{2.3.28}$$

Example Using (2.3.28), we have

$$\lim_{x\to 4}\sqrt{x^2+3} = \sqrt{\lim_{x\to 4}(x^2+3)} = \sqrt{19}.$$

Example Using the fact hat

$$\sqrt{x^2} = |x| = \begin{cases} -x, & \text{if } x < 0, \\ x, & \text{if } x \geq 0, \end{cases}$$

we have

$$\begin{aligned} \lim_{x\to\infty}\frac{4x}{\sqrt{x^2+1}} &= \lim_{x\to\infty}\frac{4}{\dfrac{\sqrt{x^2+1}}{x}} \\ &= \lim_{x\to\infty}\frac{4}{\dfrac{\sqrt{x^2+1}}{\sqrt{x^2}}} \\ &= \lim_{x\to\infty}\frac{4}{\sqrt{\dfrac{x^2+1}{x^2}}} \\ &= \lim_{x\to\infty}\frac{4}{\sqrt{1+\dfrac{1}{x^2}}} \\ &= 4 \end{aligned}$$

and

$$\begin{aligned} \lim_{x\to-\infty}\frac{4x}{\sqrt{x^2+1}} &= \lim_{x\to-\infty}\frac{4}{\dfrac{\sqrt{x^2+1}}{x}} \\ &= \lim_{x\to-\infty}\frac{4}{\dfrac{\sqrt{x^2+1}}{-\sqrt{x^2}}} \\ &= \lim_{x\to-\infty}\frac{4}{-\sqrt{\dfrac{x^2+1}{x^2}}} \\ &= \lim_{x\to-\infty}\frac{4}{-\sqrt{1+\dfrac{1}{x^2}}} \\ &= -4. \end{aligned}$$

Hence the lines $y = 4$ and $y = -4$ are both horizontal asymptotes for the graph of

$$y = \frac{4x}{\sqrt{x^2+1}}.$$

See Figure 2.3.8.

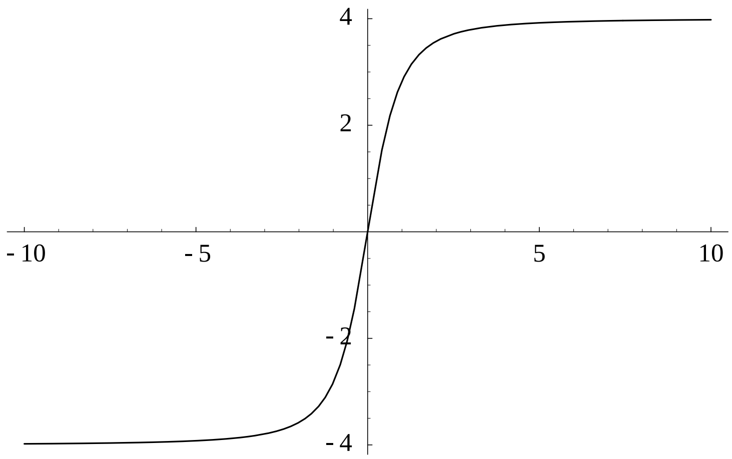


Figure 2.3.8 Graph of $y = \frac{4x}{\sqrt{x^2+1}}$

Problems

1. Evaluate the following limits.

   (a) $\lim_{x \to 2} (4x^2 - 3x)$

   (b) $\lim_{x \to 3} (x^3 - 2x + 3)$

   (c) $\lim_{t \to 1} \dfrac{t^2 - 3}{t + 5}$

   (d) $\lim_{z \to -2} \dfrac{z + 2}{z^2 + 3}$

2. Evaluate the following limits.

   (a) $\lim_{x \to 2} \dfrac{x^3 - 8}{x + 2}$

   (b) $\lim_{x \to 3} \dfrac{x^2 - x - 6}{x - 3}$

   (c) $\lim_{s \to -1} \dfrac{s^3 + 1}{s^4 - 1}$

   (d) $\lim_{x \to 1} \dfrac{\sqrt{x} - 1}{x - 1}$

   (e) $\lim_{t \to 2} \dfrac{t^3 - 8}{t - 2}$

   (f) $\lim_{x \to 1} \dfrac{x^3 - 1}{x^2 - 1}$

   (g) $\lim_{u \to 4} \dfrac{u}{(u - 4)^2}$

   (h) $\lim_{x \to -2} \dfrac{x}{x + 2}$

3. Evaluate the following limits.

   (a) $\lim_{x \to 1^+} (3x^2 + 4)$

   (b) $\lim_{x \to 3^-} \dfrac{1}{x - 3}$

   (c) $\lim_{x \to 3^+} \dfrac{1}{x - 3}$

   (d) $\lim_{t \to -2^+} \dfrac{t}{t + 2}$

   (e) $\lim_{t \to -2^-} \dfrac{t}{t + 2}$

   (f) $\lim_{x \to 10^-} \dfrac{x^2 - 9x - 10}{x^2 - 8x - 20}$

4. Evaluate the following limits.

   (a) $\lim_{x \to 2^+} \lfloor x \rfloor$

   (b) $\lim_{x \to 2^-} \lfloor x \rfloor$

   (c) $\lim_{x \to 3^-} \lceil x \rceil$

   (d) $\lim_{x \to 3^+} \lceil x \rceil$

(e) $\lim_{x \to 0^+} \lfloor \cos(x) \rfloor$

(f) $\lim_{x \to 0^+} \lceil \sin(x) \rceil$

5. Suppose
$$g(z) = \begin{cases} 3z - 1, & \text{if } z < 2, \\ 7 - z, & \text{if } z \geq 2. \end{cases}$$
   (a) Sketch the graph of g.
   (b) Find $\lim_{z \to 2^-} g(z)$.
   (c) Find $\lim_{z \to 2^+} g(z)$.
   (d) Does $\lim_{z \to 2} g(z)$ exist? If so, what is its value?

6. Suppose
$$h(t) = \begin{cases} 2t + 1, & \text{if } t \leq 1, \\ 3t - 1, & \text{if } t > 1. \end{cases}$$
   (a) Sketch the graph of h.
   (b) Find $\lim_{t \to 1^-} h(t)$.
   (c) Find $\lim_{t \to 1^+} h(t)$.
   (d) Does $\lim_{t \to 1} h(t)$ exist? If so, what is its value?

7. Evaluate the following limits.

   (a) $\lim_{x \to \infty} (3x + 4)$

   (b) $\lim_{x \to \infty} \dfrac{x^3 + 3x - 1}{2x^3 - x^2 + 21}$

   (c) $\lim_{u \to \infty} \dfrac{u^4 + 3u - 6}{3u^2 + 1}$

   (d) $\lim_{z \to \infty} \dfrac{4z^2 - 3z + 10}{2z^3 + 14z + 9}$

   (e) $\lim_{x \to -\infty} \dfrac{x^5 - 6x + 13}{x^2 + 18x - 25}$

   (f) $\lim_{x \to \infty} \dfrac{2\sqrt{x} + 3}{\sqrt{x} + 2}$

   (g) $\lim_{v \to \infty} \sqrt{\dfrac{2v + 1}{v - 2}}$

   (h) $\lim_{t \to \infty} \dfrac{\sqrt{t + 1}}{t + 3}$

   (i) $\lim_{x \to \infty} \dfrac{3x + 1}{\sqrt{4x^2 + 5}}$

   (j) $\lim_{x \to -\infty} \dfrac{3x + 1}{\sqrt{4x^2 + 5}}$

8. Let
$$f(x) = \frac{2x}{x - 4}.$$
   Find $\lim_{x \to 4^-} f(x)$, $\lim_{x \to 4^+} f(x)$, $\lim_{x \to -\infty} f(x)$, and $\lim_{x \to \infty} f(x)$. Use this information to sketch the graph of f.

9. Discuss $\lim_{x \to \frac{\pi}{2}^-} \tan(x)$, $\lim_{x \to \frac{\pi}{2}^+} \tan(x)$, and $\lim_{x \to \frac{\pi}{2}} \tan(x)$.

10. Do $\lim_{x \to \infty} \sin(\pi x)$ and $\lim_{n \to \infty} \sin(\pi n)$ denote the same thing? Discuss.

11. (a) Explain why

$$-\frac{1}{x} \le \frac{\sin(x)}{x} \le \frac{1}{x}$$

for all x .

(b) Use part (a) to find $\lim\limits_{x\to\infty} \dfrac{\sin(x)}{x}$.

Section 2.4

Continuous Functions

Given the work of the previous section, we are now in a position to state a clear definition of the notion of continuity. We will have several related definitions, but the fundamental definition is that of continuity at a point. Intuitively, continuity at a point c for a function f means that the values of f for points near c do not change abruptly from the value of f at c. Section 2.3 has shown that, mathematically, this means that as x approaches c, the value of $f(x)$ must be approaching $f(c)$. Hence we have the following basic definition.

Definition We say that a function f is *continuous at a point* c if

$$\lim_{x \to c} f(x) = f(c). \tag{2.4.1}$$

It is important to note that this definition places three conditions on the behavior of the function f near the point c. Namely, f is continuous at the point c if (1) f is defined at c, (2) $\lim_{x \to c} f(x)$ exists, and (3) $\lim_{x \to c} f(x) = f(c)$.

Corresponding to one-sided limits, we have the notions of continuity from the left and from the right.

Definition We say that a function f is *continuous from the left at a point* c if

$$\lim_{x \to c^-} f(x) = f(c). \tag{2.4.2}$$

We say that a function f is *continuous from the right at a point* c if

$$\lim_{x \to c^+} f(x) = f(c). \tag{2.4.3}$$

Simply to say that a function f is continuous, without specifying some particular point, means that the function is continuous, in the proper sense, at all points where it is defined. Here "in the proper sense" means, for example, that if f is defined only on a closed interval $[a, b]$, then we cannot ask for continuity at a or b, since it is possible to discuss only one-sided limits at these points, but it is possible to inquire about continuity from the right at a and continuity from the left at b.

Definition We say a function f is *continuous on the open interval* (a, b) if f is continuous at every point in (a, b). We say f is *continuous on the closed interval* $[a, b]$ if f is continuous on (a, b), continuous from the right at a, and continuous form the left at b.

In the previous section we saw that if f and g are polynomials and c is a point with $g(c) \neq 0$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}.$$

The following proposition restates this fact in terms of our new definitions.

Proposition If h is a rational function and h is defined at the point c, then h is continuous at c. In particular, if h is a polynomial, then h is continuous on the entire real line $(-\infty, \infty)$.

This theorem gives us a very large class of functions which we know to be continuous. As we progress, we will add many more types of functions to this class.

Example Consider the function $f(x) = 3x^3 - 6x + 3$. Since f is a polynomial, it is continuous on $(-\infty, \infty)$. That is, for any real number c, f is continuous at c.

Example Consider

$$g(t) = \frac{8t - 13t^2}{3t - 4}.$$

Then g is a rational function, and so is continuous at all points in its domain. That is, g is continuous for all real numbers c except $c = \frac{4}{3}$. Put another way, g is continuous on the intervals $(-\infty, \frac{4}{3})$ and $(\frac{4}{3}, \infty)$.

Example Suppose

$$h(z) = \begin{cases} z^2 - 2, & \text{if } z \leq 1, \\ 4z - 2, & \text{if } z > 1. \end{cases}$$

On the interval $(-\infty, 1]$, h is a polynomial; thus h is continuous on $(-\infty, 1]$. Note that this does not necessarily mean that h is continuous at 1, only that h is continuous from the left at 1. Similarly, on the interval $(1, \infty)$, h is a polynomial and hence is continuous on $(1, \infty)$. To check for continuity at 1, we note that

$$\lim_{z \to 1^-} h(z) = \lim_{z \to 1^-} (z^2 - 2) = -1,$$

while

$$\lim_{z \to 1^+} h(z) = \lim_{z \to 1^+} (4z - 2) = 2.$$

Since these limits are different, we know that $\lim\limits_{z \to 1} h(z)$ does not exist. Thus h is not continuous at 1. As we saw in Section 2.3, this behavior results in a break in the graph of h at $z = 1$. See Figure 2.4.1.

Example Suppose

$$f(s) = \begin{cases} s + 1, & \text{if } s < 0, \\ s^2 + 1, & \text{if } s \geq 0. \end{cases}$$

Similar to the situation in the previous example, f is continuous on the intervals $(-\infty, 1)$ and $[1, \infty)$ since it is a polynomial on both of these intervals. Now

$$\lim_{s \to 0^-} f(s) = \lim_{s \to 0^-} (s + 1) = 1$$

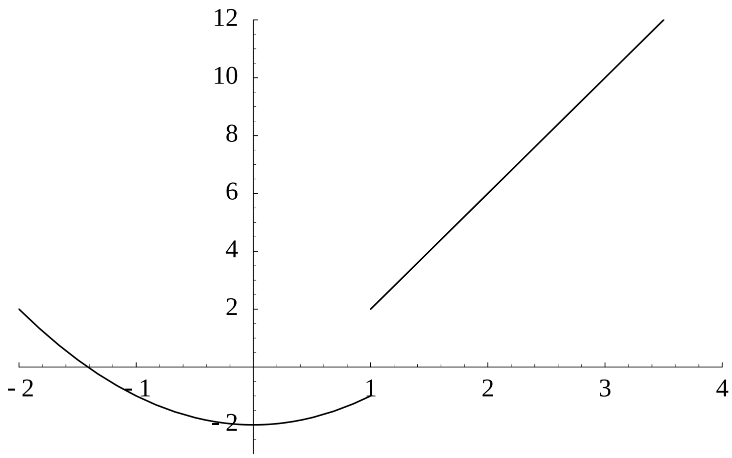


Figure 2.4.1 Graph of $y = h(z)$

and

$$\lim_{s \to 0^+} f(s) = \lim_{s \to 0^+} (s^2 + 1) = 1.$$

Thus $\lim_{s \to 0} f(s) = 1$; since $f(0) = 1$, we have

$$\lim_{s \to 0} f(s) = 1 = f(0).$$

Thus f is continuous at 1. Altogether this shows that f is continuous on the entire interval $(-\infty, \infty)$. As we see in Figure 2.4.2, the graph of f does not have a break at $s = 0$.

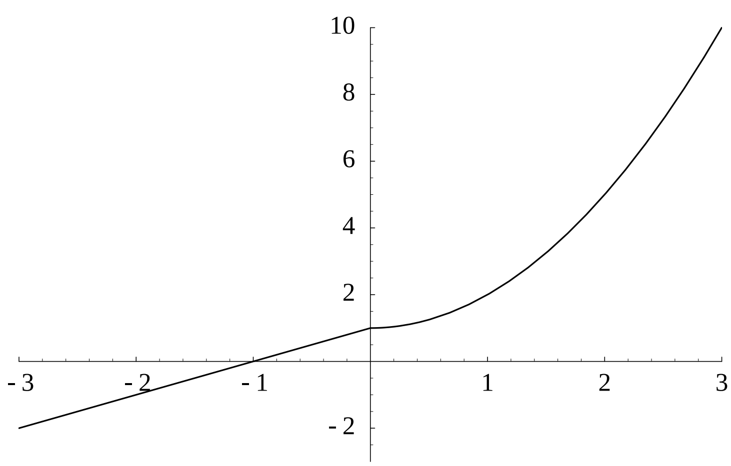


Figure 2.4.2 Graph of $y = f(s)$

Example Suppose

$$g(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & \text{if } x \neq 2, \\ 6, & \text{if } x = 2. \end{cases}$$

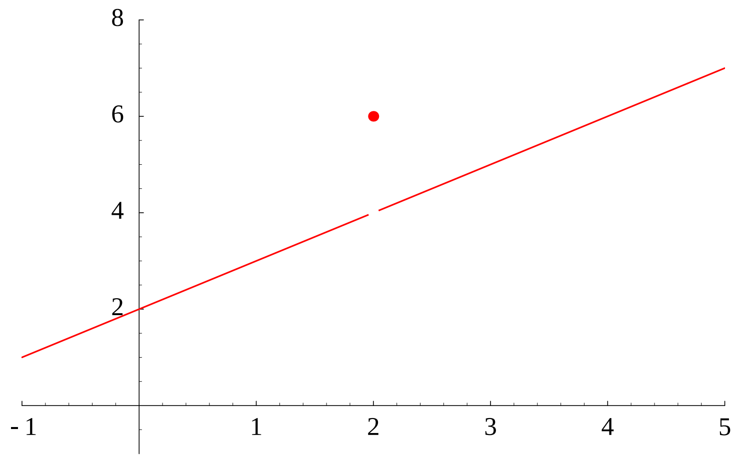


Figure 2.4.3 Graph of $y = g(x)$

Then, since g is a rational function on the intervals $(-\infty, 2)$ and $(2, \infty)$, and is defined throughout these intervals, g is continuous on the intervals $(-\infty, 2)$ and $(2, \infty)$. To check for continuity at 2, we notice that

$$\lim_{x \to 2} g(x) = \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x-2)(x+2)}{x - 2} = \lim_{x \to 2} (x + 2) = 4,$$

while $g(2) = 6$. Hence $\lim\limits_{x \to 2} g(x) \neq g(2)$, and so g is not continuous at 2. See Figure 2.4.3.

It is interesting to note that in the last example the function g could be made continuous if its value at 2 were changed from 6 to 4. In general, if, for a function f and a point c, $\lim\limits_{x \to c} f(x) = L$, but f is not continuous at c because either f is not defined at c or $f(c) \neq L$, we can define a new function h such that $h(x) = f(x)$ for all $x \neq c$ and h is continuous at c. Namely, if we let

$$h(x) = \begin{cases} f(x), & \text{if } x \neq c, \\ L, & \text{if } x = c, \end{cases}$$

then $h(x) = f(x)$ for all $x \neq c$ and

$$\lim_{x \to c} h(x) = \lim_{x \to c} f(x) = L = h(c).$$

Thus h is a function which is identical to f everywhere except at c, but, unlike f, is continuous at c. In this case we say that f has a *removable discontinuity* at c. Note that the existence of a limit at c is essential in order for a discontinuity at c to be removable.

The following proposition lists some properties of continuous functions, all of which are consequences of our results about limits in Section 2.3.

Proposition Suppose the functions f and g are both continuous at a point c and k is a constant. Then the functions which take on the following values for a variable x are also continuous at c:

$$kf(x), \tag{2.4.4}$$

$$f(x)+g(x), \tag{2.4.5}$$

$$f(x)-g(x), \tag{2.4.6}$$

$$f(x)g(x), \tag{2.4.7}$$

$$\frac{f(x)}{g(x)}, \tag{2.4.8}$$

provided $g(c) \neq 0$, and

$$(f(x))^p, \tag{2.4.9}$$

provided p is a rational number and $(f(x))^p$ is defined on an open interval containing c.

Example It follows from (2.4.9) that functions of the form $f(x) = x^p$, where p is a rational number, are continuous throughout their domain. For example, $f(x) = \sqrt{x}$ is continuous on $[0, \infty)$.

Example Using (2.4.8) and (2.4.9),

$$g(t) = \frac{\sqrt{3t+2}}{2t}$$

is continuous for all points t where $3t+2 \geq 0$ and $t \neq 0$. Thus g is continuous on the intervals $[-\frac{2}{3}, 0)$ and $(0, \infty)$.

At this point we have the tools necessary to determine questions of continuity for algebraic functions. We will now show that the sine and cosine functions are continuous on $(-\infty, \infty)$. For $0 < x < \frac{\pi}{2}$, consider the point $C = (\cos(x), \sin(x))$ on the unit circle centered at the origin. If we let $A = (0,0)$ and $B = (1,0)$, as in Figure 2.4.4, then the area of $\triangle ABC$ is

$$\frac{1}{2}\sin(x).$$

The area of the sector of the circle cut off by the arc from B to C is the fraction $\frac{x}{2\pi}$ of the area of the entire circle; hence, this area is

$$\frac{x}{2\pi}\pi = \frac{x}{2}.$$

Since this sector contains $\triangle ABC$, we have

$$0 < \frac{1}{2}\sin(x) < \frac{x}{2},$$

from which it follows that

$$0 < \sin(x) < x.$$

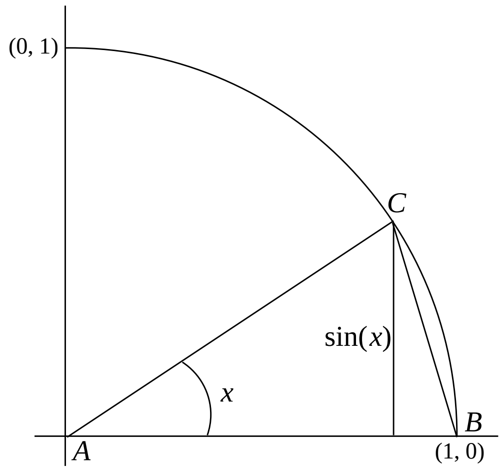


Figure 2.4.4

Since

$$\lim_{x \to 0^+} x = 0,$$

it follows that

$$\lim_{x \to 0^+} \sin(x) = 0.$$

Moreover, we also have

$$\lim_{x \to 0^-} \sin(x) = \lim_{x \to 0^+} \sin(-x) = -\lim_{x \to 0^+} \sin(x) = 0,$$

so

$$\lim_{x \to 0} \sin(x) = 0. \tag{2.4.10}$$

Since $\sin(0) = 0$, this shows that sine is continuous at 0. Now for $-\frac{\pi}{2} < x < \frac{\pi}{2}$,

$$\cos(x) = \sqrt{1 - \sin^2(x)}.$$

Hence

$$\lim_{x \to 0} \cos(x) = \lim_{x \to 0} \sqrt{1 - \sin^2(x)} = \sqrt{1 - \lim_{x \to 0} \sin^2(x)} = 1. \tag{2.4.11}$$

Since $\cos(0) = 1$, this shows that cosine is continuous at 0. For an arbitrary c, we have, using the angle addition formulas for sine and cosine,

$$\begin{aligned}
\lim_{x \to c} \sin(x) &= \lim_{h \to 0} \sin(c + h) \\
&= \lim_{h \to 0} (\sin(c)\cos(h) + \cos(c)\sin(h)) \\
&= \sin(c) \lim_{h \to 0} \cos(h) + \cos(c) \lim_{h \to 0} \sin(h) \\
&= \sin(c)(1) + \cos(c)(0) \\
&= \sin(c)
\end{aligned}$$

and

$$\begin{aligned}\lim_{x\to c}\cos(x) &= \lim_{h\to 0}\cos(c+h)\\ &= \lim_{h\to 0}(\cos(c)\cos(h)-\sin(c)\sin(h))\\ &= \cos(c)\lim_{h\to 0}\cos(h)-\sin(c)\lim_{h\to 0}\sin(h)\cdot\\ &= \cos(c)(1)-\sin(c)(0)\\ &= \cos(c)\end{aligned}$$

Thus we have the following proposition.

Proposition The sine and cosine functions are continuous on $(-\infty,\infty)$.

The next proposition is then an immediate consequence of (2.4.8).

Proposition The tangent, cotangent, secant and cosecant functions are continuous at all points in their respective domains.

We have not yet considered the composition of continuous functions. Suppose g is continuous at c and f is continuous at $g(c)$. If $\{x_n\}$ is a sequence converging to c, then we know, since g is continuous at c, that the sequence $\{g(x_n)\}$ will converge to $g(c)$. But then, since f is continuous at $g(c)$, the sequence $\{f(g(x_n))\}$ will converge to $f(g(c))$. That is,

$$\lim_{x\to c} f\circ g(x) = \lim_{x\to c} f(g(x)) = f(g(c)) = f\circ g(c). \qquad (2.4.12)$$

Hence $f\circ g$ is continuous at c.

Proposition If g is continuous at c and f is continuous at $g(c)$, then $f\circ g$ is continuous at c.

Example The function $h(t)=\cos(3t+4)$ is continuous on $(-\infty,\infty)$ since it is the composition of the functions $g(t)=3t+4$ and $f(t)=\cos(t)$, both of which are continuous on $(-\infty,\infty)$.

Example Consider the function

$$g(t)=\frac{\sin(t^2+1)}{t}.$$

Now the numerator of g is continuous on $(-\infty,\infty)$ since it is the composition of $h(t)=t^2+1$ and $f(t)=\sin(t)$, both of which are continuous on $(-\infty,\infty)$. It follows that, since the denominator of g is continuous on $(-\infty,\infty)$, g is continuous at all points for which the denominator is not equal to zero, that is, for all $t\neq 0$. Thus g is continuous on the intervals $(-\infty,0)$ and $(0,\infty)$.

In Section 2.5 we will consider two properties of continuous functions which partially explain the important role they play in calculus.

Problems

1. Discuss the continuity of the given function at the specified point.

   (a) $f(t) = 3t^2 - 6$ at $t = 2$

   (b) $f(x) = \dfrac{2x+5}{x-16}$ at $x = 17$

   (c) $f(x) = \dfrac{2x+5}{x-16}$ at $x = 16$

   (d) $h(s) = \dfrac{s^2-1}{s+1}$ at $s = 1$

   (e) $h(s) = \dfrac{s^2-1}{s+1}$ at $s = -1$

2. Discuss the continuity of the function

$$g(t) = \begin{cases} 4t - 1, & \text{if } t \leq 2, \\ t + 5, & \text{if } t > 2, \end{cases}$$

   at $t = 2$.

3. Discuss the continuity of the following functions.

   (a) $g(x) = 4x^{23} - x^{18} + 16x - 3$

   (b) $f(t) = \dfrac{t^2 - t - 6}{t+2}$

   (c) $g(t) = 32t - \dfrac{8}{t}$

   (d) $f(u) = \dfrac{8}{u^2 - 4}$

   (e) $f(t) = \sqrt{t^2 - 4}$

   (f) $g(x) = \dfrac{1}{\sqrt{9 - x^2}}$

4. Discuss the continuity of the function

$$f(x) = \begin{cases} 3x + 2, & \text{if } x < 1, \\ 3x + 1, & \text{if } x \geq 1. \end{cases}$$

5. Discuss the continuity of the function

$$h(z) = \begin{cases} z^2 - 1, & \text{if } z \leq -1, \\ z - 1, & \text{if } z > -1. \end{cases}$$

6. The function

$$f(t) = \frac{t^2 - 7t + 12}{t - 4}$$

   is not continuous at $t = 4$. Is this discontinuity removable? If it is, define a new function g which agrees with f whenever $t \neq 4$, but is continuous at 4.

7. The function

$$f(t) = \frac{t^2 - 7t + 12}{t - 5}$$

   is not continuous at $t = 5$. Is this discontinuity removable? If it is, define a new function g which agrees with f whenever $t \neq 5$, but is continuous at 5.

8. Explain why $g(x) = x^2 \sin(x^2 + 1)$ is continuous on $(-\infty, \infty)$.

9. Recall that the Heaviside function is defined by

$$H(t) = \begin{cases} 0, & \text{if } t < 0, \\ 1, & \text{if } t \geq 0. \end{cases}$$

   (a) Discuss the continuity of $f(t) = H(t^2 + 1)$. Graph f on the interval $[-5, 5]$.

   (b) Discuss the continuity of $g(t) = H(t^2 - 1)$. Graph g on the interval $[-5, 5]$.

   (c) Discuss the continuity of $h(t) = H(\sin(\pi t))$. Graph h on the interval $[-5, 5]$.

10. Discuss the continuity of $f(x) = \lfloor x \rfloor$ and $g(x) = \lceil x \rceil$.

11. Discuss the continuity of $f(x) = \lfloor \sin(x) \rfloor$ and $g(x) = \lceil \sin(x) \rceil$.

Difference Equations to Differential Equations

Section 2.5

Some Consequences Of Continuity

In this section we consider two properties of functions which are very closely connected to the notion of continuity. The first of these, the Intermediate Value Theorem, says that the graph of a continuous function is a connected continuum in the sense of our normal intuition. That is, the theorem states that as a continuous function changes from one value to another, it must take on every intermediate value. The second theorem, the Extreme Value Theorem, says that a continuous function on a closed interval attains a maximum and a minimum value on that interval. This is related to our intuitive notion that if we draw a continuous curve with definite beginning and ending points, then the curve has a point where it is higher than at any other point and a point where it is lower than at any other point. We shall not attempt formal justifications of these theorems; such justifications require inquiries into the subtleties of real numbers which are best left to more advanced courses.

We will begin with a statement of the Intermediate Value Theorem, followed by a consideration of its application to solving equations.

Intermediate Value Theorem If f is a continuous function on a closed interval $[a, b]$ and m is any number between $f(a)$ and $f(b)$, then there is a number c in the interval $[a, b]$ such that $f(c) = m$.

Example Since $f(t) = \sin(t)$ is continuous on $[0, \frac{\pi}{2}]$ with $f(0) = 0$ and $f(\frac{\pi}{2}) = 1$, the fact that $0 < \frac{5}{2\pi} < 1$ guarantees that there is a number c in $[0, \frac{\pi}{2}]$ such that

$$f(c) = \frac{5}{2\pi}.$$

Graphically, the situation is as in Figure 2.5.1. Of course, the theorem tells us neither the value of c nor how we might find it. The Intermediate Value Theorem is an existence theorem; it guarantees the existence of a certain value, but does not directly provide any method for calculating the value.

Example Suppose $f(t)$ is the height, in inches, of a certain plant t days after it first emerges from the soil. From our knowledge of how plants grow, it would be reasonable to assume that f is a continuous function. Also, we have $f(0) = 0$. Now if $f(10) = 12$, then we know, for example, that there is some time c, $0 < c < 10$, such that $f(c) = 5$. Of course, this is not surprising, and, in fact, we did not have to bring the subject of continuous functions into the problem in order to realize that between the time when the plant was 0 inches tall and the time when it was 12 inches there was a time when it was 5 inches tall. However, the point of an example like this is to emphasize that the

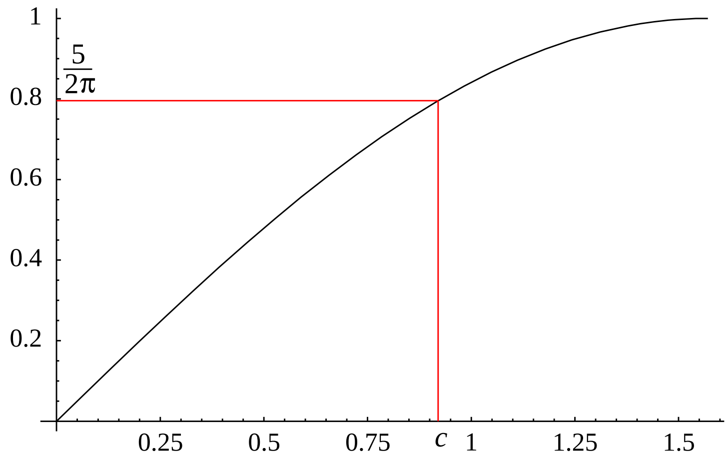


Figure 2.5.1 Intermediate Value Theorem: $\sin(c) = \frac{5}{2\pi}$

Intermediate Value Theorem simply states a property that we should expect continuous functions to have if they are to be used as mathematical models of real world processes that undergo continuous change.

As a special case, the Intermediate Value Theorem tells us that if f is a continuous function on a closed interval $[a, b]$ with $f(a)$ and $f(b)$ having opposite signs (that is, one is negative and the other positive), then there is a point c in the open interval (a, b) where $f(c) = 0$. In other words, under these conditions, the Intermediate Value Theorem guarantees that the equation $f(x) = 0$. has at least one solution in $[a, b]$. Although the theorem does not provide a method for solving the equation, it does provides a basis for constructing an algorithm for approximating a solution to any desired accuracy.

Bisection Algorithm Suppose f is continuous on $[a_1, b_1]$ and $f(a_1)f(b_1) < 0$ (an easy way to check that $f(a_1)$ and $f(b_1)$ have opposite signs). Then, as above, the equation

$$f(x) = 0 \tag{2.5.1}$$

has at least one solution in $[a_1, b_1]$. Let

$$m_1 = \frac{a_1 + b_1}{2}.$$

If $f(m_1) = 0$, then we have found a solution to (2.5.1). If $f(m_1) \neq 0$, then either

$$f(a_1)f(m_1) < 0,$$

in which case (2.5.1) has a solution in $[a_1, m_1]$, or

$$f(m_1)f(b_1) < 0,$$

in which case (2.5.1) has a solution in $[m_1, b_1]$. In the first case, let $a_2 = a_1$ and $b_2 = m_1$; in the second case, let $a_2 = m_1$ and $b_2 = b_1$. Then

$$m_2 = \frac{a_2 + b_2}{2} \tag{2.5.2}$$

will approximate a solution to (2.5.1) with an error less than

$$\frac{b_2 - a_2}{2}. \tag{2.5.3}$$

Proceed in a the same manner to define a_n, b_n, and m_n for $n = 3, 4, 5, \ldots$. That is, if we have found a_{n-1}, b_{n-1}, and m_{n-1}, and $f(m_{n-1}) \neq 0$, let

$$a_n = a_{n-1} \text{ and } b_n = m_{n-1} \text{ if } f(a_{n-1})f(m_{n-1}) < 0 \tag{2.5.4}$$

and

$$a_n = m_{n-1} \text{ and } b_n = b_{n-1} \text{ if } f(m_{n-1})f(b_{n-1}) < 0. \tag{2.5.5}$$

Then

$$m_n = \frac{a_n + b_n}{2} \tag{2.5.6}$$

will approximate a solution of (2.5.1) with an error less than

$$\frac{b_n - a_n}{2}. \tag{2.5.7}$$

Repeat the procedure as many times as necessary to obtain the desired level of accuracy.

Example Suppose we wish to find a root to the equation

$$x^5 + x = 1. \tag{2.5.8}$$

First note that solving (2.5.8) is equivalent to solving

$$x^5 + x - 1 = 0. \tag{2.5.9}$$

Letting

$$f(x) = x^5 + x - 1,$$

we may write (2.5.9) as $f(x) = 0$. To find an initial interval $[a_1, b_1]$, we graph f as in Figure 2.5.2. Noting that $f(0) = -1$ and $f(1) = 1$, we may start with $a_1 = 0$ and $b_1 = 1$. That is, (2.5.8) has a solution in the $[0, 1]$. Then

$$m_1 = \frac{0+1}{2} = 0.5.$$

Now $f(0.5) = -0.468750$, so $f(0.5)f(1) < 0$. Hence $a_2 = 0.5$, $b_2 = 1$, and

$$m_2 = \frac{0.5 + 1.0}{2} = 0.75.$$

Now $f(0.75) = -0.012695$, so $f(0.75)f(1) < 0$. Hence $a_3 = 0.75$, $b_3 = 1$, and

$$m_3 = \frac{0.75 + 1.00}{2} = 0.875.$$

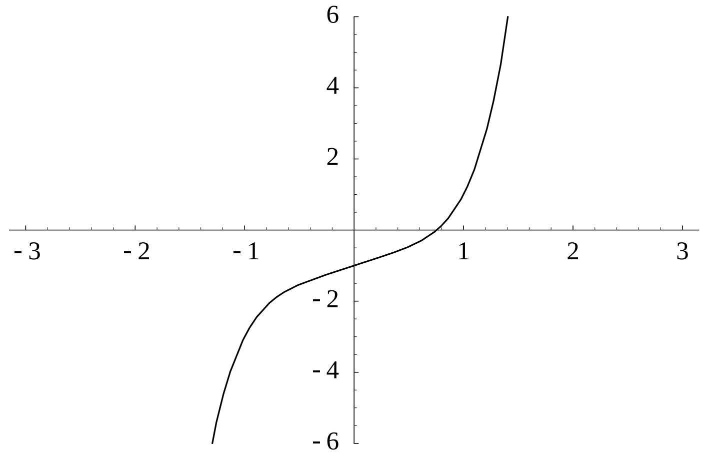


Figure 2.5.2 Graph of $f(x) = x^5 + x - 1$

At this stage we know that 0.875 is an approximation for a solution to (2.5.8) with an error of no more than

$$\frac{1.00 - 0.75}{2} = 0.125.$$

We may continue in this manner until we attain any desired level of accuracy. The following table gives the values of a_n and b_n for $n = 1, 2, 3, \ldots, 10$.

a_n	b_n	m_n	$f(a_n)$	$f(b_n)$	$f(m_n)$
0.000000000	1.000000000	0.500000000	−1.000000000	1.000000000	−0.468750000
0.500000000	1.000000000	0.750000000	−0.468750000	1.000000000	−0.012695300
0.750000000	1.000000000	0.875000000	−0.012695300	1.000000000	0.387909000
0.750000000	0.875000000	0.812500000	−0.012695300	0.387909000	0.166593000
0.750000000	0.812500000	0.781250000	−0.012695300	0.166593000	0.072288300
0.750000000	0.781250000	0.765625000	−0.012695300	0.072288300	0.028700600
0.750000000	0.765625000	0.757812500	−0.012695300	0.028700600	0.007736990
0.750000000	0.757812500	0.753906250	−0.012695300	0.007736990	−0.002544540
0.753906250	0.757812500	0.755859380	−0.002544540	0.007736990	0.002579770
0.753906250	0.755859380	0.754882815	−0.002544540	0.002579770	

Rounding to three decimal places, we see that $x = 0.755$ approximates a solution of (2.5.8) with an error of no more than, to three decimal places,

$$\frac{0.756 - 0.754}{2} = 0.001.$$

In Section 3.6 we will discuss another method, called Newton's method, for approximating a solution to an equation of the form $f(x) = 0$. At that time we will see that Newton's method is faster than the bisection algorithm. However, we will also see that there are conditions under which Newton's method will fail, whereas the bisection algorithm will always work.

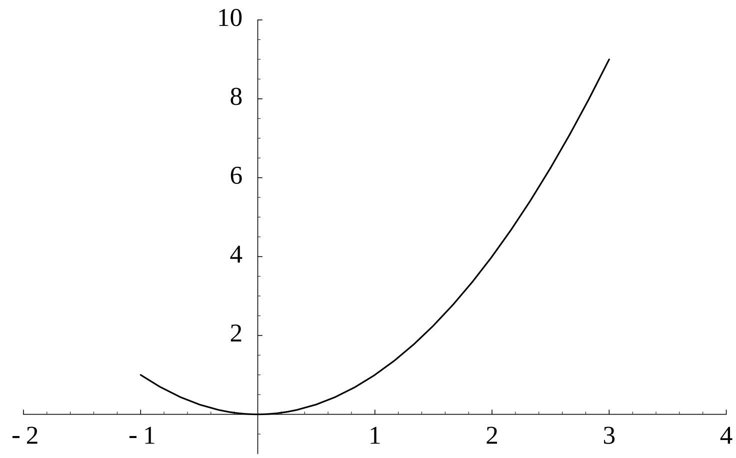


Figure 2.5.3 Graph of $f(x) = x^2$ on $[-1, 3]$

We know turn to the Extreme Value Theorem and some of its consequences.

Extreme Value Theorem If f is a continuous function on a closed interval $[a, b]$, then there exists a point c in $[a, b]$ such that $f(c) \geq f(x)$ for all values of x in $[a, b]$. Similarly, there exists a point d in $[a, b]$ such that $f(d) \leq f(x)$ for all values of x in $[a, b]$.

In other words, using the notation of the statement of the theorem, $f(c)$ is the maximum value attained by f on $[a, b]$ and $f(d)$ is the minimum value attained by f on $[a, b]$. As with the Intermediate Value Theorem, this is an existence theorem which does not indicate any method for finding the points c and d. The importance of the theorem lies in the fact that it gives conditions under which maximum and minimum values of a function are guaranteed to exist. Optimization problems, that is, problems concerned with finding the maximum and minimum values of functions, occur frequently in mathematics and in the applications of mathematics. As we shall see in Section 3.8, conditions which guarantee the existence of a solution to an optimization problem, such as those given in the Extreme Value Theorem, are often an important first step in solving such problems.

Example Consider $f(x) = x^2$ on the interval $[-1, 3]$. Since f is a continuous function on this closed interval, the Extreme Value Theorem guarantees the existence of a maximum value and a minimum value for f. In fact, from our knowledge of the behavior of this function, in particular that $f(0) = 0$, $f(x) > 0$ if $x \neq 0$, and $f(x) > f(y)$ if $|x| > |y|$, it is easy to see that $f(x)$ attains its maximum value when $x = 3$ and its minimum value when $x = 0$ (see Figure 2.5.3). Hence the maximum value of f on $[-1, 3]$ is 9 when $x = 3$ and the minimum value is 0 when $x = 0$.

Example Let A, B, and C be constants with $A > 0$. Suppose we wish to find the minimum value of the quadratic polynomial

$$f(x) = Ax^2 + Bx + C \tag{2.5.10}$$

on an interval $[a, b]$. Completing the square, we may rewrite f as

$$\begin{aligned}
f(x) &= Ax^2 + Bx + C \\
&= A\left(x^2 + \frac{B}{A}x + \frac{C}{A}\right) \\
&= A\left(\left(x + \frac{B}{2A}\right)^2 - \frac{B^2}{4A^2} + \frac{C}{A}\right) \\
&= A\left(x + \frac{B}{2A}\right)^2 + C - \frac{B^2}{4A}.
\end{aligned}$$

Since $C - \frac{B^2}{4A}$ is a constant, $f(x)$ is minimized when $A\left(x + \frac{B}{2A}\right)^2$ is minimized. This latter term is never negative and is minimized when it is 0, that is, when

$$x + \frac{B}{2A} = 0.$$

Hence the minimum value of $f(x)$ on $[a, b]$ will occur when

$$x = -\frac{B}{2A}, \tag{2.5.11}$$

unless this point is not in the interval, in which case the minimum value occurs at one of the endpoints, $x = a$ or $x = b$. Note that, geometrically, (2.5.11) is the location of the vertex of the parabola which is the graph of f. Note that if $A < 0$, then the maximum value of $f(x)$ would occur at (2.5.11) if it is in the interval $[a, b]$, and at one of the endpoints otherwise.

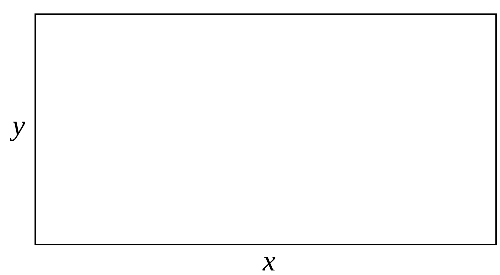


Figure 2.5.4 A field of length x and width y

Example Suppose we wish to fence in a rectangular field with 500 yards of fencing in such a way that we maximize the area of the resulting field. If, as in Figure 2.5.4, we let x denote the length of the field, y its width, and A its area, then

$$A = xy.$$

Moreover, since we only have 500 yards of fencing to work with, we know that

$$2x + 2y = 500.$$

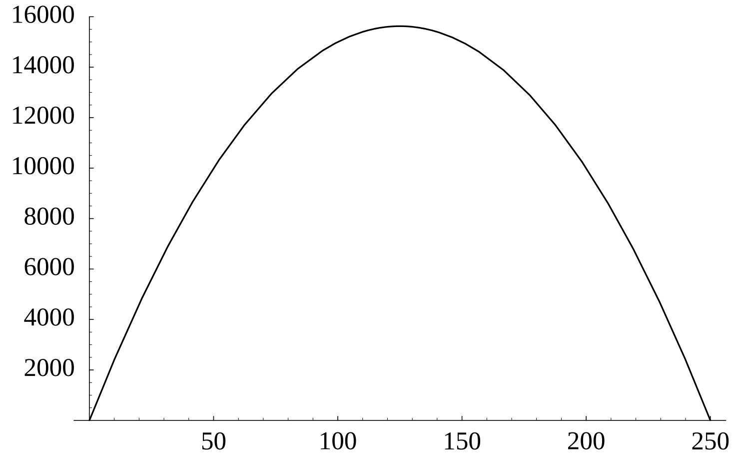


Figure 2.5.5 Graph of $A = 250x - x^2$ on $[0, 25]$

Hence

$$y = 250 - x, \tag{2.5.12}$$

from which it follows that

$$A = xy = x(250 - x) = 250x - x^2.$$

From (2.5.12), and the fact that we must have both $x \geq 0$ and $y \geq 0$, it follows that $0 \leq x \leq 250$. Thus our problem becomes one of finding the maximum value of

$$A = -x^2 + 250x$$

on the closed interval $[0, 250]$. From our previous example, the maximum value of A will occur when

$$x = -\frac{250}{(2)(-1)} = \frac{250}{2} = 125.$$

From (2.5.12), we have $y = 125$ when $x = 125$. Hence the area of the field is maximized when its dimensions are 125 yards by 125 yards. For these dimensions, the area of the field is

$$A\Big|_{x=125} = (125)(125) = 15,625 \text{ square yards.}$$

See Figure 2.5.5 for the graph of A.

Example Consider the function $f(x) = x^2 + 1$ on the open interval $(0, 1)$. Then

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x^2 + 1) = 1$$

and

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2 + 1) = 2,$$

but $0 < f(x) < 2$ for all values of x in $(0, 1)$. Hence, as x approaches 0 from the right, $f(x)$ approaches, but never reaches, 1; similarly, as x approaches 1 from the left, $f(x)$

approaches, but never reaches, 2. Thus f is an example of a continuous function on an open interval which attains neither a maximum nor a minimum value on the interval. Hence we see why the interval in the statement of the Extreme Value Theorem must be a closed interval.

Problems

1. Use the bisection algorithm to approximate a solution to each of the following equations on the given interval. Your answer should have an error of no more than 0.005.

   (a) $x^2 - 2 = 0$ on $[0, 4]$ (b) $x^5 - 6x^3 + 2x = 2$ on $[-1, 1]$

   (c) $\cos(x) = x$ on $[0, \pi]$ (d) $2\sin(x) = \sqrt{x+1}$ on $[0, 2]$

2. (a) Plot the graph of $g(t) = t^2 - \cos^2(t)$ on $[-\pi, \pi]$.

   (b) How many solutions are there to the equation $t^2 = \cos^2(t)$?

   (c) Use the bisection algorithm to estimate the solutions to the equation $t^2 = \cos^2(t)$. State your answers with an error less than 0.005.

3. Suppose if the market price for a certain product is p dollars, then the demand for that product will be
$$D(p) = \frac{50000p + 10000}{p^2} \text{ units.}$$
At the same time, suppose that at a price of p dollars producers will be willing to supply
$$S(p) = \frac{1}{3}p^2 + 2p \text{ units.}$$

   (a) Plot the graphs of D and S on the same graph.

   (b) Use the bisection algorithm to estimate the solution to the equation
$$D(p) = S(p).$$
   This point is called the *equilibrium price* because it is the price for which the consumers' demand for the product is exactly equal to the manufacturers' supply.

   (c) How many units of the product will be manufactured at the equilibrium price?

   (d) What would happen if the producers raised the price above the equilibrium price? What would happen if they lowered the price below the equilibrium price?

   (e) What would happen if the producers increased production? What would happen if they lowered production?

4. A farmer wishes to fence in a rectangular field, using a straight river for one side, with 500 yards of fencing. What should the dimensions of the field be in order to maximize the area of the field?

5. When a potter sells his pots for p dollars apiece, he can sell $D(p) = 750 - 50p$ of them. Suppose the pots cost him \$5.00 apiece to make. What price should the potter charge in order to maximize his profit?

6. Let $h(t) = t^4 - 1$.
   (a) Does h have a maximum value on $[-1, 2)$?
   (b) Does h have a minimum value on $[-1, 2)$?
   (c) Are the results of (a) and (b) consistent with the Extreme Value Theorem? Explain.

7. Recall that the Heavidside function is defined by

$$H(t) = \begin{cases} 0, & \text{if } t < 0, \\ 1, & \text{if } t \geq 0. \end{cases}$$

   (a) Note that $H(-1) = 0$ and $H(1) = 1$. Is there a point c in $(-1, 1)$ such that $H(c) = 0.5$?
   (b) Is the result of (a) consistent with the Intermediate Value Theorem?
   (c) Does H attain a maximum value on $[-1, 1]$? Does H attain a minimum value on $[-1, 1]$?
   (d) Are the results of (c) consistent with the Extreme Value Theorem?

8. Suppose g is defined on $[-1, 1]$ by

$$g(t) = \begin{cases} |t|, & \text{if } t \neq 0, \\ 1, & \text{if } t = 0. \end{cases}$$

   (a) Does g attain a maximum value on $[-1, 1]$? If so, at what points?
   (b) Does g attain a minimum value on $[-1, 1]$? If so, at what points?
   (c) Are the results of (a) and (b) consistent with the Extreme Value Theorem?

9. Suppose f and g are continuous on $[0, 1]$, $f(0) < g(0)$ and $f(1) > g(1)$. Show that there exists a point c in the open interval $(0, 1)$ such that $f(c) = g(c)$.

10. Suppose f is continuous on $[0, 1]$ and $0 \leq f(x) \leq 1$ for all x in $[0, 1]$. Show that there exists a point c in $[0, 1]$ such that $f(c) = c$.

Section 3.1

Best Affine Approximations

We are now in a position to discuss the two central problems of calculus as mentioned in Section 1.1. In this chapter we will take up the problem of finding tangent lines; in Chapter 4 we will consider the problem of finding areas. We choose this order only because the work we do in solving the tangent line problem in this chapter will be of use, through the Fundamental Theorem of Calculus, in solving area problems in the next.

We begin with some preliminary notation and terminology. If f is a function with domain contained in the set A and range contained in the set B, then we may denote this fact by writing $f : A \to B$. For example, if $g(t) = \sqrt{1-t^2}$ and $\mathbb{R}$ denotes the set of real numbers, then the statements $g : \mathbb{R} \to \mathbb{R}$, $g : [-1,1] \to \mathbb{R}$, and $g : [-1,1] \to [0,1]$ are all correct. We will work exclusively with functions of the form $f : \mathbb{R} \to \mathbb{R}$ until Chapter 7, where we will introduce functions of the form $f : \mathbb{R} \to \mathbb{C}$ and $f : \mathbb{C} \to \mathbb{C}$, where $\mathbb{C}$ denotes the set of complex numbers.

We call a function $f : \mathbb{R} \to \mathbb{R}$ *linear* if there is a constant m such that $f(x) = mx$ for all values of x. Graphically, linear functions are functions whose graphs are straight lines passing through the origin. We call a function $f : \mathbb{R} \to \mathbb{R}$ *affine* if there are constants m and b such that $f(x) = mx + b$ for all values of x. Graphically, affine functions are functions whose graphs are straight lines, not necessarily passing through the origin. Put another way, an affine function is a first degree polynomial. Thus $f(x) = 3x$ is both linear and affine, whereas $g(t) = 4t - 6$ is affine but not linear.

The problem of finding the tangent line for the graph of a given function f at a point (x_0, y_0) is really the problem of finding the affine function T which best approximates f for points close to x_0. In this section we will discuss how to solve this problem. In the remaining sections of this chapter we will consider techniques for finding best affine approximations and discuss some applications. In Chapter 5 we will see how to improve upon affine approximations by using higher degree polynomials.

The following example should help to make these ideas more concrete.

Example Consider the problem of approximating the function $f(x) = \sqrt{x}$ for values of x close to 1. For a first approximation, we might say that

$$x \approx 1$$

for x close to 1. In other words, if we let

$$T(x) = 1$$

for all x, then we are saying that the affine function T is a good approximation for f when x is close to 1. Two facts characterize this statement. First, T and f agree at 1; that is,

$$T(1) = 1 = f(1). \tag{3.1.1}$$

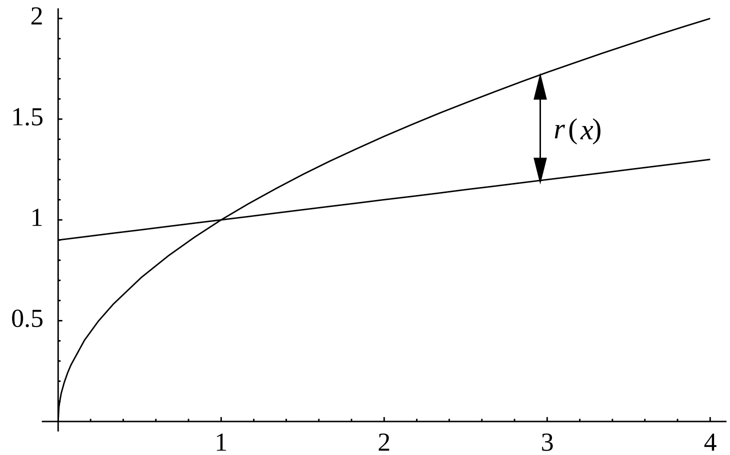


Figure 3.1.1 Graph of $f(x) = \sqrt{x}$ and an approximating affine function

Second, the error committed by approximating f by T goes to 0 as x approaches 1. That is, if we let

$$r(x) = f(x) - T(x),$$

then $r(x)$ is the error made in approximating f by T at the point x, and

$$\lim_{x\to 1} r(x) = \lim_{x\to 1}(f(x) - T(x)) = \lim_{x\to 1}(\sqrt{x} - 1) = 1 - 1 = 0. \tag{3.1.2}$$

Hence we have found an affine function which approximates our function f about $x = 1$ according to some reasonable criterion.

However, it is easy to see that any affine function T whose graph passes through $(1, 1)$ will satisfy (3.1.1) and (3.1.2). First note that if the graph of T is a straight line passing through $(1, 1)$ with slope m, then, using the point-slope form for the equation of a line,

$$T(x) = m(x - 1) + 1.$$

It then follows that

$$T(1) = 1 = f(1)$$

and, if we again let $r(x) = f(x) - T(x)$,

$$\lim_{x\to 1} r(x) = \lim_{x\to 1}(f(x) - T(x)) = \lim_{x\to 1}(\sqrt{x} - (m(x - 1) + 1)) = 1 - 1 = 0.$$

See Figure 3.1.1 for the geometrical interpretation. So now we must ask if there is a value of m which makes T, in some sense, better than any other affine function for approximating f for x near 1. In answering this question, it is convenient to let $h = x - 1$ and to define

$$R(h) = r(1 + h) = f(1 + h) - T(1 + h),$$

the amount of error committed when f is approximated by T at a point a distance h from 1. Since h approaches 0 as x approaches 1, (3.1.1) and (3.1.2) become, in terms of R,

$$R(0) = 0 \tag{3.1.3}$$

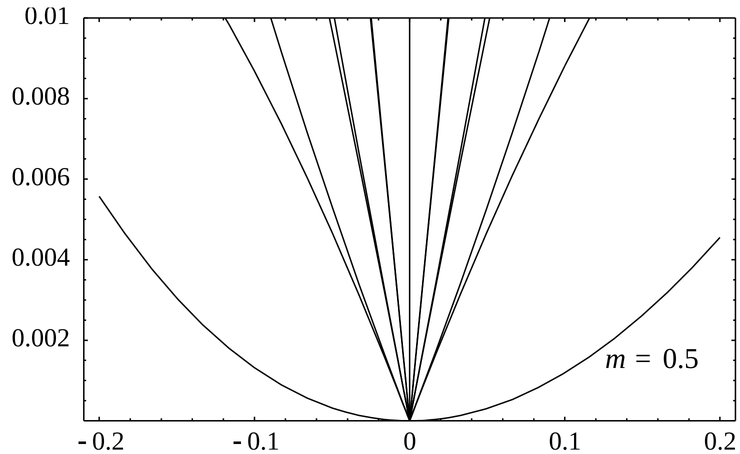


Figure 3.1.2 Graphs of $|R(h)|$ for different values of m

and

$$\lim_{h \to 0} R(h) = 0. \tag{3.1.4}$$

In this case, we have

$$R(h) = \sqrt{1+h} - (m((1+h)-1)+1) = \sqrt{1+h} - (mh+1).$$

Figure 3.1.2 shows the graphs of $|R(h)|$ on the interval $[-0.2, 0.2]$ for $m = 0.1$, 0.3, 0.4, 0.5, 0.6, 0.7, and 0.9. Note that although all these functions approach 0 as h approaches 0, one of the graphs clearly stands out from the others. Namely, when $m = 0.5$, the absolute value of the approximation error appears to approach 0 at a significantly faster rate than does the error for other values of m. To see why this is so, consider that

$$\begin{aligned} R(h) &= \sqrt{1+h} - (mh+1) \\ &= (\sqrt{1+h} - (mh+1))\frac{\sqrt{1+h} + (mh+1)}{\sqrt{1+h} + (mh+1)} \\ &= \frac{1+h-(mh+1)^2}{\sqrt{1+h}+mh+1} \\ &= \frac{1+h-(m^2h^2+2mh+1)}{\sqrt{1+h}+mh+1} \\ &= \frac{h(1-2m)-m^2h^2}{\sqrt{1+h}+mh+1}. \end{aligned}$$

Note that when $m = 0.5$, the numerator reduces to $-m^2h^2$, whereas for other values of m there is also the term $h(1-2m)$. This explains why in Figure 3.1.2 the graph for $m = 0.5$ looks parabolic while the other graphs appear more as straight lines. Moreover, since, for small values of h, h^2 is significantly smaller than h (for example, $(0.001)^2 = 0.000001$ is much smaller than 0.001), we see why the approximation errors when $m = 0.5$ are so much smaller than they are for other values of m.

Intuitively, we should think that for small values of h, $R(h)$ behaves like a multiple of h when $m \neq 0.5$ and like a multiple of h^2 when $m = 0.5$. To see this algebraically, it is useful to consider the quotient

$$\frac{R(h)}{h} = \frac{1 - 2m - m^2 h}{\sqrt{1+h} + mh + 1}.$$

Notice that

$$\lim_{h \to 0} \frac{R(h)}{h} = \frac{1-2m}{2},$$

which is 0 only when $m = 0.5$. We interpret this as an indication that $R(h)$ behaves like a multiple of h for small values of h when $m \neq 0.5$, but approaches 0 more rapidly than h when $m = 0.5$.

In our example, we saw that

$$\lim_{h \to 0} \frac{R(h)}{h} = 0$$

when $m = 0.5$, but

$$\lim_{h \to 0} \frac{R(h)}{h} \neq 0$$

for all other values of m. We distinguish the two cases by saying that in the first case $R(h)$ is $o(h)$, whereas in the second case $R(h)$ is only $O(h)$.

Definition A function f is said to be $o(h)$ if

$$\lim_{h \to 0} \frac{f(h)}{h} = 0. \tag{3.1.5}$$

Definition A function f is said to be $O(h)$ if there exist constants M and $\epsilon > 0$ such that

$$\left| \frac{f(h)}{h} \right| \leq M \tag{3.1.6}$$

whenever $-\epsilon < h < \epsilon$.

Note that if

$$\lim_{h \to 0} \frac{f(h)}{h} = L,$$

then we may find an $\epsilon > 0$ such that

$$\left| \frac{f(h)}{h} - L \right| \leq 1$$

whenever $|h| < \epsilon$. Hence

$$L - 1 \leq \frac{f(h)}{h} \leq L + 1$$

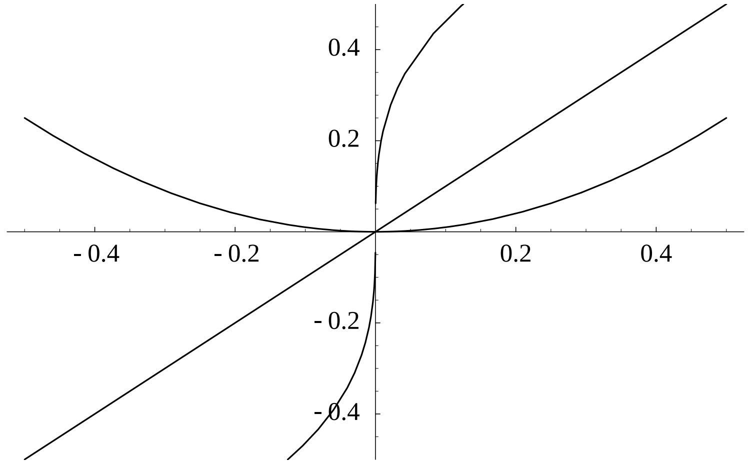


Figure 3.1.3 Rates of convergence to 0 of $f(x) = x^2$, $g(x) = x$, and $k(x) = x^{\frac{1}{3}}$

whenever $|h| < \epsilon$. If we let M be the larger of $|L - 1|$ and $|L + 1|$, then this shows that

$$\left| \frac{f(h)}{h} \right| \leq M$$

whenever $-\epsilon < h < \epsilon$. Hence we have the following proposition.

Proposition If $\displaystyle\lim_{h \to 0} \frac{f(h)}{h}$ exists, then f is $O(h)$.

Note that a function which is $o(h)$ is also $O(h)$. Intuitively, we think of a function which is $o(h)$ as approaching 0 faster than h as h goes to 0, and a function which is $O(h)$ as approaching 0 at a rate which is at least as fast as that of h.

Example Let $f(x) = x^2$, $g(x) = x$, and $k(x) = x^{\frac{1}{3}}$. Then

$$\lim_{h \to 0} f(h) = \lim_{h \to 0} h^2 = 0,$$

$$\lim_{h \to 0} g(h) = \lim_{h \to 0} h = 0,$$

and

$$\lim_{h \to 0} k(h) = \lim_{h \to 0} h^{\frac{1}{3}} = 0.$$

However,

$$\lim_{h \to 0} \frac{f(h)}{h} = \lim_{h \to 0} \frac{h^2}{h} = \lim_{h \to 0} h = 0,$$

so f is $o(h)$;

$$\lim_{h \to 0} \frac{g(h)}{h} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1,$$

so g is $O(h)$, but not $o(h)$; and

$$\lim_{h \to 0} \frac{k(h)}{h} = \lim_{h \to 0} \frac{h^{\frac{1}{3}}}{h} = \lim_{h \to 0} \frac{1}{h^{\frac{2}{3}}} = \infty,$$

so k is neither $o(h)$ nor $O(h)$. Note in Figure 3.1.3 the difference in the way in which these functions approach 0.

Example Let $f(x) = x - x^2$. Then

$$\lim_{h\to 0} \frac{f(h)}{h} = \lim_{h\to 0} \frac{h - h^2}{h} = \lim_{h\to 0}(1-h) = 1,$$

so f is $O(h)$, but not $o(h)$.

Example Let $g(x) = 2\sqrt{1+x} - x - 2$. Then

$$\begin{aligned}
\lim_{h\to 0} \frac{g(h)}{h} &= \lim_{h\to 0} \frac{2\sqrt{1+h} - h - 2}{h} \\
&= \lim_{h\to 0} \left(\frac{2\sqrt{1+h} - (h+2)}{h}\right)\left(\frac{2\sqrt{1+h} + (h+2)}{2\sqrt{1+h} + (h+2)}\right) \\
&= \lim_{h\to 0} \frac{4(1+h) - (h+2)^2}{h(2\sqrt{1+h} + (h+2))} \\
&= \lim_{h\to 0} \frac{4 + 4h - (h^2 + 4h + 4)}{h(2\sqrt{1+h} + h + 2)} \\
&= \lim_{h\to 0} \frac{-h^2}{h(2\sqrt{1+h} + h + 2)} \\
&= \lim_{h\to 0} \frac{-h}{2\sqrt{1+h} + h + 2} \\
&= \frac{0}{4} = 0.
\end{aligned}$$

Thus g is $o(h)$.

Example Returning to the problem of approximating $f(x) = \sqrt{x}$ for x close to 1, let

$$T(x) = m(x-1) + 1$$

and

$$R(h) = f(1+h) - T(1+h).$$

We saw above that

$$\lim_{h\to 0} \frac{R(h)}{h} = \frac{1-2m}{2}.$$

Thus $R(h)$ is $o(h)$ if and only if $m = 0.5$. In other words, the error in approximating $f(x) = \sqrt{x}$ by the affine function

$$T(x) = \frac{1}{2}(x-1) + 1 \tag{3.1.7}$$

goes to 0 faster as x approaches 1 than the error for any other affine function approximation. Because of this, we will call (3.1.7) the *best affine approximation* to f at 1. Moreover, we

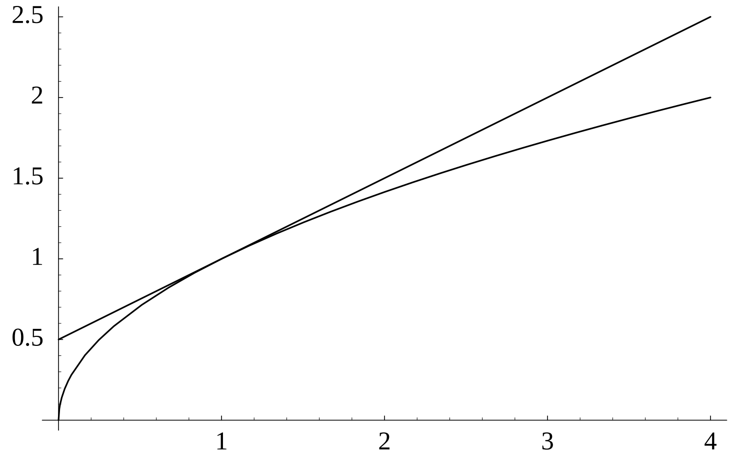


Figure 3.1.4 Graph of $f(x) = \sqrt{x}$ and its tangent line at $(1, 1)$

will call the graph of T, which is a straight line through $(1, 1)$ with slope 0.5, the *tangent line* to the graph of f at $(1, 1)$. See Figure 3.1.4.

As an example of using T to approximate f, note that, to 4 decimal places,

$$\sqrt{1.1} = 1.0488,$$

while

$$T(1.1) = \frac{1}{2}(1.1 - 1) + 1 = 1.05,$$

giving a remainder of only

$$R(0.1) = 1.0488 - 1.0500 = -0.0012.$$

This approximation is remarkably accurate considering the simplicity of the calculations used to obtain it. Of course, we expect the accuracy of the approximation to increase as h decreases. For example, to 4 decimal places,

$$\sqrt{1.05} = 1.0247,$$

while

$$T(1.05) = \frac{1}{2}(1.05 - 1) + 1 = 1.025,$$

giving a remainder of only

$$R(0.05) = 1.0247 - 1.0250 = -0.0003.$$

Note that when we decreased h from 0.1 to 0.05, a factor of $\frac{1}{2}$, the error went from -0.0012 to -0.003, a factor of $\frac{1}{4}$. This is evidence of the quadratic nature of the error, the fact that $R(h)$ is approaching 0 like h^2, not like h.

Using the ideas of this example, we may now make the following definition.

Definition Let f be a function defined in an open interval about a point c. If T is an affine function such that $T(c) = f(c)$ and

$$R(h) = f(c+h) - T(c+h)$$

is $o(h)$, then we call T the *best affine approximation* to f at c. Moreover, the graph of T is called the *tangent line* to the graph of f at $(c, f(c))$.

Using the point-slope form for the equation of a line, the equation of the tangent line at $(c, f(c))$ may be written in the form

$$y - f(c) = m(x - c)$$

for some slope m. That is,

$$y = m(x - c) + f(c),$$

or, in other words, the best affine approximation has the form

$$T(x) = m(x - c) + f(c).$$

Thus, to determine T, we need only find the value of m. Since this number m is of such importance, we give it a formal definition.

Definition If

$$T(x) = m(x - c) + f(c)$$

is the best affine approximation to f at c, then we call m, the slope of the graph of T, the *derivative* of f at c. This value is denoted by $f'(c)$.

With this notation, the best affine approximation has the form

$$T(x) = f'(c)(x - c) + f(c). \tag{3.1.8}$$

Example As a consequence of our previous example, if $f(x) = \sqrt{x}$, then

$$f'(1) = \frac{1}{2}.$$

Example Let $f(x) = x^2$ and suppose we wish to find the best affine approximation to f at 3. Then $f(3) = 9$, so we will let

$$T(x) = m(x - 3) + 9$$

and

$$R(h) = f(3+h) - T(3+h) = (3+h)^2 - (mh + 9).$$

Hence

$$R(h) = 9 + 6h + h^2 - mh - 9 = h(6 + h - m),$$

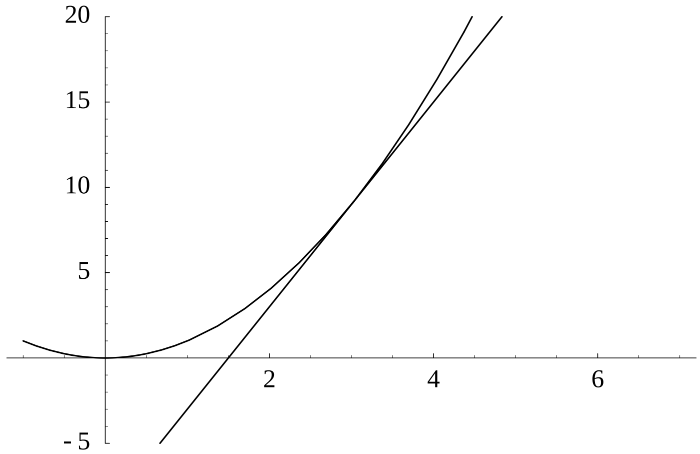


Figure 3.1.5 Graph of $f(x) = x^2$ and its tangent line at $(3, 9)$

and so

$$\lim_{h \to 0} \frac{R(h)}{h} = \lim_{h \to 0} \frac{h(6 + h - m)}{h} = \lim_{h \to 0} (6 - m + h) = 6 - m.$$

Thus $R(h)$ is $o(h)$ if and only if $m = 6$. It follows then that $f'(3) = 6$ and the best affine approximation to f at 3 is

$$T(x) = 6(x - 3) + 9.$$

The equation of the tangent line at $(3, 9)$ is

$$y = 6(x - 3) + 9,$$

or, equivalently,

$$y = 6x - 9.$$

See Figure 3.1.5.

In Sections 3.2 through 3.5 we will explore techniques which will simplify greatly the process of finding derivatives.

Problems

1. Consider the problem of finding an affine approximation for $f(x) = \sin(x)$ near 0. Since $f(0) = 0$, we let $T(x) = mx$ and

$$R(h) = f(h) - T(h) = \sin(h) - mh.$$

   (a) Plot $|R(h)|$ on the interval $[-0.2, 0.2]$ for $m = 0.0, 0.2, 0.4, \ldots, 2.0$.
   (b) Which value of m gives the smallest errors?

2. For each of the following, decide if the given function is $O(h)$, $o(h)$, or neither.

   (a) $f(x) = x^3$ (b) $f(x) = x^2 + 3x$

(c) $g(t) = 4t^3 - 3t^2$

(d) $g(x) = \sqrt{4+x} - \dfrac{x}{4} - 2$

(e) $f(t) = t^{\frac{4}{3}}$

(f) $g(t) = t - t^{\frac{3}{5}}$

3. Let $f(x) = \sqrt{x}$, $T(x) = \dfrac{1}{2}(x-9) + 3$, and $S(x) = \dfrac{1}{6}(x-9) + 3$.

   (a) Graph f, T, and S together. Note that the graphs of T and S are straight lines passing through the point $(9, 3)$ on the graph of f.

   (b) Let $R_T(h) = f(9+h) - T(9+h)$. Is $R_T(h)$ $o(h)$? Is it $O(h)$?

   (c) Let $R_S(h) = f(9+h) - S(9+h)$. Is $R_S(h)$ $o(h)$? Is it $O(h)$?

   (d) Which of T or S is the best affine approximation to f at 9?

   (e) Use the best affine approximation to f at 9 to approximate $\sqrt{10}$, $\sqrt{8.9}$, and $\sqrt{9.3}$. Compare these approximations with values from your calculator.

4. Let $g(z) = z^2$, $T(z) = 2(z-1) + 1$, and $S(z) = 3(z-1) + 1$.

   (a) Graph g, T, and S together. Note that the graphs of T and S are straight lines passing through the point $(1, 1)$ on the graph of g.

   (b) Let $R_T(h) = g(1+h) - T(1+h)$. Is $R_T(h)$ $o(h)$? Is it $O(h)$?

   (c) Let $R_S(h) = g(1+h) - S(1+h)$. Is $R_S(h)$ $o(h)$? Is it $O(h)$?

   (d) Which of T or S is the best affine approximation to g at 1?

   (e) Use the best affine approximation to g at 1 to approximate $(1.1)^2$ and $(0.999)^2$. Compare these approximations with values from your calculator.

5. Find the best affine approximation to $f(x) = 2x^2$ at 1. What is $f'(1)$?

6. Find the best affine approximation to $g(x) = \dfrac{1}{x}$ at 1. What is $g'(1)$?

7. Find the best affine approximation to $f(t) = t^2 + t - 1$ at 0. What is $f'(0)$?

Difference Equations
to
Differential Equations

Section 3.2

Best Affine Approximations, Derivatives, and Rates of Change

In this section we will take up the general question of how to find best affine approximations and also discuss an interpretation of the derivative of a function as an instantaneous rate of change. We will consider specific computational procedures for finding derivatives in Sections 3.3 through 3.5.

To begin, suppose f is a function defined on an open interval containing the point c and let T be an affine function with $T(c) = f(c)$. As in Section 3.1, we may write T in the form

$$T(x) = m(x - c) + f(c) \tag{3.2.1}$$

for some constant m. Let

$$R(h) = f(c + h) - T(c + h) = f(c + h) - mh - f(c). \tag{3.2.2}$$

Then

$$\begin{aligned} \lim_{h\to 0} \frac{R(h)}{h} &= \lim_{h\to 0} \frac{f(c+h) - T(c+h)}{h} \\ &= \lim_{h\to 0} \frac{f(c+h) - mh - f(c)}{h} \\ &= \lim_{h\to 0} \left(\frac{f(c+h) - f(c)}{h} - m \right). \end{aligned} \tag{3.2.3}$$

Hence $R(h)$ is $o(h)$, and T is the best affine approximation to f at c, if and only if

$$\lim_{h\to 0} \left(\frac{f(c+h) - f(c)}{h} - m \right) = 0, \tag{3.2.4}$$

which is true if and only if

$$\lim_{h\to 0} \frac{f(c+h) - f(c)}{h} = m. \tag{3.2.5}$$

In particular, if

$$\lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$$

exists, then f has a best affine approximation at c and

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}. \tag{3.2.6}$$

Conversely, if $T(x) = m(x-c) + f(c)$ is the best affine approximation to f at c, then it follows that

$$m = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}. \tag{3.2.7}$$

Definition We say a function f is *differentiable* at a point c if

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \tag{3.2.8}$$

exists.

In summary, if we are given a function f which is differentiable at c, then the best affine approximation to f at c exists and is given by

$$T(x) = f'(c)(x-c) + f(c), \tag{3.2.9}$$

where

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}. \tag{3.2.10}$$

Conversely, if f has a best affine approximation at a point c, then f is differentiable at c and the best affine approximation is given by (3.2.9).

Example Consider the problem of finding the best affine approximation to $f(x) = x^2$ at $x = 1$, a problem we first looked at in Section 1.1. We first need to find the derivative $f'(1)$. Using (3.2.10), we have

$$\begin{aligned} f'(1) &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h \to 0} \frac{(1+h)^2 - 1}{h} \\ &= \lim_{h \to 0} \frac{1 + 2h + h^2 - 1}{h} \\ &= \lim_{h \to 0} \frac{h(2+h)}{h} \\ &= \lim_{h \to 0} (2+h) \\ &= 2. \end{aligned}$$

From (3.2.9), it now follows that the best affine approximation to f at 1 is

$$T(x) = 2(x-1) + 1 = 2x - 1.$$

Furthermore, from our discussion in Section 3.1, the equation of the line tangent to the graph of $y = x^2$ at the point $(1, 1)$ is then

$$y = 2x - 1,$$

as shown in Figure 3.2.1.

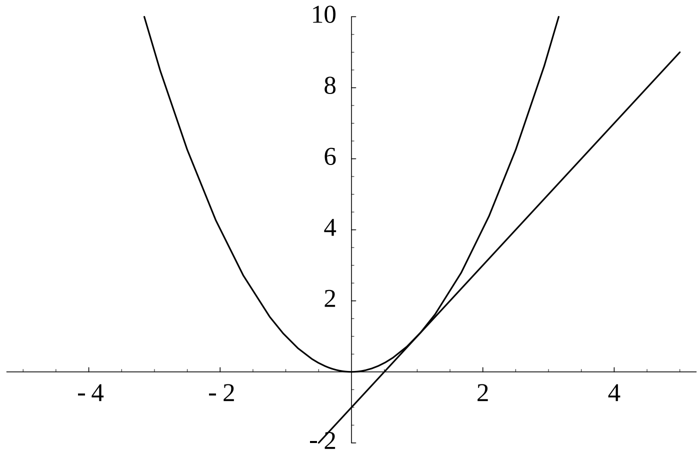


Figure 3.2.1 Graphs of $y = x^2$ and $y = 2x - 1$

Frequently we will be interested in the derivative of a function not just at a single point, but at many different points. Instead of performing the above calculation at each point separately, we try to compute the derivative at an arbitrary point, after which we can substitute in any desired point for evaluation. In fact, for any given function f, we may define a new function f' by setting

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{3.2.11}$$

for all points x at which the limit exists. This new function, f', is called the *derivative* of f. Note that the domain of f' may be smaller than the domain of f. If the open interval (a, b) is in the domain of f', we say f is *differentiable* on (a, b).

Example Let $f(x) = \sqrt{x}$. From our work in the Section 3.1 we know that

$$f'(1) = \frac{1}{2}.$$

Now we will find a general expression for $f'(x)$ at an arbitrary point x in $(0, \infty)$. Using (3.2.11), we have

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \\
&= \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} \\
&= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} \\
&= \frac{1}{\sqrt{x}+\sqrt{x}} \\
&= \frac{1}{2\sqrt{x}}.
\end{aligned}
$$

Hence f is differentiable on $(0,\infty)$. In particular, we once again have

$$f'(1) = \frac{1}{2}.$$

Moreover, it is now straightforward to find the best affine approximation to f at any point $c > 0$. For example,

$$f'(16) = \frac{1}{8},$$

so the best affine approximation to $f(x) = \sqrt{x}$ at $x = 16$ is

$$T(x) = \frac{1}{8}(x-16)+4 = \frac{1}{8}x + 2.$$

See Figure 3.2.2 for the graphs of f and T.

Example Now consider $g(t) = t^3$. Then

$$
\begin{aligned}
g'(t) &= \lim_{h\to 0} \frac{g(t+h)-g(t)}{h} \\
&= \lim_{h\to 0} \frac{(t+h)^3-t^3}{h} \\
&= \lim_{h\to 0} \frac{t^3+3t^2h+3th^2+h^3-t^3}{h} \\
&= \lim_{h\to 0} \frac{h(3t^2+3th+h^2)}{h} \\
&= \lim_{h\to 0}(3t^2+3th+h^2) \\
&= 3t^2.
\end{aligned}
$$

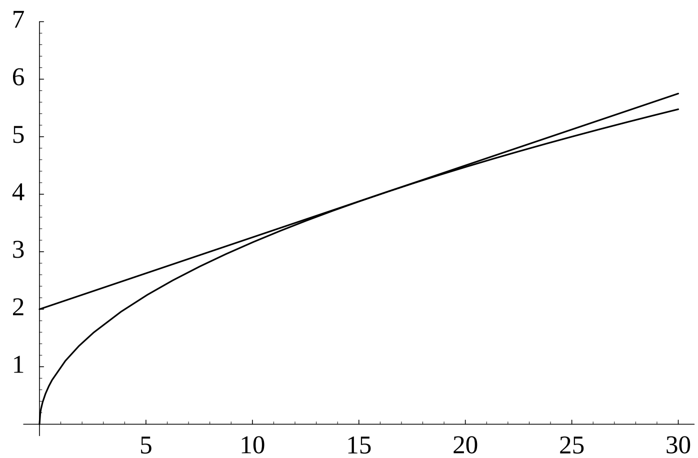


Figure 3.2.2 Graphs of $f(x) = \sqrt{x}$ and $T(x) = \frac{1}{8}x + 2$

Hence, for example, $g'(-2) = 12$, and the best affine approximation to $g(t) = t^3$ at $t = -2$ is

$$T(t) = 12(t + 2) - 8 = 12t + 16.$$

See Figure 3.2.3 for the graphs of g and T.

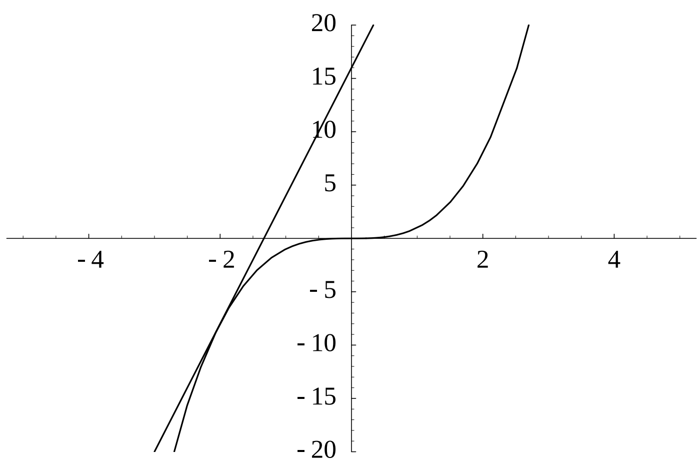


Figure 3.2.3 Graphs of $g(t) = t^3$ and $T(t) = 12t + 16$

Example Suppose we wish to find the best affine approximation to $f(x) = |x|$ at $x = 0$. To find the derivative of f at 0, we need to consider the quotient

$$\frac{f(0+h) - f(0)}{h} = \frac{|h|}{h} = \begin{cases} \dfrac{-h}{h} = -1, & \text{if } h < 0, \\ \dfrac{h}{h} = 1, & \text{if } h > 0. \end{cases}$$

Thus

$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = -1$$

and

$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = 1,$$

from which it follows that

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

does not exist. In other words, f is not differentiable at 0. Thus f does not have a best affine approximation at 0. However, for $x < 0$, the graph of f is a straight line with slope -1 and for $x > 0$, the graph of f is a straight line with slope 1. Thus

$$f'(x) = \begin{cases} -1, & \text{if } x < 0, \\ 1, & \text{if } x > 0. \end{cases}$$

Hence the domain of f' is $\{x \mid x \neq 0\}$, whereas the domain of f is the interval $(-\infty, \infty)$.

The previous example illustrates the fact that a function may be continuous at a point, as $f(x) = |x|$ is continuous at $x = 0$, without being differentiable at that point. However, it turns out that if a function is differentiable at a point, then it must be continuous at that point. To see this, note that if T is the best affine approximation to a function f at c and $r(x) = f(x) - T(x)$ is the remainder function, then

$$f(x) = T(x) + r(x). \tag{3.2.12}$$

Since T is a continuous function, $\lim\limits_{x \to c} r(x) = 0$, and $T(c) = f(c)$, we have

$$\lim_{x \to c} f(x) = \lim_{x \to c}(T(x) + r(x)) = \lim_{x \to c} T(x) + \lim_{x \to c} r(x) = T(c) + 0 = f(c), \tag{3.2.13}$$

which is what it means for f to be continuous at c.

Proposition If f is differentiable at a point c, then f is continuous at c.

Leibniz notation and rates of change

If $y = f(x)$ with $f(x) = mx + b$, then one unit change in x results in m units of change in y. That is, for a straight line, the slope of the line is the rate of change of y with respect to x. Moreover, since f is its own best affine approximation (and a straight line is its own tangent line), we have $f'(x) = m$ for all values of x. Hence, in this case, the derivative of f gives the rate of change of y with respect to x. What distinguishes this type of function from other functions, and what makes the slope easily computable, is that this rate of change is a constant. We will now use derivatives to give meaning to the rate of change of an arbitrary function at a point where it is differentiable.

If $y = f(x)$, it is common to write Δx for an increment in x and Δy for the change in y corresponding to a change in x of Δx. In our notation above, we would write

$$\Delta x = h \tag{3.2.14}$$

and

$$\Delta y = f(x + \Delta x) - f(x). \tag{3.2.15}$$

Thus we can write

$$\frac{f(x+h) - f(x)}{h} = \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{\Delta y}{\Delta x}, \tag{3.2.16}$$

from which we have

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}. \tag{3.2.17}$$

This type of notation, although not this type of reasoning, motivated Leibniz to denote $f'(x)$ by

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}. \tag{3.2.18}$$

If the derivative is to be evaluated at a point c, then we would write

$$f'(c) = \left.\frac{dy}{dx}\right|_{x=c}. \tag{3.2.19}$$

Example If $y = \sqrt{x}$, then from our result above we may write

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

and, for example,

$$\left.\frac{dy}{dx}\right|_{x=9} = \frac{1}{2\sqrt{9}} = \frac{1}{6}.$$

Now $\frac{\Delta y}{\Delta x}$ represents the average rate of change of y over the interval from x to $x + \Delta x$. That is, this ratio tells us how much y changes per unit change in x over the interval. As we let Δx go to 0, this ratio will approach a limiting value, namely, the derivative , which we may interpret as the *instantaneous rate of change* of y with respect to x. If the rate of change of y with respect to x were not to change over an interval of length 1, then y would change by an amount equal to $\frac{dy}{dx}$ over that interval.

As an example, if $s = f(t)$ gives the position of an object moving in a straight line, then $\frac{\Delta s}{\Delta t}$ is the average rate of change of position of the object with respect to time, which we call its *average velocity*. Then $\frac{ds}{dt}$, the derivative of s with respect to t, is the instantaneous rate of change of position with respect to time; that is, $\frac{ds}{dt}$ is the *instantaneous velocity*, or, simply, *velocity*, of the object. The difference between $\frac{\Delta s}{\Delta t}$ and $\frac{ds}{dt}$ is the difference between finding the average speed for a trip in a car by dividing the total miles traveled by the total time elapsed and finding the instantaneous speed at any one time during the trip by looking at the car's speedometer.

Example Galileo discovered that if an object is dropped from a initial height of 100 feet, then, ignoring the effects of air resistance, its height, in feet, above the ground after t seconds would be

$$s = 100 - 16t^2.$$

For example, at time $t = 1$ the object would be at a height of

$$s\Big|_{t=1} = 100 - 16 = 84 \text{ feet}$$

and at time $t = 2$ it would be at a height of

$$s\Big|_{t=2} = 100 - 64 = 36 \text{ feet}.$$

Hence the average velocity of the object over the time interval would be

$$\frac{\Delta s}{\Delta t} = \frac{36 - 84}{2 - 1} = -48 \text{ feet/second}.$$

Note that the average velocity over this time interval is negative because we have taken the positive direction to be up. The average speed of the object, which is the absolute value of the velocity, would be 48 feet per second. To find the instantaneous velocity at time t, we compute

$$\begin{aligned}
\frac{ds}{dt} &= \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{(100 - 16(t + \Delta t)^2) - (100 - 16t^2)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{100 - 16(t^2 + 2t\Delta t + (\Delta t)^2) - 100 - 16t^2}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{-32t\Delta t - 16(\Delta t)^2}{\Delta t} \\
&= \lim_{\Delta t \to 0} (-32t - 16\Delta t) \\
&= -32t.
\end{aligned}$$

Hence the instantaneous velocity of the object at time $t = 1$ is

$$\left.\frac{ds}{dt}\right|_{t=1} = -32 \text{ feet/second}$$

and the instantaneous velocity at time $t = 2$ is

$$\left.\frac{ds}{dt}\right|_{t=2} = -64 \text{ feet/second}.$$

Although Leibniz seems to have thought of the expression $\frac{dy}{dx}$ as a ratio, we should think of $\frac{d}{dx}$ as the operation of differentiation, which, when applied to y, yields the derivative of

y with respect to x. In other words, we should not think of $\frac{dy}{dx}$ as a ratio, but as $\frac{d}{dx}(y)$. For example, if $y = x^3$, then, using an earlier result from this section, we might write

$$\frac{dy}{dx} = \frac{d}{dx}(x^3) = 3x^2.$$

The "prime" notation for a derivative is due, not to Newton, but to Joseph Louis Lagrange (1736-1813). Newton's notation, a dot above the dependent variable, represents a derivative with respect to time, denoted by t. For example, in the previous example we may write

$$\dot{s} = -32t$$

using Newton's notation. Because of its simplicity and the frequency with which derivatives with respect to time occur, this is often a useful notation and we will make extensive use of it when we study differential equations in Chapter 8.

Problems

1. Using (3.2.10), find the derivative of each of the following functions at the indicated point.

   (a) $f(x) = x^2 + 1$ at $x = 2$
   (b) $f(t) = \dfrac{1}{t}$ at $t = 1$
   (c) $g(x) = \dfrac{1}{x^2}$ at $x = 2$
   (d) $h(t) = \sqrt{t+1}$ at $t = 3$
   (e) $f(s) = \dfrac{1}{\sqrt{s}}$ at $s = 1$
   (f) $g(z) = (z+1)^2$ at $z = -1$

2. For each of the functions in Problem 1, find the best affine approximation for the function at the indicated point. Also, find the equation of the tangent line at that point and graph the function and its tangent line together.

3. Using (3.2.11), find the derivative of each of the following functions. Note any points where the given function is not differentiable.

   (a) $f(x) = 2x^2$
   (b) $g(x) = \dfrac{1}{x}$
   (c) $f(t) = \dfrac{1}{\sqrt{t}}$
   (d) $h(z) = \dfrac{1}{3z}$
   (e) $y(t) = t^2 + 4t$
   (f) $g(s) = 2s^3 - s^2$

4. Using your result from part (c) of Problem 3, find the best affine approximation to

$$f(t) = \frac{1}{\sqrt{t}}$$

   at $t = 4$. Use it to approximate $\dfrac{1}{\sqrt{3.98}}$.

5. Let $f(x) = ax^2 + bx + c$, where a, b, and c are constants. Show that $f'(x) = 2ax + b$. Does this result agree with your results in parts (a) and (e) of Problems 3?

6. Use your result from Problem 5 to find the best affine approximation to

$$f(x) = 3x^2 - 2x + 5$$

at $x = -2$.

7. Use your result from Problem 5 to find the best affine approximation to

$$g(t) = -2t^2 + 3t - 6$$

at $t = 3$.

8. Suppose f is a function with the properties that $f(0) = 0$ and

$$\lim_{t \to 0} \frac{f(t)}{t} = 1.$$

Show that $f'(0) = 1$.

9. Suppose g is a function with the properties that $g(0) = 0$ and g is $o(h)$. Show that $g'(0) = 0$.

10. Suppose f is a function with the properties that

$$f(s + t) = f(s)f(t)$$

for all numbers s and t and

$$\lim_{t \to 0} \frac{f(t) - 1}{t} = 1.$$

Show that $f'(t) = f(t)$.

11. Suppose $f(x) = \begin{cases} 3x^2, & \text{if } x < 0 \\ x^3, & \text{if } x \geq 0 \end{cases}$. Is f differentiable at $x = 0$? If it is, find $f'(0)$.

12. Suppose $g(t) = \begin{cases} 5t, & \text{if } t < 0 \\ 3t^2, & \text{if } t \geq 0 \end{cases}$. Is g differentiable at $t = 0$? If it is, find $g'(0)$.

13. Suppose $g(x) = \begin{cases} x^2 - 2x + 2, & \text{if } x \leq 1 \\ 4x - 3, & \text{if } x > 1 \end{cases}$. Is g differentiable at $x = 1$? If it is, find $g'(1)$.

14. For each of the following, find the derivative of the dependent variable with respect to the independent variable. Denote the derivative using Leibniz's notation.

    (a) $s = 2t^3$

    (b) $z = 2\sqrt{t}$

    (c) $q = s - \dfrac{2}{s}$

    (d) $t = x^4$

15. Find $\dfrac{d}{dx}(4x^2)$ and $\dfrac{d}{du}(3\sqrt{u-1})$.

16. An object is thrown vertically into the air from an initial height of 100 meters above the ground with an initial velocity of 10 meters per second. If s represents the height, in meters, of the object above the ground after t seconds and we ignore the effects of air resistance, then
$$s = 100 + 10t - 4.9t^2.$$
   (a) What is the average velocity over the time interval $[0, 2]$? Over $[0, 1]$? Over $[1, 2]$?
   (b) Find the velocity v of the object at time t. You may use Problem 5.
   (c) What is the velocity after 1 second? After 2 seconds?
   (d) When is $v = 0$? Is v positive or negative before this time? Is v positive or negative after this time?
   (e) What is the height of the object when $v = 0$? What is the significance of this height?
   (f) The rate of change of velocity is called *acceleration*. Find the acceleration of the object; that is, find $\frac{dv}{dt}$.
   (g) What is the significance of the fact that $\frac{dv}{dt}$ is a constant?

17. Find the rate of change of the area A of a circle with respect to its radius r.

18. Find the rate of change of the volume V of a sphere with respect to its radius r.

19. Find the rate of change of the area A of a square with respect to the length x of one of its sides.

20. Find the rate of change of the volume V of a cube with respect to the length x of one of its sides.

Difference Equations to **D**ifferential Equations

Section 3.3

Differentiation of Polynomials and Rational Functions

In this section we begin the task of discovering rules for differentiating various classes of functions. By the end of Section 3.5 we will be able to differentiate any algebraic or trigonometric function as a matter of routine without reference to the limits used in Section 3.2.

Differentiation of polynomials

We first note that if f is a first degree polynomial, say, $f(x) = ax + b$ for some constants a and b, then f is an affine function and hence its own best affine approximation. Thus $f'(x) = a$ for all x. In particular, if f is a constant function, say, $f(x) = b$ for all x, then $f'(x) = 0$ for all x.

Next we consider the case of a monomial $f(x) = x^n$, where n is a positive integer greater than 1. Then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}. \tag{3.3.1}$$

Now

$$(x+h)^n = x^n + nx^{n-1}h + R(h), \tag{3.3.2}$$

where $R(h)$ represents the remaining terms in the expansion. Since every term in $R(h)$ has a factor of h raised to a power greater than or equal to 2, it follows that $R(h)$ is $o(h)$. Hence we have

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{x^n + nx^{n-1}h + R(h) - x^n}{h} \\ &= \lim_{h \to 0} \frac{nx^{n-1}h + R(h)}{h} \\ &= \lim_{h \to 0} \left(nx^{n-1} + \frac{R(h)}{h} \right) \\ &= nx^{n-1} + \lim_{h \to 0} \frac{R(h)}{h} \\ &= nx^{n-1}. \end{aligned}$$

Since from our previous result $f'(x) = 1$ when $f(x) = x$, this formula also works in the case $n = 1$. Hence we have the following proposition.

Proposition For any positive integer n,

$$\frac{d}{dx} x^n = nx^{n-1}. \tag{3.3.3}$$

Example If $f(x) = x^3$, then $f'(x) = 3x^2$, as we saw in an example in Section 3.2.

Example Similarly,

$$\frac{d}{dt}t^5 = 5t^4.$$

Hence, for example, the equation of the line tangent to the curve $x = t^5$ at $(-1, -1)$ is

$$x = 5(t+1) - 1,$$

or

$$x = 5t + 4.$$

Once we establish results for the derivative of a constant times a function and for the derivative of the sum of two functions, similar to the results we have for limits, we will be able to easily differentiate any polynomial. So suppose f is a differentiable function and let $k(x) = cf(x)$, where c is any constant. Then

$$\begin{aligned} k'(x) &= \lim_{h \to 0} \frac{k(x+h) - k(x)}{h} \\ &= \lim_{h \to 0} \frac{cf(x+h) - cf(x)}{h} \\ &= \lim_{h \to 0} \frac{c(f(x+h) - f(x))}{h} \\ &= c \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= cf'(x). \end{aligned}$$

That is, the derivative of a constant times a function is the constant times the derivative of the function.

Proposition If f is differentiable and c is any constant, then

$$\frac{d}{dx}(cf(x)) = c\frac{d}{dx}f(x). \tag{3.3.4}$$

Example If $f(x) = 14x^3$, then

$$f'(x) = (14)(3x^2) = 42x^2.$$

Now suppose f and g are both differentiable functions and let $k(x) = f(x) + g(x)$. Then

$$\begin{aligned} k'(x) &= \lim_{h \to 0} \frac{k(x+h) - k(x)}{h} \\ &= \lim_{h \to 0} \frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h} \\ &= \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right) \\ &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\ &= f'(x) + g'(x). \end{aligned}$$

Hence the derivative of the sum of two functions is the sum of their derivatives. A similar argument would show that the derivative of the difference of two functions is the difference of their derivatives.

Proposition If f and g are both differentiable, then

$$\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x) \tag{3.3.5}$$

and

$$\frac{d}{dx}(f(x)-g(x)) = \frac{d}{dx}f(x) - \frac{d}{dx}g(x). \tag{3.3.6}$$

Putting the preceding results together, we are now in a position to easily differentiate any polynomial, as the next examples will illustrate.

Example Suppose $f(x) = 3x^5 - 6x^2 + 2x - 16$. Then

$$\begin{aligned} f'(x) &= \frac{d}{dx}(3x^5 - 6x^2 + 2x - 16) \\ &= \frac{d}{dx}(3x^5) - \frac{d}{dx}(6x^2) + \frac{d}{dx}(2x) - \frac{d}{dx}(16) \\ &= 3\frac{d}{dx}x^5 - 6\frac{d}{dx}x^2 + 2\frac{d}{dx}x - 0 \\ &= (3)(5x^4) - (6)(2x) + (2)(1) \\ &= 15x^4 - 12x + 2. \end{aligned}$$

Example Of course, it is not necessary to write out in detail all the steps in differentiating a polynomial as we did in the preceding example. For example, if $g(t) = 3t^{12} - 6t^2 + t$, then

$$g'(t) = (3)(12t^{11}) - (6)(2t) + 1 = 36t^{11} - 12t + 1.$$

In particular, since $g(1) = -2$ and $g'(1) = 25$, the best affine approximation to g at $t = 1$ is

$$T(t) = 25(t-1) - 2 = 25t - 27.$$

Differentiation of rational functions

We next consider the problem of differentiating the quotient of two functions whose derivatives are already known. In particular, combining this result with our result for polynomials will enable us to easily differentiate any rational function. We might hope that, analogous to the last two results and the related results for limits, the derivative of the quotient of two functions would be equal to the quotient of their derivatives. This turns out not to be true; nevertheless, there is a nice rule for differentiating quotients.

Suppose f and g are both differentiable functions and let $k(x) = \frac{f(x)}{g(x)}$. Then, at all points where $g(x) \neq 0$,

$$\begin{aligned}
k'(x) &= \lim_{h\to 0} \frac{k(x+h) - k(x)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{f(x+h)}{g(x+h)} - \dfrac{f(x)}{g(x)}}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{g(x)f(x+h) - g(x+h)f(x)}{g(x+h)g(x)}}{h} \\
&= \lim_{h\to 0} \frac{g(x)f(x+h) - g(x+h)f(x)}{hg(x)g(x+h)}.
\end{aligned}$$

It turns out that by adding and subtracting the term $g(x)f(x)$ (a standard mathematical trick of adding 0) in the numerator, we can simplify this limit into a form that we can evaluate. That is,

$$\begin{aligned}
k'(x) &= \lim_{h\to 0} \frac{g(x)f(x+h) - g(x+h)f(x)}{hg(x)g(x+h)} \\
&= \lim_{h\to 0} \frac{g(x)f(x+h) - g(x)f(x) + g(x)f(x) - g(x+h)f(x)}{hg(x)g(x+h)} \\
&= \lim_{h\to 0} \frac{g(x)(f(x+h) - f(x)) - f(x)(g(x+h) - g(x))}{hg(x)g(x+h)} \\
&= \lim_{h\to 0} \frac{g(x)\left(\dfrac{f(x+h) - f(x)}{h}\right) - f(x)\left(\dfrac{g(x+h) - g(x)}{h}\right)}{g(x)g(x+h)}.
\end{aligned}$$

Now

$$\lim_{h\to 0} g(x)\left(\frac{f(x+h) - f(x)}{h}\right) = g(x)\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = g(x)f'(x), \tag{3.3.7}$$

$$\lim_{h\to 0} f(x)\left(\frac{g(x+h) - g(x)}{h}\right) = f(x)\lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = f(x)g'(x), \tag{3.3.8}$$

and

$$\lim_{h\to 0} g(x)g(x+h) = g(x)\lim_{h\to 0} g(x+h) = g(x)g(x) = (g(x))^2, \tag{3.3.9}$$

where the limits in (3.3.7) and (3.3.8) follow from the differentiability of f and g, while the limit in (3.3.9) follows from the continuity of g (which is a consequence of the differentiability of g). Putting everything together, we have

$$k'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}, \tag{3.3.10}$$

a result known as the *quotient rule.*

Quotient Rule If f and g are both differentiable, then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{(g(x))^2} \tag{3.3.11}$$

at all points where $g(x) \neq 0$.

Example Suppose $f(x) = \dfrac{2x+1}{x-2}$. Then

$$\begin{aligned} f'(x) &= \frac{(x-2)\frac{d}{dx}(2x+1) - (2x+1)\frac{d}{dx}(x-2)}{(x-2)^2} \\ &= \frac{(x-2)(2) - (2x+1)(1)}{(x-2)^2} \\ &= \frac{2x-4-2x-1}{(x-2)^2} \\ &= -\frac{5}{(x-2)^2}. \end{aligned}$$

Hence, for example, $f(3) = 7$ and $f'(3) = -5$, so the equation of the line tangent to the graph of f at $(3, 7)$ is

$$y = -5(x-3) + 7,$$

or

$$y = -5x + 22.$$

Example Suppose $g(z) = \dfrac{1}{z^2}$. Then

$$g'(z) = \frac{z^2\frac{d}{dz}(1) - (1)\frac{d}{dz}(z^2)}{z^4} = \frac{(z^2)(0) - 2z}{z^4} = -\frac{2}{z^3}.$$

Note that we may write this result in the form

$$\frac{d}{dz}z^{-2} = -2z^{-3},$$

which is consistent with our previous result

$$\frac{d}{dz}z^n = nz^{n-1}.$$

However, we derived the latter under the assumption that n was a positive integer. We will now show that we can extend this result to the case of negative integer exponents.

Suppose $f(x) = x^n$, where n is a negative integer. Then, using the quotient rule and the fact that $-n > 0$,

$$\begin{aligned} f'(x) &= \frac{d}{dx}x^n \\ &= \frac{d}{dx}\left(\frac{1}{x^{-n}}\right) \\ &= \frac{x^{-n}\frac{d}{dx}(1) - (1)\frac{d}{dx}(x^{-n})}{x^{-2n}} \\ &= \frac{(x^{-n})(0) - (-nx^{-n-1})}{x^{-2n}} \\ &= \frac{nx^{-n-1}}{x^{-2n}} \\ &= nx^{-n-1+2n} \\ &= nx^{n-1}. \end{aligned}$$

We can now state the more general result.

Proposition For any integer $n \neq 0$,

$$\frac{d}{dx}x^n = nx^{n-1}. \tag{3.3.12}$$

Example If $f(x) = \frac{1}{x}$, then

$$f'(x) = \frac{d}{dx}x^{-1} = -x^{-2} = -\frac{1}{x^2}.$$

Example Similarly,

$$\frac{d}{dx}\left(\frac{5}{x^3}\right) = \frac{d}{dx}(5x^{-3}) = -15x^{-4} = -\frac{15}{x^4}.$$

We will eventually see that (3.3.12) holds for rational and irrational exponents as well. We will consider the rational case in Section 3.4, but we will not have the tools for handling the irrational case until we discuss exponential and logarithm functions in Chapter 6.

Differentiation of products

We will close this section with a discussion of a rule for differentiating the product of two functions. Since the product of two rational functions is again a rational function, this will not extend the class of functions that we know how to differentiate routinely. However, this rule will be very useful in the future and, even at the present point, can help simplify some problems.

Suppose f and g are both differentiable and $k(x) = f(x)g(x)$. Then

$$k'(x) = \lim_{h \to 0} \frac{k(x+h) - k(x)}{h} = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}. \tag{3.3.13}$$

Adding and subtracting $f(x+h)g(x)$ in the numerator (again, the mathematical trick of adding 0 in a useful manner) will help simplify this limit. Namely,

$$\begin{aligned} k'(x) &= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\ &= \lim_{h \to 0} \frac{f(x+h)(g(x+h) - g(x)) + g(x)(f(x+h) - f(x))}{h} \\ &= \lim_{h \to 0} \left(f(x+h) \left(\frac{g(x+h) - g(x)}{h} \right) + g(x) \left(\frac{f(x+h) - f(x)}{h} \right) \right). \end{aligned}$$

Now

$$\lim_{h \to 0} f(x+h) \left(\frac{g(x+h) - g(x)}{h} \right) = \lim_{h \to 0} f(x+h) \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = f(x)g'(x) \tag{3.3.14}$$

and

$$\lim_{h \to 0} g(x) \left(\frac{f(x+h) - f(x)}{h} \right) = g(x) \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = g(x)f'(x), \tag{3.3.15}$$

where, as with the derivation of the quotient rule, we have used the differentiability of f and g as well as the continuity of f in evaluating the limits. Putting everything together, we have

$$k'(x) = f(x)g'(x) + g(x)f'(x). \tag{3.3.16}$$

a result known as the *product rule.*

Product Rule If f and g are both differentiable, then

$$\frac{d}{dx} f(x)g(x) = f(x) \frac{d}{dx} g(x) + g(x) \frac{d}{dx} f(x). \tag{3.3.17}$$

Example If

$$f(x) = (x^4 - 3x^2 + 6x - 3)(6x^3 + 2x + 5),$$

then

$$\begin{aligned} f'(x) &= (x^4 - 3x^2 + 6x - 3) \frac{d}{dx}(6x^3 + 2x + 5) + (6x^3 + 2x + 5) \frac{d}{dx}(x^4 - 3x^2 + 6x - 3) \\ &= (x^4 - 3x^2 + 6x - 3)(18x^2 + 2) + (6x^3 + 2x + 5)(4x^3 - 6x + 6). \end{aligned}$$

Of course, in this example, f is just a polynomial so we could also find f' by multiplying out the two factors of f and differentiating the polynomial term by term as usual. However, the product rule gives us a quicker route to the derivative. Although the result is not simplified into the standard form of a polynomial, for most applications this form is just as useful as any other.

It is worth noting that although we can now differentiate any rational function in theory, in practice our methods may not be the most useful. For example, the function

$$f(x) = (x^2+1)^{567}$$

is a polynomial, and so we know how to differentiate it. However, at this point the only way we could perform the differentiation would be to expand $f(x)$ into standard polynomial form and then differentiate term by term. In Section 3.4 we will learn how to handle this problem more directly. At the same time we will extend the class of functions that we can differentiate routinely to include all algebraic functions.

Problems

1. Find the derivative of each of the following functions.

   (a) $f(x) = x^3 + 6x$
   (b) $g(x) = 13x^5 - 6x^2 + 13$
   (c) $g(t) = 3t - 6t^2$
   (d) $y(t) = 4t^3 - 18t + 3$
   (e) $f(t) = (3t-6)^2$
   (f) $f(x) = (4x+5)(6x^2-1)$

2. Find the derivative of each of the following functions.

   (a) $f(x) = (2x+1)^2$
   (b) $g(t) = (t^2-3)^3$
   (c) $g(x) = \dfrac{x-3}{2x+5}$
   (d) $h(s) = \dfrac{2s-s^2}{s^2+1}$
   (e) $f(t) = \dfrac{3t^4-8t+1}{2t^3+6}$
   (f) $x(t) = \dfrac{3}{t^3} - 16t^2$
   (g) $h(t) = \dfrac{3}{t}$
   (h) $f(x) = \dfrac{41}{3x^7}$
   (i) $h(z) = 8z^3 - \dfrac{1}{2z}$
   (j) $f(s) = \dfrac{31}{s^3} + \dfrac{1}{2s^2} - 16s$

3. For each of the following, make use of the product rule in finding the derivative of the dependent variable with respect to the independent variable.

   (a) $s = (t^2-6t+3)(8t^4+6t^2-7)$
   (b) $q = (13t^4+5t)(3t^5+4t^3+16t-31)$
   (c) $y = (x^2-2x+3)(2x^2+13x-6)(3x^2-4x+1)$
   (d) $z = \dfrac{(x^2-3x+6)(8x^2+3x-2)}{x^2-6}$

4. Suppose $f(2) = -2$, $f'(2) = 6$, $g(2) = 3$, and $g'(2) = -4$. Find $k'(2)$ for each of the following.

   (a) $k(x) = f(x)g(x)$

   (b) $k(x) = \dfrac{f(x)}{g(x)}$

   (c) $k(x) = f(x)(g(x))^2$

   (d) $k(x) = \dfrac{f(x) - f(x)g(x)}{g(x)}$

5. Suppose an object moves along the x-axis so that its position at time t is $x = -t + \dfrac{t^3}{6}$.

   (a) Find the velocity, $v(t) = \dot{x}(t)$, of the object.

   (b) What is $v(0)$? What does this say about the direction of motion of the object at time $t = 0$?

   (c) When is the object at the origin? What is the velocity of the object when it is at the origin?

   (d) For what values of t is the object moving toward the right?

   (e) For what values of t is the object moving toward the left?

   (f) What is happening at the points where $v(t) = 0$?

   (g) Find the acceleration of the object, $a(t) = \dot{v}(t)$.

   (h) When is the acceleration positive? When is it negative?

   (i) Notice that $v(1) < 0$ and $a(1) > 0$. What does this say about the motion at time $t = 1$?

6. (a) Using only the product rule and the fact that $\dfrac{d}{dx}x = 1$, show that $\dfrac{d}{dx}x^2 = 2x$.

   (b) Now use the product rule to show that $\dfrac{d}{dx}x^3 = 3x^2$.

   (c) Let $n > 1$ and suppose we know that

   $$\frac{d}{dx}x^m = mx^{m-1}$$

   for all $m < n$. Use the product rule to show that

   $$\frac{d}{dx}x^n = nx^{n-1}.$$

Difference Equations to Differential Equations

Section 3.4

Differentiation of Compositions of Functions

In this section we will consider the relationship between the derivative of the composition of two functions and the derivatives of the individual functions being composed. We shall see that the resulting differentiation rule, known as the *chain rule*, will be useful in a variety of situations in our later work. The following example will set the stage.

Example Consider a spherical balloon which is being inflated so that its radius is increasing at a rate of 2 centimeters per second. If we let r denote the radius of the balloon in centimeters, t denote time in seconds, and V denote the volume of the balloon in cubic centimeters, then we know that $r = 2t$ and

$$V = \frac{4}{3}\pi r^3.$$

Moreover, we can see that, as a function of t,

$$V = \frac{4}{3}\pi(2t)^3 = \frac{32}{3}\pi t^3.$$

At time $t = 5$, the rate of change of the radius with respect to time is

$$\left.\frac{dr}{dt}\right|_{t=5} = 2 \text{ centimeters per second,}$$

the rate of change of the volume with respect to the radius is

$$\left.\frac{dV}{dr}\right|_{r=10} = \left.4\pi r^2\right|_{r=10} = 400\pi \text{ centimeters per centimeter,}$$

and the rate of change of the volume with respect to time is

$$\left.\frac{dV}{dt}\right|_{t=5} = \left.32\pi t^2\right|_{t=5} = 800\pi \text{ cubic centimeters per second,}$$

where $\frac{dV}{dr}$ is evaluated at $r = 10$ since this is the value of r when $t = 5$. It follows that

$$\left.\frac{dV}{dt}\right|_{t=5} = \left.\frac{dV}{dr}\right|_{r=10}\left.\frac{dr}{dt}\right|_{t=5}.$$

That is, the overall rate of change of V with respect to t is the product of the rate of change of V with respect to r and the rate of change of r with respect to t. This is an example of the chain rule. Viewed in this manner, the chain rule is saying that if V changes 400π times as fast as r and r changes 2 times as fast as t, then V changes $(400\pi)(2) = 800\pi$ times as fast as t.

Another interesting special case of the chain rule arises with the composition of two affine functions. Specifically, if $f(x) = ax + b$ and $g(x) = cx + d$, where a, b, c, and d are all constants, then

$$f \circ g(x) = f(g(x)) = f(cx + d) = a(cx + d) + b = acx + ad + b.$$

Thus the slope of graph of $f \circ g$ is ac, the product of the slopes of the graphs of f and g. In terms of derivatives, this says that

$$(f \circ g)'(x) = ac = f'(g(x))g'(x).$$

The chain rule says this relationship holds for all differentiable functions.

For the general case, suppose g is differentiable at a point c and f is differentiable at $g(c)$. We wish to compute the value of the derivative of $f \circ g$ at c. We have

$$(f \circ g)'(c) = \lim_{h \to 0} \frac{f \circ g(c+h) - f \circ g(c)}{h} = \lim_{h \to 0} \frac{f(g(c+h)) - f(g(c))}{h}. \tag{3.4.1}$$

As with our demonstrations of the quotient and product rules, we need to manipulate (3.4.1) into a form which allows us to evaluate the limit in terms of what we already know. The trick that works this time is to multiply and divide by $g(c+h) - g(c)$. However, we must be aware of one possible complication: In order to divide by $g(c+h) - g(c)$ we must be assured that $g(c+h) - g(c) \neq 0$ for all h in some interval about 0. We will assume that this is the case. If in fact this were not the case, then one can show that both $(f \circ g)'(c) = 0$ and $g'(c) = 0$, giving us the desired result that

$$(f \circ g)'(c) = f'(g(c))g'(c).$$

With our assumption, we have

$$(f \circ g)'(c) = \lim_{h \to 0} \left(\frac{f(g(c+h)) - f(g(c))}{g(c+h) - g(c)} \right) \left(\frac{g(c+h) - g(c)}{h} \right). \tag{3.4.2}$$

Since g is differentiable at c, we have

$$\lim_{h \to 0} \frac{g(c+h) - g(c)}{h} = g'(c). \tag{3.4.3}$$

Since f is differentiable at $g(c)$, if we let $s = g(c+h) - g(c)$, then

$$\lim_{h \to 0} \frac{f(g(c+h)) - f(g(c))}{g(c+h) - g(c)} = \lim_{s \to 0} \frac{f(g(c)+s) - f(g(c))}{s} = f'(g(c)), \tag{3.4.4}$$

where we have used the continuity of g at c to ascertain that s goes to 0 as h goes to 0. Putting (3.4.2), (3.4.3) and (3.4.4) together, we now have

$$(f \circ g)'(c) = f'(g(c))g'(c), \tag{3.4.5}$$

which is our desired result.

Chain Rule If f and g are differentiable, then

$$(f \circ g)'(x) = f'(g(x))g'(x). \tag{3.4.6}$$

Example Suppose $h(x) = (1 + x^2)^{10}$. Then $h(x) = f \circ g(x)$ where $g(x) = 1 + x^2$ and $f(x) = x^{10}$. Now

$$g'(x) = 2x$$

and

$$f'(x) = 10x^9,$$

so

$$h'(x) = (f \circ g)'(x) = f'(g(x))g'(x) = f'(1 + x^2)(2x) = 10(1 + x^2)^9(2x) = 20x(1 + x^2)^9.$$

Note that the preceding example is a particular case of the following general example. If g is a differentiable function, $n \neq 0$ is an integer, and $h(x) = (g(x))^n$, then $h(x) = f \circ g(x)$ where $f(x) = x^n$. Then we have

$$f'(x) = nx^{n-1},$$

and so

$$h'(x) = (f \circ g)'(x) = f'(g(x))g'(x) = n(g(x))^{n-1}g'(x).$$

That is,

$$\frac{d}{dx}(g(x)^n) = n(g(x))^{n-1}g'(x). \tag{3.4.7}$$

Example To illustrate the previous comments,

$$\frac{d}{dx}(3x - 2)^6 = 6(3x - 2)^5\frac{d}{dx}(3x - 2) = 6(3x - 2)^5(3) = 18(3x - 2)^5.$$

Example For another illustration, if

$$f(x) = \frac{3}{(x^3 + 4)^5},$$

then

$$f'(x) = (-5)(3)(x^3 + 4)^{-6}\frac{d}{dx}(x^3 + 4) = -15(x^3 + 4)^{-6}(3x^2) = -\frac{45x^2}{(x^3 + 4)^6}.$$

If we translate the chain rule into the notation of Leibniz, we obtain a formulation like that of the first example. Specifically, if we let $y = f(x)$ and $x = g(t)$, then

$$\left.\frac{dy}{dt}\right|_{t=c} = (f \circ g)'(c) = f'(g(c))g'(c) = \left.\frac{dy}{dx}\right|_{x=g(c)} \left.\frac{dx}{dt}\right|_{t=c}. \tag{3.4.8}$$

For short, we write

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}. \tag{3.4.9}$$

This formula is easy to remember, but at the same time care must be taken to remember that if we want to evaluate $\frac{dy}{dt}$ at $t = c$, then we must evaluate $\frac{dy}{dx}$ at $x = g(c)$.

Example Suppose that for a certain city, when the population of the city is p, the total amount of waste deposited in the city landfill every day is given by $W = 5\sqrt{p}$ pounds per day. Moreover, suppose that the population of the city is growing so that t years from now the population will be

$$p = 100,000(1 + 0.04t + 0.008t^2).$$

To find the rate of change of W with respect to t five years from now, we note that $p = 140,000$ when $t = 5$ and then compute

$$\left.\frac{dW}{dp}\right|_{p=140,000} = \left.\frac{5}{2\sqrt{p}}\right|_{p=140,000} = \frac{5}{2\sqrt{140,000}}$$

and

$$\left.\frac{dp}{dt}\right|_{t=5} = 100,000(0.04 + 0.016t)\Big|_{t=5} = 12,000.$$

Hence the rate of increase of the number of pounds of waste in the landfill after five years is, in pounds per day per year,

$$\left.\frac{dW}{dt}\right|_{t=5} = \left.\frac{dW}{dp}\right|_{p=140,000} \left.\frac{dp}{dt}\right|_{t=5} = \left(\frac{5}{2\sqrt{140,000}}\right)(12,000) = \frac{30,000}{\sqrt{140,000}} = 80.12,$$

where the final answer is rounded to 2 decimal places.

Differentiation of algebraic functions

At this point the only thing keeping us from routinely differentiating any algebraic function is that we do not have a rule for handling exponents which are rational numbers, but not integers. We now consider this problem. Suppose $y = x^n$, where $n = \frac{p}{q}$ for nonzero integers p and q. Then

$$y^q = \left(x^{\frac{p}{q}}\right)^q = x^p. \tag{3.4.10}$$

Differentiating the left-hand side of (3.4.9) with respect to x gives us

$$\frac{d}{dx}y^q = qy^{q-1}\frac{dy}{dx}, \tag{3.4.11}$$

where the factor $\frac{dy}{dx}$ is a consequence of the special case of the chain rule in (3.4.7). Of course,

$$\frac{d}{dx}x^p = px^{p-1}. \tag{3.4.12}$$

We may equate (3.4.11) and (3.4.12) (by (3.4.10) they are the derivatives of equal functions) to obtain

$$qy^{q-1}\frac{dy}{dx} = px^{p-1}. \tag{3.4.13}$$

Solving for $\frac{dy}{dx}$, we have

$$\frac{dy}{dx} = \frac{px^{p-1}}{qy^{q-1}} = \frac{p}{q}x^{p-1}y^{1-q}. \tag{3.4.14}$$

Recalling that $y = x^{\frac{p}{q}}$ and $n = \frac{p}{q}$, (3.4.14) becomes

$$\frac{dy}{dx} = \frac{p}{q}x^{p-1}\left(x^{\frac{p}{q}}\right)^{1-q} = \frac{p}{q}x^{p-1}x^{\frac{p}{q}-p} = \frac{p}{q}x^{\frac{p}{q}-1} = nx^{n-1}. \tag{3.4.15}$$

Hence we may now state the following proposition as an extension of our previous results.

Proposition If $n \neq 0$ is a rational number, then

$$\frac{d}{dx}x^n = nx^{n-1}. \tag{3.4.16}$$

Example We have

$$\frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{\frac{1}{2}} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}},$$

in agreement with our result in Section 3.2.

Example If

$$f(x) = \frac{3}{\sqrt{x^2+1}},$$

then

$$f'(x) = \frac{d}{dx}3(x^2+1)^{-\frac{1}{2}} = -\frac{3}{2}(x^2+1)^{-\frac{3}{2}}(2x) = -\frac{3x}{(x^2+1)^{\frac{3}{2}}}.$$

Implicit differentiation

The technique used in the demonstration of the last proposition is of general use. Any equation involving two variables, such as $f(x, y) = 0$, determines a curve in the plane consisting of the set of all ordered pairs (x, y) which satisfy the equation. Such a curve need not be the graph of a function. For example, the curve associated with $x^2+y^2-25 = 0$, or, more simply, $x^2 + y^2 = 25$, is a circle of radius 5 centered at the origin, which is not the graph of any function. However, for a specified point on the curve, it may be the case that a segment of the curve containing that point is the graph of some function; hence the

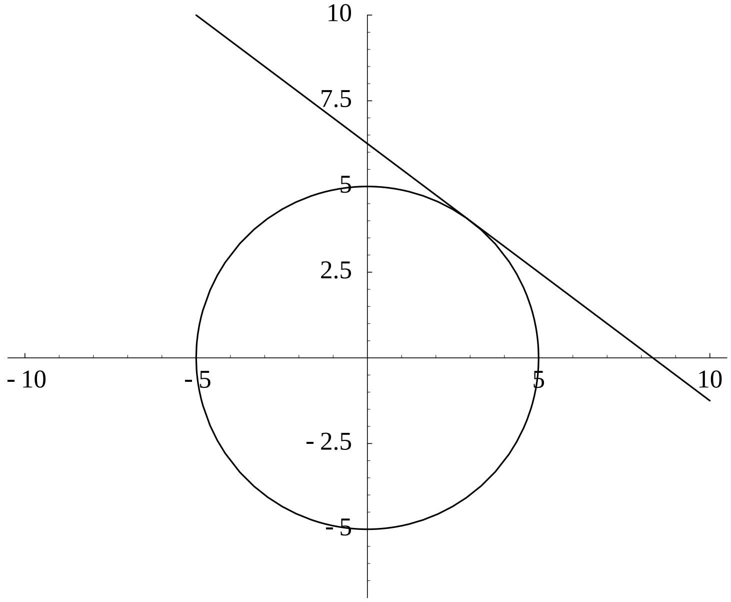


Figure 3.4.1 Tangent line to the circle $x^2 + y^2 = 25$ at $(3, 4)$

curve may have a tangent line at this point. For example, $(3, 4)$ is a point on the curve $x^2 + y^2 = 25$ which lies on the half of the circle lying above the x-axis and, considered by itself, this piece of the circle is the graph of a function, namely, the function $y = \sqrt{25 - x^2}$. To find the slope of the tangent line at such a point on the curve, we may borrow the technique we used in demonstrating the previous proposition. That is, we differentiate both sides of the equation, treating one variable as a function of the other. If we treat y as a function of x, then, differentiating with respect to x and using the chain rule, we obtain an equation involving $\frac{dy}{dx}$ which we can then solve for $\frac{dy}{dx}$. For the equation $x^2 + y^2 = 25$, we have

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}25.$$

Since

$$\frac{d}{dx}(x^2 + y^2) = 2x + 2y\frac{dy}{dx}$$

and

$$\frac{d}{dx}25 = 0,$$

we have

$$2x + 2y\frac{dy}{dx} = 0.$$

Solving for $\frac{dy}{dx}$, we have

$$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$$

at all points (x, y) for which $y \neq 0$. Now we have

$$\left.\frac{dy}{dx}\right|_{(x,y)=(3,4)} = -\frac{3}{4},$$

and so the equation of the tangent line at $(3,4)$ is

$$y = -\frac{3}{4}(x-3)+4.$$

The circle with equation $x^2+y^2=25$ and the tangent line at $(3,4)$ are shown in Figure 3.4.1. Note that our procedure would not work to find the tangent lines to the circle at $(-5,0)$ and $(5,0)$. However, the tangents lines at these points are vertical, and, hence, do not have a slope. Although it is beyond the scope of this book to provide a justification, it is in fact the case that the technique outlined in this example will work to find the slope of the tangent line at all points on the curve that have a tangent line with a slope.

This technique for finding derivatives is called *implicit differentiation* because we did not use an explicit formula for y in terms x. In this case we could have obtained the same result by first solving for y in terms of x for values close to $(3,4)$, giving us $y=\sqrt{25-x^2}$, and then evaluating the derivative of this function at $x=3$. However, this is not always possible or desirable; in many cases implicit differentiation is significantly simpler even if an explicit solution is possible.

Example Consider the problem of finding the best affine approximation to the curve with equation

$$y^3+3xy^2-xy+x=7$$

near the point $(2,1)$. To find $\frac{dy}{dx}$, we compute

$$\frac{d}{dx}(y^3+3xy^2-xy+x)=\frac{d}{dx}7,$$

which give us

$$\frac{d}{dx}y^3+3x\frac{d}{dx}y^2+3y^2\frac{d}{dx}x-\left(x\frac{dy}{dx}+y\frac{d}{dx}x\right)+\frac{d}{dx}x=0.$$

Computing the derivatives on the left-hand side gives us

$$3y^2\frac{dy}{dx}+3x\left(2y\frac{dy}{dx}\right)+3y^2(1)-x\frac{dy}{dx}-y(1)+1=0.$$

Hence

$$3y^2\frac{dy}{dx}+6xy\frac{dy}{dx}+3y^2-x\frac{dy}{dx}-y+1=0,$$

from which it follows that

$$\frac{dy}{dx}(3y^2+6xy-x)=y-3y^2-1.$$

Solving for $\frac{dy}{dx}$, we have

$$\frac{dy}{dx}=\frac{y-3y^2-1}{3y^2+6xy-x},$$

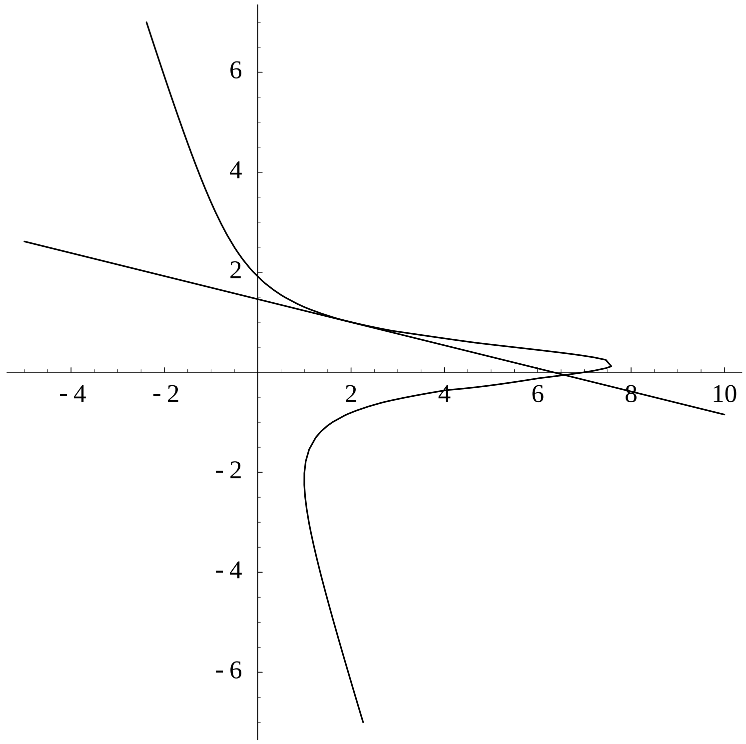


Figure 3.4.2 Curve with equation $y^3 + 3xy^2 - xy + x = 7$ and tangent line at $(2, 1)$

which holds at all points for which the denominator is not 0. Thus

$$\left.\frac{dy}{dx}\right|_{(x,y)=(2,1)} = \frac{1 - 3 - 1}{3 + 12 - 2} = -\frac{3}{13}.$$

So the best affine approximation at $(2, 1)$ is given by

$$T(x) = -\frac{3}{13}(x - 2) + 1.$$

The equation in this example does not specify y as a function of x (in fact, in Figure 3.4.2 we can see that there are at least two other values of y that correspond to $x = 2$), but there is a segment of the curve through $(2, 1)$ which is the graph of some function. For this function, which we have not explicitly found, T is the best affine approximation at $x = 2$. For example, if we denote this unknown function by h, we know that

$$h(2.05) \approx T(2.05) = -\frac{3}{13}(0.05) + 1 = 0.9885,$$

where we have rounded the result to four decimal places. Put another way, the point $(2.05, 0.9885)$ is an approximate solution to the equation

$$y^3 + 3xy^2 - xy + x = 7.$$

At this point we can routinely find the derivative of any algebraic function. In the next section we will consider the derivatives of the trigonometric functions.

Problems

1. Find the derivative of each of the following functions.

   (a) $f(x) = (4x+5)^4$

   (b) $g(x) = 13x(x^2+2)^5$

   (c) $h(t) = \dfrac{3}{2(6t-2)^2}$

   (d) $f(s) = \dfrac{3s-4}{(s^3+2)^4}$

   (e) $g(z) = (3z+4)^3(2z^2+z)^2$

   (f) $f(x) = \dfrac{(3x+4)^3(8x-13)^4}{(2x+3)}$

2. For each of the following, find the derivative of the dependent variable with respect to the independent variable.

   (a) $s = 4t^2(t^2-1)^2$

   (b) $z = -\dfrac{s^2(4s-3)^2}{s^2+1}$

   (c) $q = \sqrt{3t^3-4t}$

   (d) $y = \dfrac{3x}{\sqrt{3x+4}}$

   (e) $x = 8t(4t+5)^{-2}$

   (f) $u = 3(v^2+4)^{-\frac{2}{3}}$

   (g) $y = (3x-1)^{\frac{1}{5}}$

   (h) $v = \sqrt{u^2+(3u-2)^2}$

3. Find the best affine approximation to the function

$$f(x) = \frac{3x}{(x^2+1)^2}$$

   at $x = 2$.

4. (a) Find the best affine approximation to $f(x) = (1+x)^h$ at $x = 0$, where $h \neq 0$ is a constant.

   (b) Use your result from (a) to approximate $\sqrt{1.06}$ and compare with the value obtained from a calculator.

   (c) Use your result from (a) to approximate $\sqrt[3]{1.06}$ and compare with the value obtained from a calculator.

   (d) Use your result from (a) to approximate $\sqrt[5]{1.06}$ and compare with the value obtained from a calculator.

5. Find the equation of the line tangent to each of the following curves at the indicated point.

   (a) $x^2+3y^2 = 21$ at $(3,2)$

   (b) $x^2-3y^2 = 4$ at $(4,2)$

   (c) $x^2+3xy+y^2 = 11$ at $(2,1)$

   (d) $y^5+2x^2y^2-x^2 = 10$ at $(3,1)$

   (e) $x^5+xy+y^5 = 3$ at $(1,1)$

   (f) $4x^2-3xy-2xy^2 = 26$ at $(-2,1)$

6. Suppose values for $f(x)$, $f'(x)$, $g(x)$, and $g'(x)$ are as given in the following table.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	1	2	2	3
1	2	-1	0	2
2	0	3	1	-2

Find $k'(0)$ for each of the following.

(a) $k(x) = f \circ g(x)$
(b) $k(x) = g \circ f(x)$
(c) $k(x) = g \circ g(x)$
(d) $k(x) = f \circ f(x)$
(e) $k(x) = (f \circ g) \circ f(x)$
(f) $k(x) = g(f(x))f(x)$

7. Show that if $g'(c) = 0$, then $(g \circ g)'(c) = 0$.

8. Suppose the sides of cube are increasing at a rate of 3 centimeters per minute. At what rate is the volume of the cube increasing when the length of one of the sides is 10 centimeters?

9. A pebble is dropped in a pond of water. Suppose that the resulting circular wave has a radius given by $r = 20\sqrt{t}$ centimeters after t seconds. Find the rate of change of the area of the wave with respect to time after 5 seconds.

10. The volume of a balloon is increasing at a rate of 50 cubic centimeters per second. At what rate is the radius increasing when the radius is 10 centimeters?

11. The kinetic energy of an object moving in a straight line is given by

$$K = \frac{1}{2}mv^2,$$

where m is the mass of the object and v is its velocity. If the acceleration of the object, $a = \frac{dv}{dt}$, is a constant 9.8 meters per second per second, find $\frac{dK}{dt}$ when $v = 10$ meters per second.

12. Ship A passes a buoy at 10:00 a.m. and heads north at 20 miles per hour. Ship B passes the same buoy at 11:00 a.m. and heads east at 25 miles per hour. If s is the distance between the ships, what is $\frac{ds}{dt}$ at noon?

13. Suppose the height of a rectangle is growing at a rate of 0.1 inches per second while its length is growing at a rate of 0.2 inches per second. When the height of the rectangle is 4 inches and its length is 8 inches, at what rate is the area of the rectangle increasing?

14. The work force of a certain factory is growing at rate of 2 per month while the average productivity of a worker is growing at a rate of 4 units per month. If the work force is currently 100 and the average productivity per month is 200 units, at what rate is the total productivity per month of the factory increasing?

15. (a) What happens in Problem 14 if the work force is declining by 2 per month?
    (b) What happens in Problem 14 if the average productivity is decreasing by 5 per month?

16. A circular oil slick is 0.03 feet thick and has a radius which is increasing at a rate of 2 feet per hour. When the radius is 100 feet, at what rate is the volume of the oil slick increasing?

17. Oil is being added to a circular oil slick at the rate of 100 cubic feet per minute. If the oil slick is 0.05 feet thick, at what rate is the radius of the oil slick increasing when the radius is 400 feet?

18. In Section 2.2 we mentioned that the period of a pendulum of length b centimeters undergoing small oscillations is given by

$$T = 2\pi\sqrt{\frac{b}{g}} \text{ seconds,}$$

where $g = 980$ centimeters per second per second. Suppose the length of the pendulum changes as a function of temperature τ so that

$$\frac{db}{d\tau} = 0.08 \text{ centimeters per degree Celsius.}$$

(a) Find $\frac{dT}{d\tau}$ when $b = 20$ centimeters.

(b) Use (a) to approximate the effect on T of a 1°C increase in temperature. Do the same for a 2°C increase and a 2°C decrease.

Difference Equations to Differential Equations

Section 3.5

Differentiation of Trigonometric Functions

We now take up the question of differentiating the trigonometric functions. We will start with the sine function. From Section 3.2, we know that

$$\frac{d}{dx}\sin(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}. \tag{3.5.1}$$

From the addition formula for sine we have

$$\sin(x+h) = \sin(x)\cos(h) + \sin(h)\cos(x), \tag{3.5.2}$$

and so (3.5.1) becomes

$$\frac{d}{dx}\sin(x) = \lim_{h \to 0} \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h}. \tag{3.5.3}$$

Now

$$\begin{aligned} \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h} &= \frac{\sin(x)(\cos(h) - 1) + \cos(x)\sin(h)}{h} \\ &= \sin(x)\left(\frac{\cos(h) - 1}{h}\right) + \cos(x)\left(\frac{\sin(h)}{h}\right). \end{aligned}$$

Thus

$$\frac{d}{dx}\sin(x) = \sin(x)\lim_{h \to 0} \frac{\cos(h) - 1}{h} + \cos(x)\lim_{h \to 0} \frac{\sin(h)}{h}. \tag{3.5.4}$$

Our problem then comes down to evaluating the two limits in (3.5.4). The second of these turns out to be the key, so we will begin with it.

For $0 < h < \frac{\pi}{2}$, consider the point $C = (\cos(h), \sin(h))$ on the unit circle centered at the origin. We first repeat an argument from Section 2.4 to show that $\sin(h) < h$: If we let $A = (0,0)$ and $B = (1,0)$, as in Figure 3.5.1, then the area of $\triangle ABC$ is

$$\frac{1}{2}\sin(h).$$

The area of the sector of the circle cut off by the arc from B to C is the fraction $\frac{h}{2\pi}$ of the area of the entire circle; hence, this area is

$$\frac{h}{2\pi}\pi = \frac{h}{2}.$$

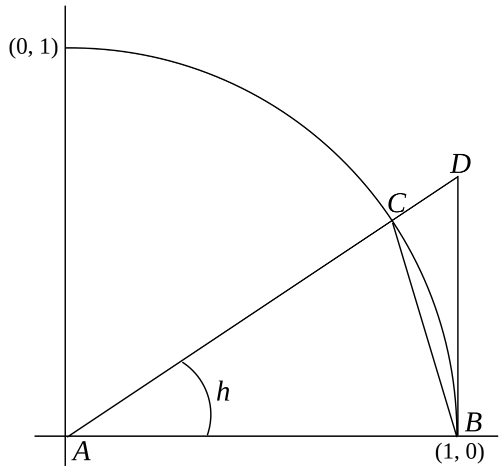


Figure 3.5.1

Since $\triangle ABC$ is contained in this section, we have

$$\frac{1}{2}\sin(h) < \frac{h}{2}, \tag{3.5.5}$$

or simply

$$\sin(h) < h. \tag{3.5.6}$$

Now let $D = (1, \tan(h))$, the point where the line passing through A and C intersects the line perpendicular to the x-axis passing through B. Then $\triangle ABD$ has area

$$\frac{1}{2}\tan(h) = \frac{\sin(h)}{2\cos(h)}.$$

Since $\triangle ABD$ contains the sector of the circle considered above, we have

$$\frac{h}{2} < \frac{\sin(h)}{2\cos(h)}, \tag{3.5.7}$$

or

$$h < \frac{\sin(h)}{\cos(h)}. \tag{3.5.8}$$

Putting inequalities (3.5.6) and (3.5.8) together gives us

$$\sin(h) < h < \frac{\sin(h)}{\cos(h)}. \tag{3.5.9}$$

Dividing through by $\sin(h)$ yields

$$1 < \frac{h}{\sin(h)} < \frac{1}{\cos(h)}, \tag{3.5.10}$$

which, after taking reciprocals, gives us

$$1 > \frac{\sin(h)}{h} > \cos(h). \tag{3.5.11}$$

Now, finally, we can see where all of this has been heading. Since

$$\lim_{h \to 0^+} \cos(h) = 1,$$

(3.5.11) implies that we must have

$$\lim_{h \to 0^+} \frac{\sin(h)}{h} = 1. \tag{3.5.12}$$

To check the limit from the other side, we make use of the identity $\sin(-x) = -\sin(x)$. Letting $t = -h$, we have

$$\lim_{h \to 0^-} \frac{\sin(h)}{h} = \lim_{h \to 0^-} \frac{-\sin(h)}{-h} = \lim_{h \to 0^-} \frac{\sin(-h)}{-h} = \lim_{t \to 0^+} \frac{\sin(t)}{t} = 1. \tag{3.5.13}$$

Together (3.5.12) and (3.5.13) give us the following proposition.

Proposition

$$\lim_{h \to 0} \frac{\sin(h)}{h} = 1. \tag{3.5.14}$$

With this result, we may now compute

$$\begin{aligned}
\lim_{h \to 0} \frac{1 - \cos(h)}{h} &= \lim_{h \to 0} \left(\frac{1 - \cos(h)}{h}\right)\left(\frac{1 + \cos(h)}{1 + \cos(h)}\right) \\
&= \lim_{h \to 0} \frac{1 - \cos^2(h)}{h(1 + \cos(h))} \\
&= \lim_{h \to 0} \frac{\sin^2(h)}{h(1 + \cos(h))} \\
&= \lim_{h \to 0} \left(\frac{\sin(h)}{h}\right)\left(\frac{\sin(h)}{1 + \cos(h)}\right) \\
&= \lim_{h \to 0} \frac{\sin(h)}{h} \lim_{h \to 0} \frac{\sin(h)}{1 + \cos(h)} \\
&= (1)\left(\frac{0}{2}\right) = 0.
\end{aligned}$$

Proposition

$$\lim_{h \to 0} \frac{1 - \cos(h)}{h} = 0. \tag{3.5.15}$$

Of course, from (3.5.15) we have

$$\lim_{h\to 0}\frac{\cos(h)-1}{h} = -\lim_{h\to 0}\frac{1-\cos(h)}{h} = 0. \tag{3.5.16}$$

Putting (3.5.14) and (3.5.16) into (3.5.4) gives us

$$\begin{aligned}\frac{d}{dx}\sin(x) &= \sin(x)\lim_{h\to 0}\frac{\cos(h)-1}{h} + \cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}\\ &= \sin(x)(0)+\cos(x)(1)\\ &= \cos(x).\end{aligned}$$

Proposition The function $f(x) = \sin(x)$ is differentiable for all x in $(-\infty,\infty)$ with

$$\frac{d}{dx}\sin(x) = \cos(x). \tag{3.5.17}$$

The derivatives of the other trigonometric functions now follow with the help of some basic identities. Since $\cos(x) = \sin(x+\frac{\pi}{2})$ and $\cos(x+\frac{\pi}{2}) = -\sin(x)$, it follows that

$$\frac{d}{dx}\cos(x) = \frac{d}{dx}\sin\left(x+\frac{\pi}{2}\right) = \cos\left(x+\frac{\pi}{2}\right)\frac{d}{dx}\left(x+\frac{\pi}{2}\right) = \cos\left(x+\frac{\pi}{2}\right) = -\sin(x).$$

The other four derivatives are as follows:

$$\begin{aligned}\frac{d}{dx}\tan(x) &= \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right)\\ &= \frac{\cos(x)\frac{d}{dx}\sin(x)-\sin(x)\frac{d}{dx}\cos(x)}{\cos^2(x)}\\ &= \frac{\cos(x)\cos(x)-\sin(x)(-\sin(x))}{\cos^2(x)}\\ &= \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)}\\ &= \frac{1}{\cos^2(x)}\\ &= \sec^2(x),\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}\cot(x) &= \frac{d}{dx}\left(\frac{\cos(x)}{\sin(x)}\right)\\ &= \frac{\sin(x)\frac{d}{dx}\cos(x)-\cos(x)\frac{d}{dx}\sin(x)}{\sin^2(x)}\end{aligned}$$

$$\begin{aligned}
&= \frac{\sin(x)(-\sin(x)) - \cos(x)\cos(x)}{\sin^2(x)} \\
&= \frac{-(\sin^2(x) + \cos^2(x))}{\sin^2(x)} \\
&= -\frac{1}{\sin^2(x)} \\
&= -\csc^2(x),
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}\sec(x) &= \frac{d}{dx}(\cos(x))^{-1} \\
&= -(\cos(x))^{-2}\frac{d}{dx}\cos(x) \\
&= \frac{\sin(x)}{\cos^2(x)} \\
&= \left(\frac{1}{\cos(x)}\right)\left(\frac{\sin(x)}{\cos(x)}\right) \\
&= \sec(x)\tan(x),
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dx}\csc(x) &= \frac{d}{dx}(\sin(x))^{-1} \\
&= -(\sin(x))^{-2}\frac{d}{dx}\sin(x) \\
&= -\frac{\cos(x)}{\sin^2(x)} \\
&= -\left(\frac{1}{\sin(x)}\right)\left(\frac{\cos(x)}{\sin(x)}\right) \\
&= \csc(x)\cot(x).
\end{aligned}$$

The next proposition summarizes these results.

Proposition The derivatives of the trigonometric functions are as follows:

$$\frac{d}{dx}\sin(x) = \cos(x) \tag{3.5.18}$$

$$\frac{d}{dx}\cos(x) = -\sin(x) \tag{3.5.19}$$

$$\frac{d}{dx}\tan(x) = \sec^2(x) \tag{3.5.20}$$

$$\frac{d}{dx}\cot(x) = -\csc^2(x) \tag{3.5.21}$$

$$\frac{d}{dx}\sec(x) = \sec(x)\tan(x) \tag{3.5.22}$$

$$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x) \tag{3.5.23}$$

Example Using the chain rule, we have

$$\frac{d}{dx}\sin(2x) = \cos(2x)\frac{d}{dx}(2x) = 2\cos(2x).$$

Example Using the product rule followed by the chain rule, we have

$$\begin{aligned}\frac{d}{dx}(3\sin(5x)\cos(4x)) &= 3\sin(5x)\frac{d}{dx}\cos(4x) + 3\cos(4x)\frac{d}{dx}\sin(5x)\\ &= 3\sin(5x)(-\sin(4x)\frac{d}{dx}(4x)) + 3\cos(4x)\cos(5x)\frac{d}{dx}(5x)\\ &= -12\sin(5x)\sin(4x) + 15\cos(4x)\cos(5x).\end{aligned}$$

Example Using the chain rule twice, we have

$$\begin{aligned}\frac{d}{dx}\sin^2(3x) &= 2\sin(3x)\frac{d}{dx}\sin(3x)\\ &= 2\sin(3x)\cos(3x)\frac{d}{dx}(3x)\\ &= 6\sin(3x)\cos(3x).\end{aligned}$$

Example Using the product rule followed by the chain rule, we have

$$\begin{aligned}\frac{d}{dt}(t^2\tan(2t)) &= t^2\frac{d}{dt}\tan(2t) + \tan(2t)\frac{d}{dt}t^2\\ &= t^2\sec^2(2t)\frac{d}{dt}(2t) + 2t\tan(2t)\\ &= 2t^2\sec^2(2t) + 2t\tan(2t).\end{aligned}$$

Example Using the chain rule twice, we have

$$\begin{aligned}\frac{d}{dz}\sec^3(3z) &= 3\sec^2(3z)\frac{d}{dz}\sec(3z)\\ &= 3\sec^2(3z)\sec(3z)\tan(3z)\frac{d}{dz}(3z)\\ &= 9\sec^3(3z)\tan(3z).\end{aligned}$$

Example If $f(x) = 8\cot^4(3x^2)$, then

$$\begin{aligned}f(x) &= 32\cot^3(3x^2)\frac{d}{dx}\cot(3x^2)\\ &= 32\cot^3(3x^2)(-\csc^2(3x^2)\frac{d}{dx}(3x^2))\\ &= -192x\cot^3(3x^2)\csc^2(3x^2).\end{aligned}$$

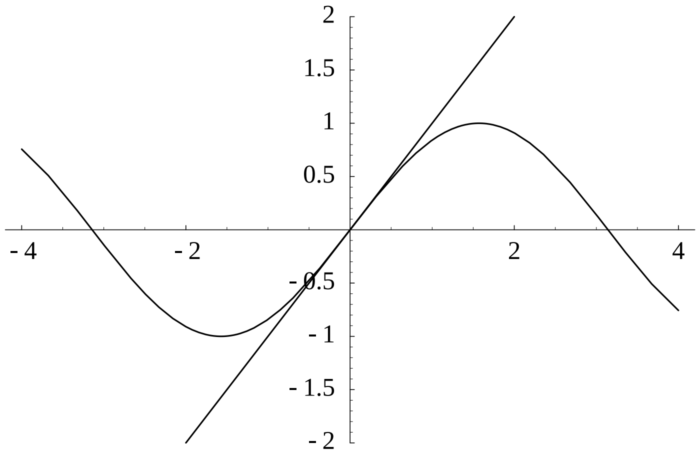


Figure 3.5.2 Graphs of $y = \sin(x)$ and $y = x$

Example If $f(x) = \sin(x)$, then $f(0) = \sin(0) = 0$ and $f'(0) = \cos(0) = 1$. Hence the best affine approximation to $f(x) = \sin(x)$ at $x = 0$ is

$$T(x) = x.$$

This says that for small values of x, $\sin(x) \approx x$ This fact is very useful in many applications where an equation cannot be solved exactly because of the presence of a sine term, but can be solved exactly once the approximation $\sin(x) \approx x$ is made. For example, the formula mentioned in Section 2.2 for the motion of a pendulum undergoing small oscillations was derived after making this approximation. Without this approximation the underlying equation cannot be solved exactly. See Figure 3.5.2 for the graphs of $y = \sin(x)$ and $y = x$.

Final comments on rules of differentiation

With the work of the last three sections we can now routinely differentiate any algebraic function or any combination of an algebraic function with a trigonometric function. In fact, the rules of these last three sections provide algorithms for differentiation which may be incorporated into computer programs. Programs that are capable of performing differentiation in this manner, as well as other types of algebraic procedures, are called symbolic manipulation programs or computer algebra systems. These programs are very useful when working with procedures that require exact knowledge of the formula for the derivative of a given function.

Contrasted to symbolic differentiation is numerical differentiation. Numerical differentiation is performed when we approximate the derivative of a function at a specific point. That is, whereas symbolic differentiation finds a formula for the derivative of a function, which may then be evaluated at any point in its domain to find specific values, numerical differentiation finds a single number which is used as an approximation to the value of the derivative at one given point. For example, if we wish to approximate the derivative of a function f at a point c, we might pick a small value of h, positive or negative, and compute

$$f'(c) \approx \frac{f(c+h) - f(c)}{h}. \tag{3.5.24}$$

Of course, we need some procedure for deciding when h is small enough for (3.5.24) to give an accurate estimate for $f'(c)$. One technique is to use (3.5.24) repeatedly, cutting h in half each time, until the result does not change through the desired number of decimal places. This method is subject to serious roundoff errors due to the loss of significant digits in the numerator when two nearly equal numbers are subtracted (see Problem 13). Hence the numerical approximation of derivatives is not recommended unless it cannot be avoided. Problem 10 suggests an alternative to (3.5.24) which is both more stable for computations and more accurate for a given value of h.

Problems

1. Find the derivative of each of the following functions.

   (a) $f(x) = x^2 \sin(x)$ (b) $g(x) = \cos(4x)$

   (c) $g(t) = 3t\cos(2t)$ (d) $h(s) = \sin^2(s)\cos(s)$

   (e) $f(t) = \sin(3t)\cos(4t)$ (f) $g(z) = \sin^3(4z)$

2. Find the derivative of the dependent variable with respect to the independent variable for each of the following.

   (a) $y = \dfrac{\sin(2x)}{x}$ (b) $x = 3\tan(2t)$

   (c) $x = \sin(4t^2 + 1)$ (d) $y = 4\theta\tan(\theta^2 - 1)$

   (e) $z = \dfrac{1}{\cos(2t)}$ (f) $q = \sec^3(3t)$

   (g) $y = x^2\csc(2x)$ (h) $s = 3t\cot(2t)$

3. Evaluate each of the following.

   (a) $\dfrac{d}{dx}(\sin^2(2x)\cos^2(3x))$ (b) $\dfrac{d}{dx}(\sec(x)\tan(x))$

   (c) $\dfrac{d}{dq}\sec^3(q^2)$ (d) $\dfrac{d}{dt}\left(\dfrac{\sin^2(t)}{\cos(t)}\right)$

   (e) $\dfrac{d}{dz}\sqrt{1 + \sin^2(z)}$ (f) $\dfrac{d}{dr}(r^2\cos(3r^2))$

4. Find the best affine approximation to $f(x) = \tan(2x)$ at 0.

5. Find the best affine approximation to $g(t) = \cos(t)$ at 0.

6. Find the best affine approximation to $f(t) = \sin^2(t)$ at 0.

7. (a) Find the best affine approximation S to $f(x) = \sqrt{1 + x}$ at 0.

   (b) Find the best affine approximation T to $g(x) = \sin(4x)$ at 0.

   (c) Find the best affine approximation U to $h(x) = \sqrt{1 + \sin(4x)}$ at 0.

   (d) What is the relationship between f, g, and h? Is their a similar relationship between S, T, and U?

8. Evaluate the following limits.

(a) $\lim_{x \to 0} \frac{\sin(2x)}{x}$

(b) $\lim_{x \to 0} \frac{\sin(2x)}{\sin(3x)}$

(c) $\lim_{x \to 0} \frac{\tan(x)}{x}$

(d) $\lim_{h \to 0} \frac{\tan(2h)}{\sin(3h)}$

(e) $\lim_{x \to 0} \frac{\sin^2(x)}{x}$

(f) $\lim_{t \to 0} \frac{1 - \cos(t)}{t^2}$

(g) $\lim_{t \to 0} \frac{\sin^2(3t)}{t^2}$

(h) $\lim_{\theta \to 0} \frac{\tan^2(5\theta)}{\sin^2(3\theta)}$

9. For each of the following, decide whether or not the given function is $o(h)$ and whether or not it is $O(h)$.

(a) $f(x) = \sin(x)$

(b) $f(x) = \sin^2(x)$

(c) $g(t) = \tan(t)$

(d) $h(t) = \tan^2(t)$

(e) $f(t) = 1 - \cos(t)$

(f) $g(t) = 1 - \cos^2(t)$

10. Given a function f which is differentiable at the point c, define

$$D(h) = \frac{f(c+h) - f(c)}{h}.$$

Then, for small values of h, $f'(c) \approx D(h)$.

(a) Let $h > 0$. A better approximation for $f'(c)$ than $D(h)$ is given by averaging $D(h)$ and $D(-h)$. Show that if we define

$$D_1(h) = \frac{D(h) + D(-h)}{2},$$

then

$$D_1(h) = \frac{f(c+h) - f(c-h)}{2h}.$$

What is $D_1(h)$ geometrically?

(b) Let $h > 0$. Another approximation that is sometimes used for $f'(c)$ is

$$D_2(h) = \frac{4}{3} D_1\left(\frac{h}{2}\right) - \frac{1}{3} D_1(h).$$

Show that

$$D_2(h) = \frac{f(c-h) - 8f\left(c - \frac{h}{2}\right) + 8f\left(c + \frac{h}{2}\right) - f(c+h)}{6h}.$$

11. Using $h = 0.00001$, approximate the derivatives of the following functions using $D(h)$, $D_1(h)$, and $D_2(h)$ (from Problem 10) at the indicated points. Compare your answers with the exact values.

    (a) $f(x) = x^2$ at $x = 2$

    (b) $f(x) = \dfrac{1}{x}$ at $x = 2$

    (c) $f(x) = \sin(x)$ at $x = 0$

    (d) $f(x) = 3\sin(x^2)\cos(4x)$ at $x = 0$

12. Compute $D(h)$, $D_1(h)$, and $D_2(h)$ (from Problem 10) for the function $f(x) = |x|$ at $x = 0$. Use $h = 0.001$. Are your answers reasonable? Can you explain them?

13. For $f(x) = x^2$ and $c = 2$, compute the values of

$$e_n = |4 - D(10^{-n})|$$

(see Problem 10) for $n = 1, 2, \ldots, 15$. Note that you are computing the absolute value of the error in approximating $f'(c)$ by $D(h)$ for different values of h. Plot the ordered pairs (n, e_n). Does the absolute value of the error decrease as h decreases? Can you explain your results?

Difference Equations to Differential Equations

Section 3.6

Newton's Method

Many problems in mathematics involve, at some point or another, solving an equation for an unknown quantity. An equation of the form $f(x) = 0$ may be solved for x by simple algebra if f is an affine function and by the quadratic formula if f is a quadratic polynomial. There are formulas similar to the quadratic formula for both cubic and quartic polynomials, but they are, in general, very cumbersome. One of the most interesting results of mathematics, due to Niels Henrik Abel (1802-1829), is that there does not exist an analogue of the quadratic formula for quintic polynomials. For this and other reasons, it turns out that in many situations solving an equation $f(x) = 0$ for x requires using a method which can approximate the solutions to a predetermined level of accuracy.

In Section 2.5 we discussed one such method, the bisection algorithm, for approximating the solutions of an equation. The strong point of the bisection algorithm is that, once an appropriate starting interval has been found, the method will always find a solution to any desired level of accuracy; its weakness lies in the slowness with which the successive approximations approach the solution. In this section we will discuss another method, known as *Newton's method*, for approximating solutions to an equation. In distinction to the bisection algorithm, Newton's method does not always work, but, when it does, it is in general remarkably fast.

Suppose we wish to find a solution to the equation $f(x) = 0$ for a given function f. Recall that, geometrically, this corresponds to finding the point where the curve $y = f(x)$ crosses the x-axis. To start Newton's method, we must first have an initial guess x_0. Frequently, we find the initial guess by graphing the curve $y = f(x)$ and letting x_0 be a point close to where the curve crosses the x-axis. Given the initial guess x_0, let T_0 be the best affine approximation to f at x_0. That is, define T_0 by

$$T_0(x) = f'(x_0)(x - x_0) + f(x_0). \tag{3.6.1}$$

The idea behind Newton's method is to obtain an improved estimate of a solution to $f(x) = 0$ by replacing the equation $f(x) = 0$ with the simpler equation $T_0(x) = 0$. If we let x_1 denote the solution to the latter equation, then we have $T_0(x_1) = 0$, that is

$$f'(x_0)(x_1 - x_0) + f(x_0), \tag{3.6.2}$$

from which it follows that

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}. \tag{3.6.3}$$

Geometrically, x_1 is the point at which the line tangent to f at $(x_0, f(x_0)$ crosses the x-axis, as shown in Figure 3.6.1. To improve upon this approximation, we solve the equation

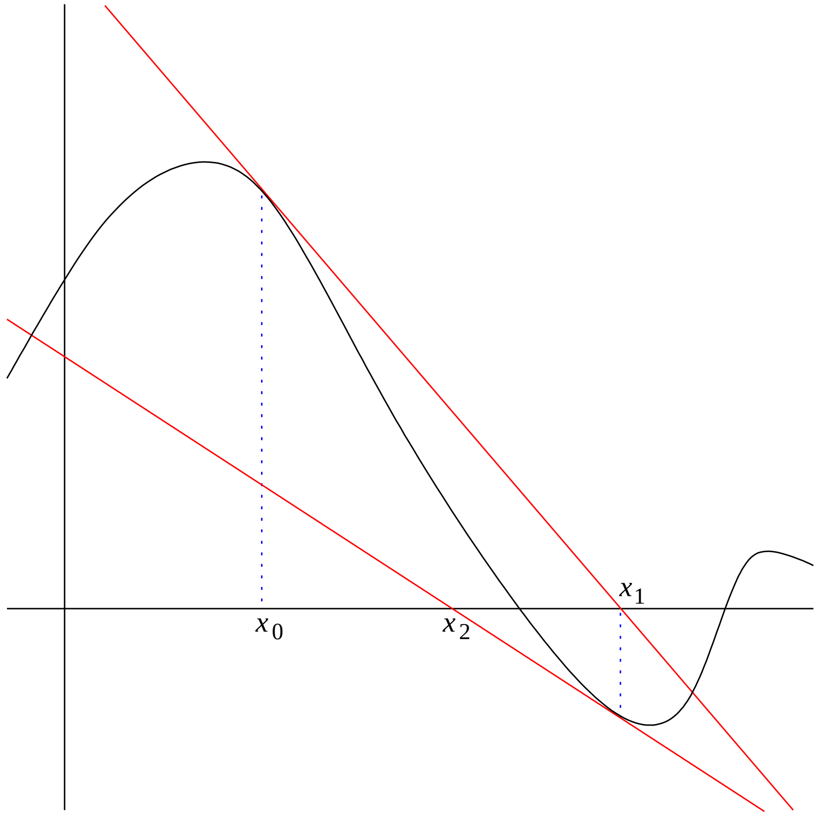


Figure 3.6.1 Two iterations of Newton's method

$T_1(x) = 0$, where T_1 is the best affine approximation to f at x_1. If we let x_1 denote the solution to this equation, then

$$f'(x_1)(x_2 - x_1) + f(x_1) = 0, \tag{3.6.4}$$

and so

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}. \tag{3.6.5}$$

We continue in this manner to generate a sequence of approximations x_0, x_1, x_2, x_3, ..., until we reach the desired degree of accuracy. Specifically, if we have found x_n, we find x_{n+1} by solving the equation $T_n(x) = 0$, where T_n is the best affine approximation to f at x_n. Hence we have

$$f(x_n)(x_{n+1} - x_n) + f(x_n) = 0, \tag{3.6.6}$$

which implies

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{3.6.7}$$

In other words, beginning with an initial guess x_0 , Newton's method generates a sequence $\{x_n\}$ using the difference equation (3.6.7). In most cases (although certainly not all), if x_0 is a good initial guess, $\lim_{n\to\infty} x_n = r$, where r is a solution of $f(x) = 0$, that is, $f(r) = 0$.

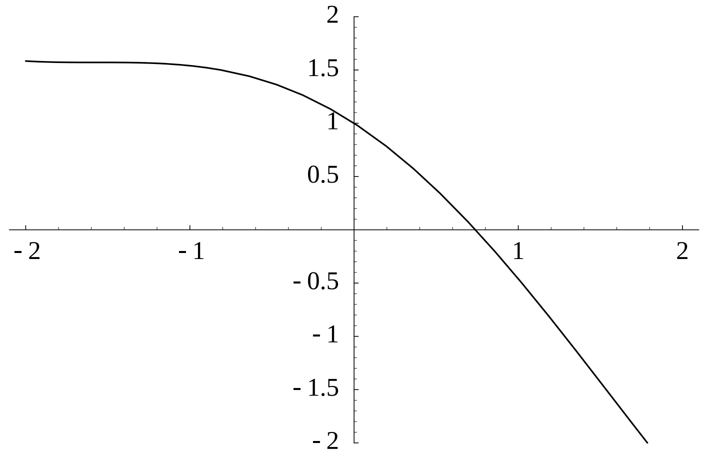


Figure 3.6.2 Graph of $f(x) = \cos(x) - x$

In any practical case we need to know when to stop generating successive approximations using (3.6.7). Since we do not know the exact solutions to the equation (if we did, we would not be using Newton's method to start with), we can never know for sure how far a given approximation is from a solution. What is done in practice is to generate terms until the difference between successive terms is less than a predetermined tolerance level. That is, if we decide that we want our approximation to be off by no more than ϵ, then we stop when $|x_{n+1} - x_n| < \epsilon$.

Newton's method To approximate a solution to an equation $f(x) = 0$ to within a tolerance of ϵ beginning with an initial guess x_0, compute the sequence of approximations x_0, x_1, x_2, x_3, ..., using the difference equation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{3.6.8}$$

stopping when $|x_{n+1} - x_n| < \epsilon$.

Example Suppose we wish to find a solution to the equation $\cos(x) = x$ with an error of no more than 0.0001. Then we should let $f(x) = \cos(x) - x$ and look for solutions to $f(x) = 0$. Since $\cos(x)$ will always be between -1 and 1, we know that any solution to $\cos(x) = x$ must lie in the interval $[-1, 1]$. Moreover, from the graph of f in Figure 3.6.2, we can see that the equation $f(x) = 0$ has only one solution, and this solution lies between 0 and 1. Alternatively, we could note that

$$f(0) = 1 > 0$$

and

$$f(1) = \cos(1) - 1 < 0,$$

which imply, by the Intermediate Value Theorem, that there is a solution in the interval $[0, 1]$. In either case, we will use $x_0 = 0.5$ for our initial guess. Now

$$f'(x) = -\sin(x) - 1,$$

so

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.5 - \frac{f(0.5)}{f'(0.5)} = 0.755222,$$

where we have rounded the result to 6 decimal places. Substituting this back into (3.6.8) gives us

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.75222 - \frac{f(0.755222)}{f'(0.755222)} = 0.739142.$$

Since

$$|x_2 - x_1| = 0.016080 > 0.0001,$$

we continue and compute

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.739142 - \frac{f(0.739142)}{f'(0.739142)} = 0.739085.$$

Now we have

$$|x_3 - x_2| = 0.000057 < 0.0001,$$

so we stop and use 0.7391 as our approximation to the solution of $\cos(x) = x$. For comparison, with the bisection algorithm starting from the initial interval $[0, 1]$, we would have had to iterate 13 times before obtaining an approximation to the solution with an error less than 0.0001.

Example As an example of where Newton's method goes wrong, consider the equation $f(x) = 0$, where

$$f(x) = \frac{x}{1 + x^2}.$$

Clearly, $x = 0$ is the only solution to this equation. However, beginning with an initial guess $x_0 = 0.75$, Newton's method yields the following sequence (where we have rounded each value to 5 digits):

$$\begin{aligned}
x_0 &= 0.75000 \\
x_1 &= -1.9286 \\
x_2 &= -5.2756 \\
x_3 &= -10.944 \\
x_4 &= -22.073 \\
x_5 &= -44.237 \\
x_6 &= -88.518 \\
x_7 &= -177.06 \\
x_8 &= -354.13 \\
x_9 &= -708.27 \\
x_{10} &= -1416.5
\end{aligned}$$

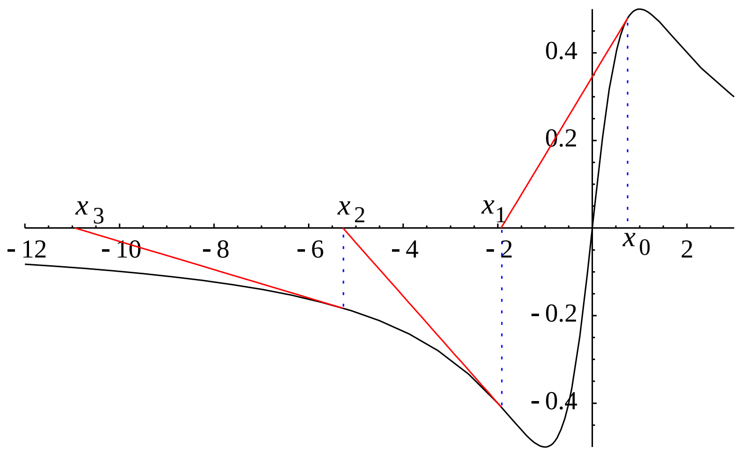


Figure 3.6.3 Newton's method diverging from a solution

Instead of converging to the solution at 0, this sequence seems to be diverging toward $-\infty$. In fact, geometrically this appears to be exactly the case, as can be seen in Figure 3.6.3. Here the problem comes from the fact that the graph of f approaches 0 asymptotically as x goes to $-\infty$. Newton's method is following the curve as it approaches 0, as it should, but, since there is no solution in this direction, the result is that the iterates are getting farther and farther away from the solution at $x = 0$.

Note that in the last example,

$$f'(x) = \frac{1 - x^2}{(1 + x^2)^2},$$

giving us $f'(1) = 0$. Hence if we had started with an initial guess of $x_0 = 1$, an application of the difference equation (3.6.8) would require a division by 0, which, of course, cannot be done. Geometrically, the tangent line to the graph of f at $(1, 0.5)$ is horizontal, and hence never crosses the x-axis, implying that there are no solutions to the equation $T(x) = 0$ if T is the best affine approximation to f at 1. Thus we must avoid starting Newton's method at a point where the derivative is 0.

Problems

1. For each of the following equations, use a graph to obtain initial guesses for solutions and then apply Newton's method to locate the solutions within 0.0001.

   (a) $x^5 - 6x^3 + 2x = 2$
   (b) $\sin(x) = x^2$
   (c) $\cos(t) = t^2$
   (d) $\cos^2(t) - t^2 = 0$
   (e) $2\sin(x) = \sqrt{x + 1}$
   (f) $6x^4 - 12x^3 + 4x - 1 = 0$

2. Even though we know that for a positive number c the equation $x^2 - c = 0$ has the exact solutions $-\sqrt{c}$ and $\sqrt{c}$, we may use Newton's methods to find decimal approximations to these square roots.

(a) Show that the sequence x_0, x_1, x_2, $\ldots$, of Newton's method approximations to a solution of $x^2 - c = 0$ satisfies the difference equation

$$x_{n+1} = \frac{x_n + \dfrac{c}{x_n}}{2}$$

for $n = 0, 1, 2, \ldots$.

(b) Use the difference equation from (a) to approximate $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{11}$ with an error of less than 0.00001.

(c) Can you see an intuitive reason why, starting with a positive initial guess, the sequence defined by the difference equation in part (a) might converge to $\sqrt{c}$?

(d) Assuming that $L = \lim_{n\to\infty} x_n$ for the sequence defined in (a), show that either $L = -\sqrt{c}$ or $L = \sqrt{c}$.

3. Use Newton's method to approximate $\sqrt[3]{2}$ with an error less than 0.00001.

4. Use Newton's method to approximate $\sqrt[7]{5}$ with an error less than 0.00001.

5. The method outlined in Problem 2 for approximating square roots was known to the Greeks and perhaps to the Babylonians. For an account of this and other aspects of Babylonian algebra, read Chapter 3 of *Mathematics in Civilization* by H. L. Resnikoff and R. O. Wells, Jr. (Dover Publications, Inc., New York, 1984).

6. What happens when you apply Newton's method to find solutions to the equation

$$x^3 - 5x = 0$$

starting with an initial guess of $x_0 = 1$? Explain this geometrically with a graph.

7. We know that when solving an equation $f(x) = 0$ using Newton's method, different initial guesses may lead to different solutions, and some may not converge to a solution at all. The problem of determining which initial guesses converge to a specified solution is surprisingly complicated, involving what mathematicians call *fractals*. For an account of this phenomenon, read pages 217-220 of *Chaos* by James Gleick (Viking Penguin, Inc., New York, 1987). Also, see the picture on the sixth color plate following page 114 in the same book.

Difference Equations to Differential Equations

Section 3.7

Rolle's Theorem and the Mean Value Theorem

The two theorems which are at the heart of this section draw connections between the instantaneous rate of change and the average rate of change of a function. The Mean Value Theorem, of which Rolle's Theorem is a special case, says that if f is differentiable on an interval, then there is some point in that interval at which the instantaneous rate of change of the function is equal to the average rate of change of the function over the entire interval. For example, if f gives the position of an object moving in a straight line, the Mean Value Theorem says that if the average velocity over some interval of time is 60 miles per hour, then at some time during that interval the object was moving at exactly 60 miles per hour. This is not a surprising fact, but it does turn out to be the key to understanding many useful applications.

Before we turn to a consideration of Rolle's theorem, we need to establish another fundamental result. Suppose an object is thrown vertically into the air so that its position at time t is given by $f(t)$ and its velocity by $v(t) = f'(t)$. Moreover, suppose it reaches its maximum height at time t_0. On its way up, the object is moving in the positive direction, and so $v(t) > 0$ for $t < t_0$; on the way down, the object is moving in the negative direction, and so $v(t) < 0$ for $t > t_0$. It follows, by the Intermediate Value Theorem and the fact that v is a continuous function, that we must have $v(t_0) = 0$. That is, at time t_0, when $f(t)$ reaches its maximum value, we have $f'(t_0) = 0$. This is an extremely useful fact which holds in general for differentiable functions, not only at maximum values but at minimum values as well. Before providing a general demonstration, we first need a few definitions.

Definition A function f is said to have a *local maximum* at a point c if there exists an open interval I containing c such that $f(c) \geq f(x)$ for all x in I. A function f is said to have a *local minimum* at a point c if there exists an open interval I containing c such that $f(c) \leq f(x)$ for all x in I. If f has either a local maximum or a local minimum at c, then we say f has a *local extremum* at c.

In short, f has a local maximum at a point c if the value of f at c is at least as large as the value of f at any nearby point, and f has a local minimum at a point c if the value of f at c is at least as small as the value of f at any nearby point. The next example provides an illustration.

Example Looking at the graph of the function $f(x) = x^3 - 3x$ in Figure 3.7.1, it appears that f has a local maximum of 2 at $x = -1$ and a local minimum of -2 at $x = 1$. We will confirm this observation in Section 3.8.

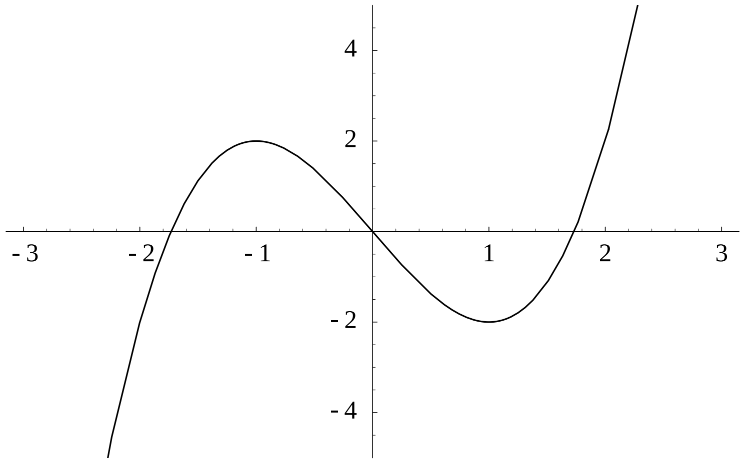


Figure 3.7.1 Graph of $f(x) = x^3 - 3x$

Now suppose f has local maximum at a point c and suppose f is differentiable at c. For small enough $h > 0$, $f(c+h) \leq f(c)$, so

$$f(c+h) - f(c) \leq 0;$$

thus

$$\frac{f(c+h) - f(c)}{h} \leq 0. \tag{3.7.1}$$

Clearly, if each term in a sequence is less than or equal to 0, and the sequence has a limit, then the limit of the sequence must be less than or equal to 0. Hence

$$f'(c) = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \leq 0. \tag{3.7.2}$$

Also, for $h < 0$ with $|h|$ small enough, we have $f(c+h) \leq f(c)$, and so

$$f(c+h) - f(c) \leq 0.$$

However, now, since $h < 0$, we have

$$\frac{f(c+h) - f(c)}{h} \geq 0, \tag{3.7.3}$$

from which it follows that

$$f'(c) = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} \geq 0. \tag{3.7.4}$$

Note that (3.7.1) is saying that secant lines to the right of a local maximum have negative slope, while (3.7.3) is saying that secant lines to the left of a local maximum have positive slope. Now the only way that both (3.7.2) and (3.7.4) can hold at the same time is if $f'(c) = 0$; that is, the only number which is both less than or equal to 0 and greater than

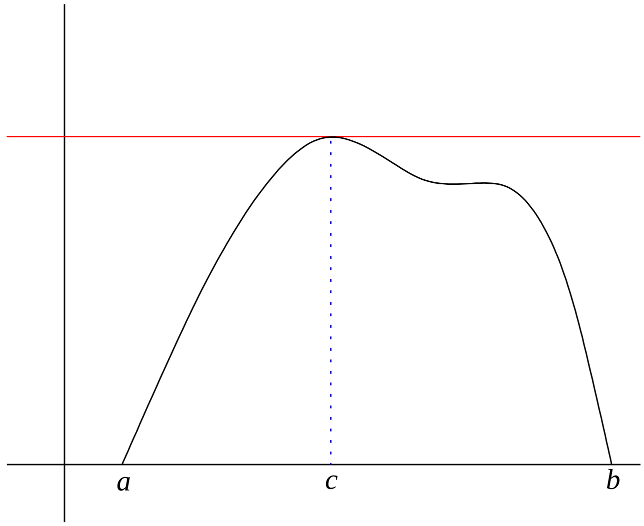


Figure 3.7.2 Illustration of Rolle's Theorem

or equal to 0 is 0 itself. Note that our argument here is just a refinement of our comments about velocity in the previous paragraph.

A similar argument gives the same result if f has a local minimum at c. The following proposition puts these together into one statement.

Proposition If f has a local extremum at c and f is differentiable at c, then $f'(c) = 0$.

Example For $f(x) = x^3 - 3x$, as in the previous example, $f'(x) = 3x^2 - 3$, and so $f'(-1) = 0$ and $f'(1) = 0$, consistent with our observation that f has a local maximum at $x = -1$ and local minimum at $x = 1$. Note, however, that this does not prove that f has local extrema at $x = -1$ and $x = 1$. Indeed, the proposition works in the other direction: if f has a local extremum at c, then $f'(c) = 0$.

This result will be very useful in our work in the next section when we consider the problem of finding the maximum and minimum values of a given function. For our present purpose, consider a function f which is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b), with $f(a) = f(b) = 0$. An example of such a function is shown in Figure 3.7.2. By the Extreme Value Theorem of Section 2.5, we know that f must have both a minimum value m and a maximum value M on $[a, b]$. If $m = M = 0$, then $f(x) = 0$ for all x in (a, b), and so $f'(x) = 0$ for all x in (a, b). If either $m \neq 0$ or $M \neq 0$, then f has a local extremum at some point c in (a, b), namely, either a point c for which $f(c) = m$ or a point c for which $f(c) = M$. Hence, by the previous proposition, $f'(c) = 0$. We have thus established the following theorem, credited originally to Michel Rolle (1652-1719).

Rolle's Theorem If f is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b) = 0$, then there exists a point c in (a, b) such that $f'(c) = 0$.

Put another way, Rolle's theorem says that if f is a differentiable function, then between any two solutions of the equation $f(x) = 0$ there is a point c where $f'(c) = 0$. Used

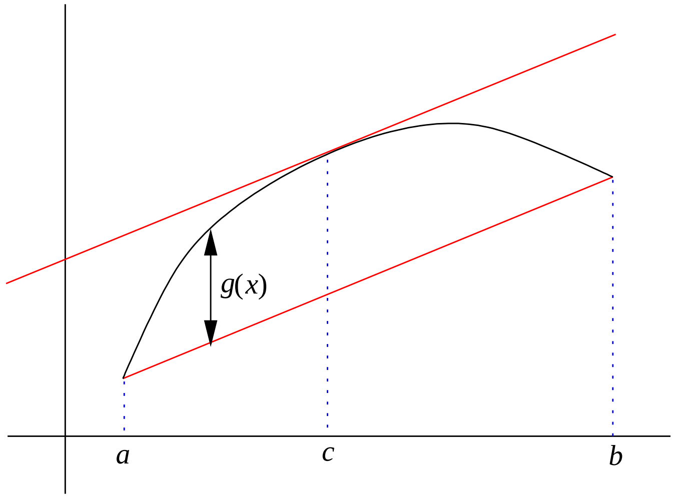


Figure 3.7.3 Illustration of the Mean Value Theorem

in conjunction with the Intermediate Value Theorem, this result can help identify intervals where an equation has a unique solution.

Example Solving the equation

$$x^5 + x^4 = 1 \tag{3.7.5}$$

is equivalent to solving the equation $f(x) = 0$ where $f(x) = x^5 + x^4 - 1$. Since $f(0) = -1$ and $f(1) = 1$, the Intermediate Value Theorem tells us that $f(x) = 0$ has at least one solution in $(0, 1)$. Moreover,

$$f'(x) = 5x^4 + 4x^3,$$

so $f'(x) > 0$ for all x in $(0, 1)$; in particular, there does not exist a point c in $(0, 1)$ such that $f'(c) = 0$. Hence, by Rolle's Theorem, there cannot be two solutions to $f(x) = 0$ in $(0, 1)$. That is, using the Intermediate Value Theorem and Rolle's Theorem together, we are able to conclude that there is exactly one solution to (3.7.5) in the interval $(0, 1)$. We may now use either the bisection algorithm or Newton's method to locate this solution.

Geometrically, Rolle's theorem says if f is a function which satisfies the conditions of the theorem on an interval $[a, b]$, then there is a point c in (a, b) such that the line tangent to the graph of f at $(c, f(c))$ is horizontal. In this case, that means that the line tangent to the graph of f at $(c, f(c))$ is parallel to the line passing through the points $(a, f(a))$ and $(b, f(b))$, as is seen in Figure 3.7.2. Certainly, if we took this picture and rotated or shifted the points $(a, f(a))$ and $(b, f(b))$, rigidly moving the graph with these points, then this conclusion would still follow. That is, if f is continuous on the closed interval $[a, b]$ and differentiable on (a, b), then there must exist a point c in (a, b) such that the line tangent to the graph of f at $(c, f(c))$ is parallel to the line passing through the points $(a, f(a))$ and $(b, f(b))$ (see Figure 3.7.3). In other words, there must be a point c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \tag{3.7.6}$$

This is the content of the Mean Value Theorem.

Although the above argument for this result seems plausible, we will present a more precise argument. Define a new function g by

$$g(x) = f(x) - S(x), \tag{3.7.7}$$

where

$$S(x) = \left(\frac{f(b) - f(a)}{b - a}\right)(x - a) + f(a). \tag{3.7.8}$$

Geometrically, the graph of S is a line passing through the points $(a, f(a))$ and $(b, f(b))$, and $g(x)$ is the distance from the graph of f to the graph of S above the point x (see Figure 3.7.3). Now g is continuous on $[a, b]$ and differentiable on (a, b); moreover,

$$g(a) = f(a) - S(a) = f(a) - f(a) = 0 \tag{3.7.9}$$

and

$$g(b) = f(b) - S(b) = f(b) - f(b) = 0. \tag{3.7.10}$$

Thus g satisfies the conditions of Rolle's theorem. Hence there exists a point c in (a, b) such that $g'(c) = 0$. But

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}, \tag{3.7.11}$$

and so $g'(c) = 0$ implies

$$f'(c) - \frac{f(b) - f(a)}{b - a} = 0, \tag{3.7.12}$$

that is,

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \tag{3.7.13}$$

Mean Value Theorem If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists a point c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \tag{3.7.14}$$

Increasing and decreasing functions

Similar to the situation with the Intermediate Value Theorem and the Extreme Value Theorem, the Mean Value Theorem is an existence theorem. The point of interest is the existence of c, not in being able to compute a value for c. Although not immediately useful for computations, we will see that the Mean Value Theorem has many important consequences. The first of these, which we will consider now, involves determining when a function is increasing and when it is decreasing.

Definition We say a function f defined on an interval I is *increasing* on I if for every two points u and v in I with $u < v$, $f(u) < f(v)$. We say a function f defined on an interval I is *decreasing* on I if for every two points u and v in I with $u < v$, $f(u) > f(v)$.

Example The function $f(x) = x^2$ is increasing on the interval $[0, \infty)$ since for any two numbers u and v with $0 \leq u < v$, $f(u) = u^2 < v^2 = f(v)$. Moreover, f is decreasing on $(-\infty, 0]$ since for any two numbers u and v with $u < v \leq 0$, $f(u) = u^2 > v^2 = f(v)$.

Now suppose f is differentiable on an interval (a, b) with $f'(x) > 0$ for all x in (a, b). If u and v are two points in (a, b) with $u < v$, then, by the Mean Value Theorem, there exists a point c with $u < c < v$ such that

$$f'(c) = \frac{f(v) - f(u)}{v - u}. \tag{3.7.15}$$

Since c is in (a, b), $f'(c) > 0$, so, using (3.7.15),

$$f(v) - f(u) = f'(c)(v - u) > 0. \tag{3.7.16}$$

Hence $f(v) > f(u)$ and f is increasing on the interval (a, b). Similarly, if $f'(x) < 0$ for all x in (a, b), then we would have $f'(c) < 0$, from which it would follow that $f(v) > f(u)$ and, hence, that f is decreasing on (a, b). In short, to determine the intervals on which a differentiable function is increasing and those on which it is decreasing, we need to look only for the intervals on which the derivative is positive and those on which it is negative, respectively.

Proposition If f is differentiable on (a, b) and $f'(x) > 0$ for all x in (a, b), then f is increasing on (a, b). If f is differentiable on (a, b) and $f'(x) < 0$ for all x in (a, b), then f is decreasing on (a, b).

Geometrically, this proposition is saying that a function is increasing where it has positive slope and decreasing where it has negative slope. This should seem intuitively clear, but it is the Mean Value Theorem which makes the connection between average rates of change and instantaneous rates of change necessary for establishing the result.

Example Suppose $f(x) = 2x^3 + 3x^2 - 12x + 1$. To determine where f is increasing and where it is decreasing, we first find

$$f'(x) = 6x^2 + 6x - 12 = 6(x + 2)(x - 1). \tag{3.7.17}$$

Hence $f'(x) = 0$ only when $x = -2$ or $x = 1$. Since f' is continuous, the Intermediate Value Theorem implies that f' cannot change sign on the intervals $(-\infty, -2)$, $(-2, 1)$, and $(1, \infty)$. Since $f'(-3) = 24 > 0$, it follows that $f'(x) > 0$ for all x in $(-\infty, -2)$. Similarly, since $f'(0) = -12 < 0$, $f'(x) < 0$ for all x in $(-2, 1)$; and, since $f'(2) = 24 > 0$, $f'(x) > 0$ for all x in $(1, \infty)$. It now follows from the previous proposition that f is increasing on the intervals $(-\infty, -2)$ and $(1, \infty)$ and decreasing on the interval $(-2, 1)$.

Note that we could obtain the same information about f' directly from (3.7.17) without evaluating f' and without invoking the Intermediate Value Theorem. Namely, from the facts that $x + 2 < 0$ and $x - 1 < 0$ whenever $x < -2$, we may conclude from the (3.7.17) that $f'(x) > 0$ for all x in $(-\infty, -2)$. Similarly, whenever $-2 < x < 1$, we have $x + 2 > 0$ and $x - 1 < 0$, implying that $f'(x) < 0$ for all x in $(-2, 1)$; and whenever $x > 1$ we have $x + 2 > 0$ and $x - 1 > 0$, implying that $f'(x) > 0$ for all x in $(1, \infty)$.

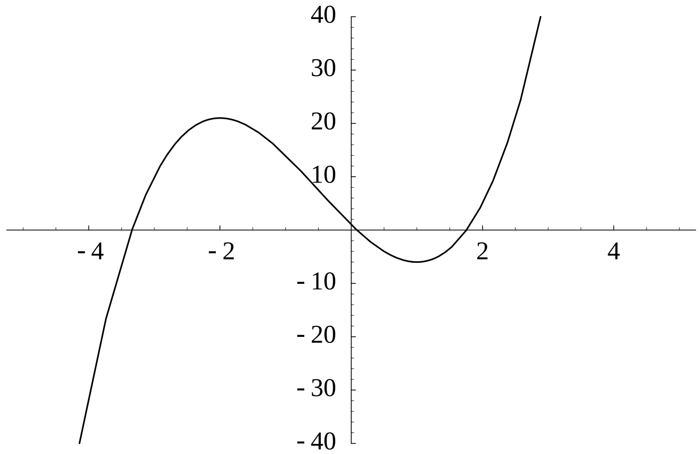


Figure 3.7.4 Graph of $f(x) = 2x^3 + 3x^2 - 12x + 1$

Combining our information on intervals where f is increasing and intervals where f is decreasing with the facts that $f(-3) = 10$, $f(-2) = 21$, $f(0) = 1$, $f(1) = -6$, $f(2) = 5$,

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} x^3\left(2 + \frac{3}{x} - \frac{12}{x^2} + \frac{1}{x^3}\right) = -\infty,$$

and

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x^3\left(2 + \frac{3}{x} - \frac{12}{x^2} + \frac{1}{x^3}\right) = \infty,$$

we can understand why the graph of f looks as it does in Figure 3.7.4.

Example Now consider $f(x) = x^5 - x^3$. Then

$$f'(x) = 5x^4 - 3x^2 = x^2(5x^2 - 3), \tag{3.7.18}$$

so $f'(x) = 0$ when

$$x = -\sqrt{\frac{3}{5}}, x = 0, \text{ or } x = \sqrt{\frac{3}{5}}.$$

Now when

$$x < -\sqrt{\frac{3}{5}},$$

both $x^2 > 0$ and $5x^2 - 3 > 0$, implying, from (3.7.18), that $f'(x) > 0$. For

$$-\sqrt{\frac{3}{5}} < x < 0,$$

$x^2 > 0$, but $5x^2 - 3 < 0$, so $f'(x) < 0$; for

$$0 < x < \sqrt{\frac{3}{5}},$$

$x^2 > 0$ and $5x^2 - 3 < 0$, so $f'(x) < 0$; and for

$$x > \sqrt{\frac{3}{5}},$$

$x^2 > 0$ and $5x^2 - 3 > 0$, so $f'(x) > 0$. Hence f is increasing on

$$\left(-\infty, -\sqrt{\frac{3}{5}}\right)$$

and

$$\left(\sqrt{\frac{3}{5}}, \infty\right),$$

and decreasing on

$$\left(-\sqrt{\frac{3}{5}}, 0\right)$$

and

$$\left(0, \sqrt{\frac{3}{5}}\right).$$

Note that we could have determined the sign of f' on these four intervals by evaluating f' at a point in each interval and then applying the Intermediate value Theorem. For example,

$$f'(-1) = 2 > 0,$$

$$f'\left(-\frac{1}{5}\right) = -\frac{14}{125} < 0,$$

$$f'\left(\frac{1}{5}\right) = -\frac{14}{125} < 0,$$

and

$$f'(1) = 2 > 0.$$

As in the previous example, if we combine this information with the facts

$$f(-1) = 0,$$

$$f\left(-\sqrt{\frac{3}{5}}\right) = \frac{6}{25}\sqrt{\frac{3}{5}},$$

$$f(0) = 0,$$

$$f\left(\sqrt{\frac{3}{5}}\right) = -\frac{6}{25}\sqrt{\frac{3}{5}},$$

$$f(1) = 0,$$

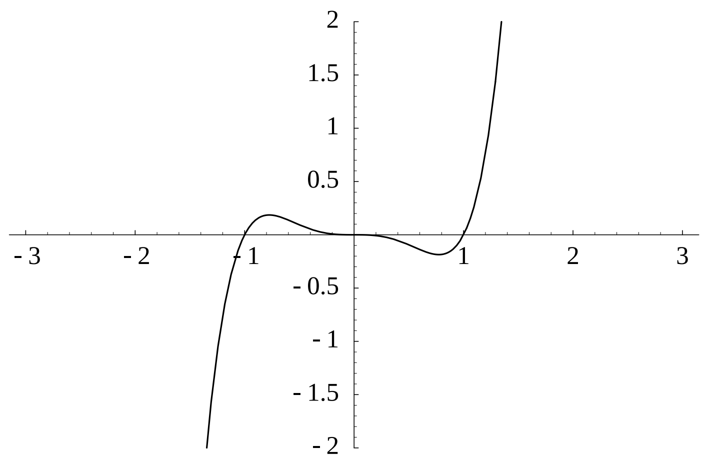


Figure 3.7.5 Graph of $f(x) = x^5 - x^3$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} x^5 \left(1 - \frac{1}{x^2}\right) = -\infty,$$

and

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} x^5 \left(1 - \frac{1}{x^2}\right) = \infty,$$

we can understand why the graph of f looks as it does in Figure 3.7.5.

Antiderivatives

We will close this section with a look at one more important application of the Mean Value Theorem. Although not needed for our current discussion, our result will be very useful in the next chapter. We begin with a definition.

Definition If F and f are functions defined on an open interval (a, b) such that $F'(x) = f(x)$ for all x in (a, b), then we call F an *antiderivative* of f.

In other words, an antiderivative of a function f is another function whose derivative is f. Although a given function f has at most one derivative, it is possible to have more than one antiderivative, as the next example demonstrates.

Example $F(x) = x^3$ is an antiderivative of $f(x) = 3x^2$ on $(-\infty, \infty)$. However, note that $G(x) = x^3 + 4$ is also an antiderivative of f. In fact, given any constant k,

$$H(x) = x^3 + k \tag{3.7.19}$$

is an antiderivative of f. This should not be too surprising since specifying the derivative of a function fixes only the slope of its graph, and the graphs of the functions in (3.7.19) are all in a sense parallel to each other.

This example shows that a given function may have an infinite number of antiderivatives. However, note that the difference of any two these antiderivatives is a constant. We will now show that this is always the case, and, in particular, that (3.7.19) specifies all possible antiderivatives of $f(x) = 3x^2$.

First consider a function F defined on an open interval (a,b) for which $F'(x) = 0$ for all x in (a,b). That is, F is an antiderivative of the function which is 0 for all values of x in (a,b). Now if u and v are any two points in (a,b), then, by the Mean Value Theorem,

$$\frac{F(v) - F(u)}{v - u} = F'(c) \tag{3.7.20}$$

for some c in (a,b). But $F'(c) = 0$, so

$$\frac{F(v) - F(u)}{v - u} = 0, \tag{3.7.21}$$

which implies $F(u) = F(v)$. If we let $k = F(u)$ for a fixed u in (a,b), we now have $F(v) = F(u) = k$ for all v in (a,b). In other words, if the derivative of a function is 0 on an open interval, then the function must be constant on that interval.

Now suppose F and G are two functions defined on an open interval (a,b) for which $F'(x) = G'(x)$ for all x in (a,b). Let $H(x) = F(x) - G(x)$ for all x in (a,b). Then

$$H'(x) = F'(x) - G'(x) = 0 \tag{3.7.22}$$

for all x in (a,b). Then, by what we have just shown, there exists a constant k for which

$$k = H(x) = F(x) - G(x) \tag{3.7.23}$$

for all x in (a,b). Hence if F an G have the same derivative on an open interval, that is, are antiderivatives of the same function, then they can differ only by a constant.

Proposition If F and G are both antiderivatives of f on an open interval (a,b), then there exists an constant k such that

$$F(x) = G(x) + k \tag{3.7.24}$$

for all x in (a,b).

Example Since

$$\frac{d}{dx}\sin(x) = \cos(x),$$

we know that $G(x) = \sin(x)$ is an antiderivative of $f(x) = \cos(x)$ on $(-\infty, \infty)$. Thus if F is any antiderivative of f, then

$$F(x) = \sin(x) + k \tag{3.7.25}$$

for some constant k. In other words, functions of the form given in (3.7.25) are the only antiderivatives of $f(x) = \cos(x)$. Figure 3.7.6 shows the graphs of (3.7.25) for nine different values of k. Although each of these graphs is the graph of a different function, they are

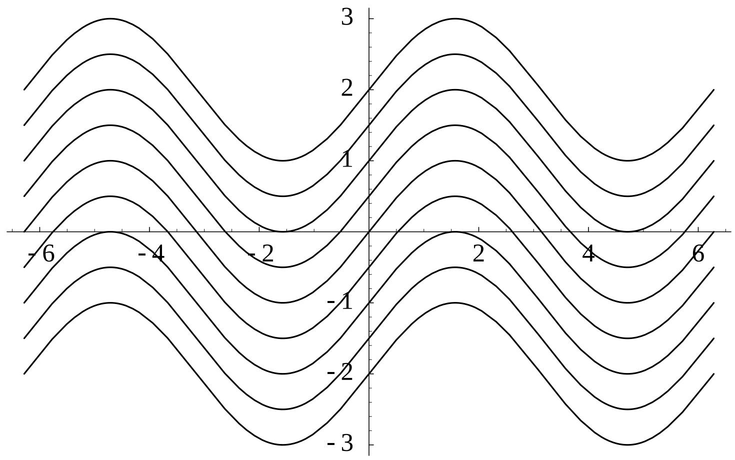


Figure 3.7.6 Graphs of $F(x) = \sin(x) + k$ for different values of k

parallel to one another in the sense that they all have the same slope at any given value of x, namely, $\cos(x)$.

Problems

1. Explain why the equation $\cos(x) = x$ has exactly one solution in the interval $[0, 1]$.
2. Explain why the equation $x^4 - 2x^2 = 2$ has exactly one solution in the interval $[1, 2]$ and exactly one solution in the interval $[-2, -1]$.
3. Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Moreover, suppose there is a constant M such that $|f'(x)| \leq M$ for all x in (a, b). Show that

$$|f(v) - f(u)| \leq M|v - u|$$

   for all u and v in $[a, b]$.
4. Use Problem 3 to show that $|\sin(x) - \sin(y)| \leq |x - y|$ for all values of x and y.
5. For each of the following functions, identify the intervals where the function is increasing and where it is decreasing. Use this information to sketch the graph.

   (a) $f(x) = x^2 - 3$ (b) $g(t) = 3t^2 + t - 6$

   (c) $h(z) = \dfrac{1}{z-1}$ (d) $f(x) = \dfrac{1}{x^2+1}$

   (e) $f(t) = \dfrac{t}{t^2+1}$ (f) $g(x) = x^4 - x^3$

   (g) $y(t) = \dfrac{t}{t^2-4}$ (h) $f(x) = 4x^5 - 15x^4 - 20x^3 + 110x^2 - 120x$
6. Let $f(x) = |x|$. Then

$$\frac{f(1) - f(-1)}{1 - (-1)} = \frac{0}{2} = 0,$$

but there does not exist a point c in $(-1, 1)$ such that $f'(c) = 0$. Does this contradict the Mean Value Theorem?

7. Suppose f is continuous on $[a, b]$ and differentiable on (a, b).
   (a) Show that if $f'(x) > 0$ for all x in (a, b), then f is increasing on $[a, b]$.
   (b) Show that if $f'(x) < 0$ for all x in (a, b), then f is decreasing on $[a, b]$.

8. Suppose f and g are continuous on $[a, b]$, differentiable on (a, b), $f(a) = g(a)$, and $f'(x) < g'(x)$ for all x in (a, b). Show that $f(b) < g(b)$.

9. Show that $\sqrt{1+x} < 1 + \frac{1}{2}x$ for $x > 0$.

10. Show that $a + \frac{1}{a} < b + \frac{1}{b}$ whenever $1 < a < b$.

11. Find antiderivatives for the following functions.
    (a) $f(x) = 2x$
    (b) $g(t) = t^2$
    (c) $g(x) = \sin(x)$
    (d) $f(z) = \sin(2z)$
    (e) $h(x) = x^2 - 3x$
    (f) $f(x) = 3\cos(4x)$

12. Find all antiderivatives of $f(x) = 3x^2 - 3$ and plot the graphs of six of them.

13. Find all antiderivatives of $g(t) = \sin(2t)$ and plot the graphs of six of them.

14. If $f(x) = -\sin^2(x)$ and $g(x) = \cos^2(x)$, then $f'(x) = g'(x)$. What does this imply about the relationship between the functions f and g?

15. If $f(t) = \tan^2(t)$ and $g(t) = \sec^2(t)$, then $f'(t) = g'(t)$. Thus $f(t) = g(t) + k$ for some constant k. Evaluate f and g at $t = 0$ in order to determine k.

16. Suppose f is differentiable on an open interval containing the closed interval $[a, b]$.
    (a) Show that for any x in (a, b),
    $$f(x) = f(a) + f'(c)(x - a)$$
    for some point c with $a < c < x$.
    (b) Let f'' denote the *second derivative* of f. That is,
    $$f''(x) = \frac{d}{dx}f'(x).$$
    Assuming that f' is continuous on $[a, b]$ and differentiable on (a, b), show that there exists a point d with $a < d < c$ such that
    $$f(x) = f(a) + f'(a)(x - a) + f''(d)(c - a)(x - a).$$
    (c) Compare the results in (a) and (b) to the statement that $f(x) \approx T(x)$ for x close to a, where T is the best affine approximation to f at a.

(d) Let $h = x - a$. Show that

$$f(a+h) - T(a+h) = f''(d)(c-a)h.$$

(e) Assuming f'' is continuous on $[a, b]$, show that

$$\left| \frac{f(a+h) - T(a+h)}{h^2} \right| \leq M$$

for some constant M. This statement means that the remainder function

$$R(h) = f(a+h) - T(a+h)$$

is $O(h^2)$. That is, $R(h)$ goes to 0 as least as fast as h^2. Note that this is a stronger statement than the statement that $R(h)$ is $o(h)$.

Section 3.8

Finding Maximum and Minimum Values

Problems involving finding the maximum or minimum value of a quantity occur frequently in mathematics and in the applications of mathematics. A company may want to maximize its profit or minimize its costs; a farmer may want to maximize the yield from his crop or minimize the amount of irrigation equipment needed to water his fields; an airline may want to maximize its fuel efficiency or minimize the length of its routes. Methods for solving some optimization problems are so computationally intense that they challenge, and sometimes even go beyond, the fastest computers currently available. An example of such a problem is the famous traveling salesman problem, in which a salesman wishes to visit a certain set of cities using the shortest possible route. In this section we will not consider problems of this type, but rather we will confine ourselves to problems involving continuous functions of a single independent variable.

Closed intervals

We will start with the simplest case. Suppose f is a continuous function on a closed interval $[a, b]$. From the Extreme Value Theorem we know that f attains both a maximum value and a minimum value on the interval. We now look for candidates at which these values might occur. To start, an extreme value could occur at one of the endpoints. For example, the maximum value of $f(x) = x^2$ on $[0, 1]$ occurs at $x = 1$. If an extreme value occurs in the open interval (a, b) at a point c where f is differentiable, then f has a local extremum at c and so, from our work in Section 3.7, we know that $f'(c) = 0$. For example, the minimum value of $f(x) = x^2$ on $[-1, 1]$ occurs at $x = 0$ and $f'(0) = 0$. Finally, the only other candidates for the locations of extreme values would be points where f' is undefined. For example, the minimum value of $f(x) = |x|$ on $[-1, 1]$ occurs at $x = 0$, where f' is not defined. Hence we are led to the following conclusion: The extreme values of a continuous function f on a closed interval are located either at the endpoints of the interval, at points where f' is 0, or at points where f' is undefined. The following terminology will help us state this more easily.

Definition If f is differentiable at c and $f'(c) = 0$, then we call c a *critical point* or *stationary point* of f. A point c at which the derivative of f is not defined is called a *singular point* of f.

Thus we know that the candidates for the location of the extreme values of a continuous function on a closed interval fall into three categories: (a) endpoints of the interval, (b) critical points, and (c) singular points. To determine the extreme values of such a function f, we identify all these points, evaluate f at each one, and identify the largest and smallest values.

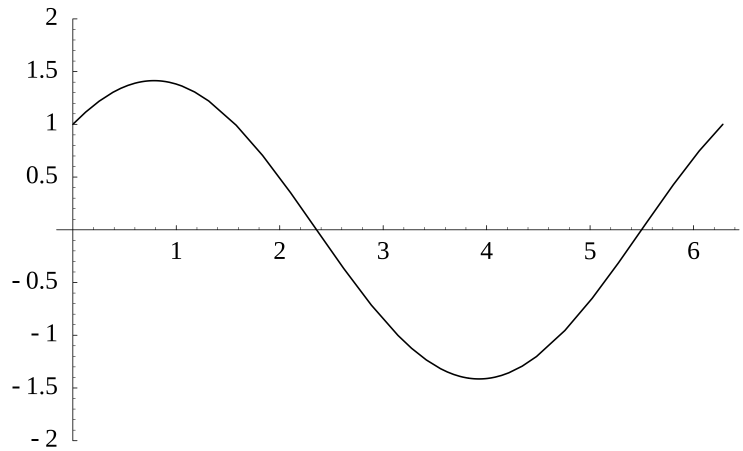


Figure 3.8.1 Graph of $g(t) = \cos(t) + \sin(t)$ on $[0, 2\pi]$

Example Suppose we wish to find the maximum and minimum values of

$$g(t) = \cos(t) + \sin(t)$$

on the interval $[0, 2\pi]$. Then

$$g'(t) = -\sin(t) + \cos(t),$$

so $g'(t) = 0$ when

$$\cos(t) = \sin(t). \tag{3.8.1}$$

Now $\cos(t)$ and $\sin(t)$ are never simultaneously 0, so we may divide both sides of (3.8.1) by $\cos(t)$ to see that $g'(t) = 0$ when

$$\tan(t) = 1.$$

Considering only the interval $[0, 2\pi]$, this implies $t = \frac{\pi}{4}$ or $t = \frac{5\pi}{4}$. Since there are no singular points, we evaluate g at the endpoints and at the critical points:

$$\begin{aligned} &g(0) = 1 \\ &g\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2} \\ &g\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2} \\ &g(2\pi) = 1 \end{aligned}$$

Thus g has a maximum value of $\sqrt{2}$ at $t = \frac{\pi}{4}$ and a minimum value of $-\sqrt{2}$ at $t = \frac{5\pi}{4}$. See Figure 3.8.1.

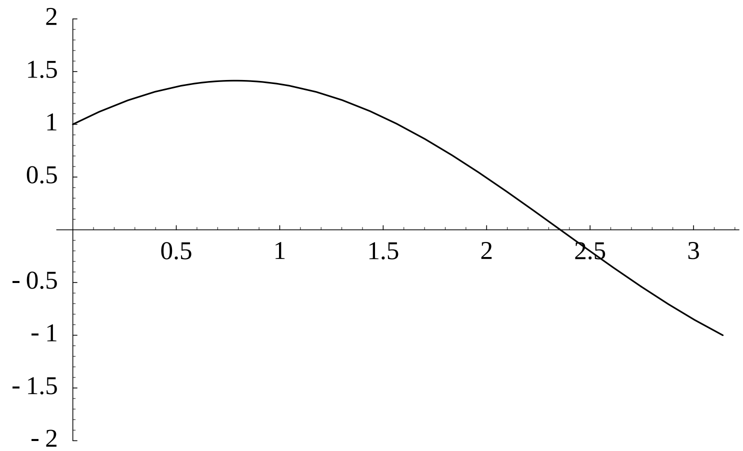


Figure 3.8.2 Graph of $g(t) = \cos(t) + \sin(t)$ on $[0, \pi]$

Example Note that if the interval in the previous example had been $[0, \pi]$, then the only critical point would be $\frac{\pi}{4}$. In this case we would evaluate:

$$g(0) = 1$$
$$g\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$$
$$g(\pi) = -1$$

Hence the maximum value of g on $[0, \pi]$ is, as before, $\sqrt{2}$ at $x = \frac{\pi}{4}$, but the minimum value of g on $[0, \pi]$ is -1 at $t = \pi$. See Figure 3.8.2.

Example Consider the function $f(x) = x^{\frac{2}{3}}$ on the interval $[-1, 1]$. Then

$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{x^{\frac{1}{3}}},$$

which is never 0, but is undefined at 0. Thus f has a singular point at 0, but no critical points in $[-1, 1]$. To find the extreme values of f, we evaluate:

$$f(-1) = 1$$
$$f(0) = 0$$
$$f(1) = 1$$

Hence f has a minimum value of 0 at $x = 0$ and a maximum value of 1 which occurs at both $x = -1$ and $x = 1$. See Figure 3.8.3.

Example A quality control engineer wishes to determine the probability that a certain type of light bulb will fail within 1000 hours of use. To do so, she tests 100 such light bulbs for 1000 hours each and finds that 20 of them failed within that time period. If p is the

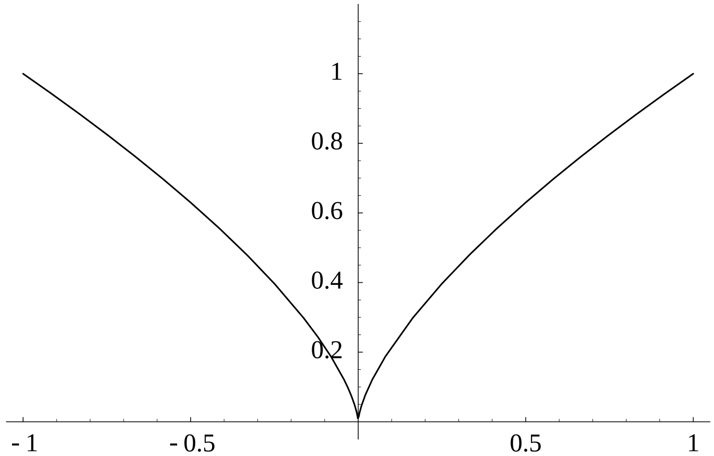


Figure 3.8.3 Graph of $f(x) = x^{\frac{2}{3}}$ on $[-1, 1]$

probability that a single light bulb fails within 1000 hours, the probability of the observed sequence of successes and failures is given by

$$L(p) = p^{20}(1-p)^{80},$$

with $0 \leq p \leq 1$. Note here that $1-p$ is the probability that a single light bulb does not fail in the 1000 hour test. If we think of p as representing the proportion of all such light bulbs that will fail within 1000 hours, then $L(p)$ represents the proportion of times that a sequence of 100 tests will yield the observed sequence of successes and failures. The quality control engineer would like to use this information to estimate p, the true probability of failure for this particular type of light bulb. One common procedure is to estimate p by the value $\hat{p}$ which maximizes the probability of the observed sequence; that is, $\hat{p}$ is the value of p that makes the given observations most likely to occur. Hence we want to maximize the function L on the interval $[0, 1]$. To find the critical points, we compute

$$\begin{aligned} L'(p) &= p^{20}(80(1-p)^{79}(-1)) + (1-p)^{80}(20p^{19}) \\ &= -80p^{20}(1-p)^{79} + 20p^{19}(1-p)^{80} \\ &= 20p^{19}(1-p)^{79}(-4p + (1-p)) \\ &= 20p^{19}(1-p)^{79}(1-5p). \end{aligned}$$

Thus $L(p) = 0$ when $p = 0$, $p = 0.2$, or $p = 1$, and so the only critical point in the interval $(0, 1)$ is 0.2. Evaluating L at the endpoints and at the critical point yields:

$$\begin{aligned} &L(0) = 0 \\ &L(0.2) = (0.2)^{20}(0.8)^{80} = 1.853 \times 10^{-22} \\ &L(1) = 0 \end{aligned}$$

Thus the quality control engineer would take the value $\hat{p} = 0.2$ as her estimate of the probability that this type of light bulb will fail within the first 1000 hours of use. See Figure 3.8.4.

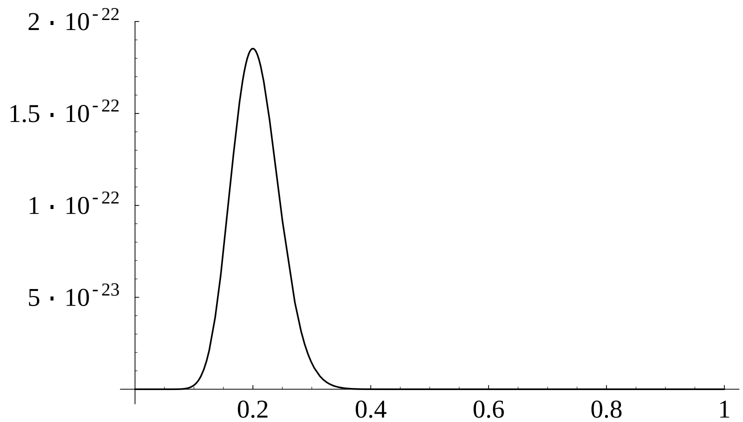


Figure 3.8.4 Graph of $L(p) = p^{20}(1-p)^{80}$ on $[0,1]$

The answer in the previous example should not be surprising: Given that 20 out of 100 light bulbs in the sample failed within 1000 hours, it seems evident that our best guess for the probability that a randomly chosen light bulb manufactured by this process will fail in less than 1000 hours is $\frac{20}{100} = \frac{1}{5}$. However, our example illustrates a general technique, called *maximum likelihood estimation*, which is widely used in applications to estimate statistical parameters. Moreover, there are other methods which would yield different answers, one popular alternative being $\frac{21}{102}$.

Open intervals

Finding the extreme values of a continuous function f on an interval I which is not closed introduces some new problems. Foremost among these is that there is no guarantee, like the Extreme Value Theorem, that extreme values even exist. However, the following special case arises frequently in practice and may be handled routinely. Suppose $I = (a, b)$ is an open interval, f and f' are continuous on I, f has a critical point at a point c in I which has been determined to be the location of a local minimum, and f has no other critical points in I. Then, by the Intermediate Value Theorem, f' cannot change sign on (a, c). Hence, since there is a local minimum at c, $f'(x) < 0$ for all x in (a, c). Similarly, we must have $f'(x) > 0$ for all x in (c, b). Thus f is decreasing on (a, c) and increasing on (c, b), so $f(c)$ must be the minimum value of f on (a, b). An analogous argument shows that if, under these conditions, f has a local maximum at c, then $f(c)$ is the maximum value of f on (a, b). The following proposition summarizes these statements.

Proposition Suppose f and f' are continuous on an open interval (a, b) and c is the only critical point of f in (a, b). If f has a local minimum at c, then $f(c)$ is the minimum value of f on (a, b); if f has a local maximum at c, then $f(c)$ is the maximum value of f on (a, b).

Of course, to make use of the proposition, we must first have a method for determining the location of local extreme values. Given a differentiable function f on an open interval I, we know that the local extreme values will occur only at critical points; hence, the first step in determining the location of local extreme values is to find all the critical points of

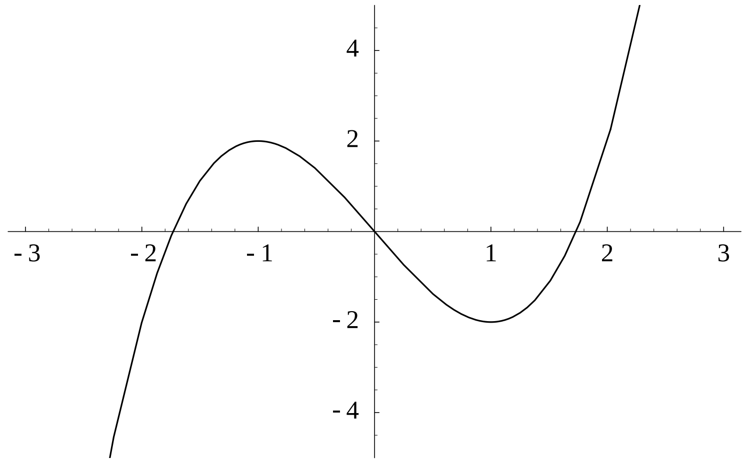


Figure 3.8.5 Graph of $f(x) = x^3 - 3x$

f in I. A given critical point c may then be classified as the location of a local maximum, a local minimum, or neither by examining the behavior of f' on either side of c. That is, if f' is negative to the left of c and positive to the right of c, then f is decreasing before c and increasing after c, making c the location of a local minimum. Conversely, if f' is positive to the left of c and negative to the right of c, then f is increasing before c and decreasing after c, making c the location of a local maximum. If f' is either negative on both sides of c or positive on both sides of c, then f has neither a local minimum nor a local maximum at c. The procedure just described is sometimes referred to as the *first derivative test for local extrema.*

Example Suppose $f(x) = x^3 - 3x$. Then

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x-1)(x+1),$$

so $f'(x) = 0$ when $x = -1$ or $x = 1$. Now $x^2 - 1 < 0$ when $-1 < x < 1$ and $x^2 - 1 > 0$ for all other values of x. Hence $f'(x) > 0$ when $x < -1$ or $x > 1$, and $f'(x) < 0$ when $-1 < x < 1$. Hence f is increasing on $(-\infty, -1)$ and on $(1, \infty)$, and f is decreasing on $(-1, 1)$. Thus f changes from increasing to decreasing at $x = -1$, implying that f has a local maximum at this point, and f changes from decreasing to increasing at $x = 1$, implying that f has a local minimum at this point. Since $f(-1) = 2$ and $f(1) = -2$, we conclude that f has a local maximum of 2 at $x = -1$ and a local minimum of -2 at $x = 1$. Note that this verifies the claim we made in Section 3.7 after looking at the graph of f (which is repeated in Figure 3.8.5).

Our next step will require the introduction of the derivative of f', called the *second derivative* of f, and denoted f''.

Example If $f(x) = x^3 - 3x^2$, then

$$f'(x) = 3x^2 - 6x$$

and

$$f''(x) = 6x - 6.$$

In Leibniz notation, if $y = f(x)$, then $f''(x)$ is denoted by $\frac{d^2y}{dx^2}$, which may be thought of as

$$\frac{d^2}{dx^2}y = \frac{d}{dx}\left(\frac{d}{dx}y\right).$$

In Newton's notation, if $x = f(t)$, then $\ddot{x} = f''(t)$.

Example If $y = x\sin(3x)$, then

$$\frac{dy}{dx} = 3x\cos(3x) + \sin(3x),$$

so

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}(3x\cos(3x) + \sin(3x)) \\ &= -9x\sin(3x) + 3\cos(3x) + 3\cos(3x) \\ &= -9x\sin(3x) + 6\cos(3x).\end{aligned}$$

Example If $x = 3t^6 - 2\cos(6t)$, then

$$\dot{x} = 18t^5 + 12\sin(6t)$$

and

$$\ddot{x} = 90t^4 + 72\cos(6t).$$

Now suppose f, f', and f'' are all continuous on an interval (a, b) containing a critical point c. Moreover, suppose $f''(c) < 0$. Since f'' is continuous, this assumption in fact implies that $f''(c) < 0$ on some open interval about c, and hence that f' is a decreasing function on some open interval about c. But $f'(c) = 0$, so for f' to be decreasing it must be the case that f' is positive to the left of c and negative to the right of c. This means, by the first derivative test discussed above, that f must have a local maximum at c. Similarly, if $f''(c) > 0$, then f' is negative to the left of c and positive to the right of c, showing that f has a local minimum at c. This important result is known as the *second derivative test for local extrema*

Second Derivative Test Suppose f, f', and f'' are all continuous on an open interval (a, b) and that c is critical point of f in (a, b). Then f has a local maximum at c if $f''(c) < 0$ and f has a local minimum at c if $f''(c) > 0$.

Example Consider the function $g(x) = \dfrac{x}{1 + x^2}$. Then

$$g'(x) = \frac{(1 + x^2)(1) - (x)(2x)}{(1 + x^2)^2} = \frac{1 - x^2}{(1 + x^2)^2},$$

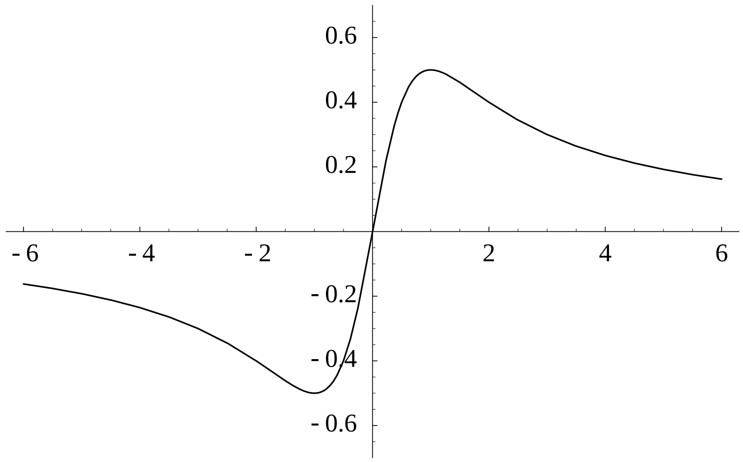


Figure 3.8.6 Graph of $g(x) = \dfrac{x}{1+x^2}$

so $g'(x) = 0$ when $1 - x^2 = 0$. Thus g has two critical points, $x = -1$ and $x = 1$. Now

$$\begin{aligned} g''(x) &= \frac{(1+x^2)^2(-2x) - (1-x^2)(2(1+x^2)(2x))}{(1+x^2)^4} \\ &= \frac{-2x(1+x^2) - 4x(1-x^2)}{(1+x^2)^3} \\ &= \frac{-6x + 2x^3}{(1+x^2)^3} \\ &= \frac{2x(x^2-3)}{(1+x^2)^3}. \end{aligned}$$

Thus $g''(-1) = 0.5 > 0$ and $g(1) = -1 < 0$, implying that g has a local minimum at $x = -1$ and a local maximum at $x = 1$. Since $g(-1) = -0.5$ and $g(1) = 0.5$, we conclude that g has a local minimum of -0.5 at $x = -1$ and a local maximum of 0.5 at $x = 1$.

Using the facts that the critical points of g are -1 and 1 and that g has a local minimum at $x = -1$, we may conclude that g must be decreasing on $(-\infty, -1)$ and increasing on $(-1, 1)$. Similarly, g having a local maximum at 1 implies that g must be increasing on $(-1, 1)$ and decreasing on $(1, \infty)$. Moreover,

$$\lim_{x \to -\infty} g(x) = \lim_{x \to -\infty} \frac{x}{1+x^2} = \lim_{x \to -\infty} \frac{\frac{1}{x}}{\frac{1}{x^2} + 1} = \frac{0}{1} = 0$$

and

$$\lim_{x \to \infty} g(x) = \lim_{x \to \infty} \frac{x}{1+x^2} = \lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{1}{x^2} + 1} = \frac{0}{1} = 0,$$

showing that the x-axis is a horizontal asymptote for the graph of g. Putting these observations together, we can see why the graph of g looks as it does in Figure 3.8.6.

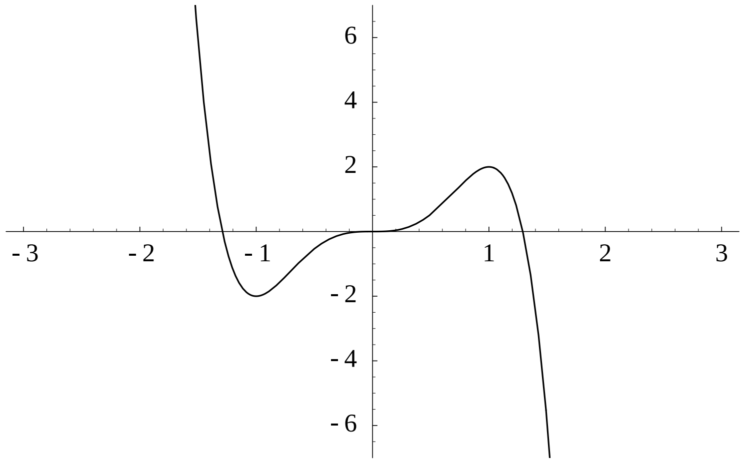


Figure 3.8.7 Graph of $f(x) = 5x^3 - 3x^5$

Example Suppose $f(x) = 5x^3 - 3x^5$. Then

$$f'(x) = 15x^2 - 15x^4 = 15x^2(1 - x^2) = 15x^2(1 - x)(1 + x),$$

from which we see that the critical points of f are -1, 0, and 1. Now

$$f''(x) = 30x - 60x^3,$$

so $f''(-1) = 30 > 0$, $f''(0) = 0$, and $f''(1) = -30 < 0$. Thus f has a local minimum of -2 at $x = -1$ and a local maximum of 2 at $x = 1$. Unfortunately, $f''(0) = 0$, so the second derivative test gives us no information about the nature of the critical point 0. However, since f has a local minimum at $x = -1$, f must be decreasing on $(-\infty, -1)$ and increasing on $(-1, 0)$; moreover, since f has a local maximum at $x = 1$, f must be increasing on $(0, 1)$ and decreasing on $(1, \infty)$. Thus f is increasing on both $(-1, 0)$ and $(1, 0)$, from which we conclude that f has neither a local minimum nor a local maximum at $x = 0$. If we add in that $f(0) = 0$,

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} x^5 \left(\frac{5}{x^2} - 3 \right) = \infty,$$

and

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} x^5 \left(\frac{5}{x^2} - 3 \right) = -\infty,$$

we can see why the graph of f look as it does in Figure 3.8.7.

It is worth emphasizing that, for a critical point c, $f''(c) = 0$ gives us no information about the nature of the critical point. The point may be the location of a local minimum, as 0 is for $f(x) = x^4$; a local maximum, as 0 is for $f(x) = -x^4$; or neither, as 0 is in the previous example. Thus, if $f''(c) = 0$ for a critical point c, the second derivative test is not applicable and some other method, such as the first derivative test, must be used to determine the nature of the point.

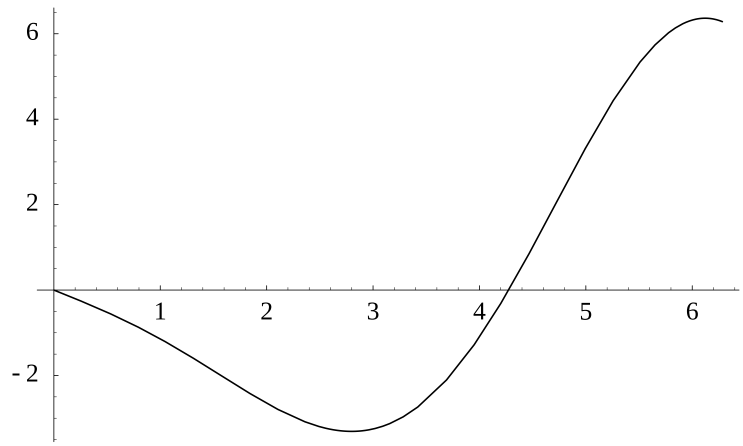


Figure 3.8.8 Graph of $g(t) = t\cos(t) - 2\sin(t)$

Now that we have techniques for determining the location of local minimums and maximums, we can return to our original problem of determining the maximum or minimum value of a function on an open interval.

Example Suppose we want to find the minimum value of

$$f(t) = \frac{\sin(t)}{t^2}$$

on the interval $(0, 2\pi)$. Then

$$f'(t) = \frac{t^2\cos(t) - \sin(t)(2t)}{t^4} = \frac{t\cos(t) - 2\sin(t)}{t^3},$$

so $f'(t) = 0$ when $t\cos(t) - 2\sin(t) = 0$. We cannot solve this equation exactly, but, from the graph of $g(t) = t\cos(t) - 2\sin(t)$ in Figure 3.8.8, we can see that it has only one solution in the interval $(0, 2\pi)$. Applying Newton's method with initial guess $t_0 = 4$, we obtain the approximation 4.2748, to four decimal places. Hence f has exactly one critical point in $(0, 2\pi)$, namely, 4.2748. Now

$$\begin{aligned} f''(t) &= \frac{t^3(-t\sin(t) + \cos(t) - 2\cos(t)) - (t\cos(t) - 2\sin(t))(3t^2)}{t^6} \\ &= \frac{-t^4\sin(t) - 4t^3\cos(t) + 6t^2\sin(t)}{t^6} \\ &= \frac{6\sin(t) - 4t\cos(t) - t^2\sin(t)}{t^4}, \end{aligned}$$

so $f''(4.2748) = 0.05499 > 0$. Thus f has a local minimum at $t = 4.2748$. Moreover, since 4.2748 is the only critical point in $(0, 2\pi)$ and $f(4.2748) = -0.04957$, we may conclude that the minimum value of f on $(0, 2\pi)$ is -0.04957, and this value occurs at $t = 4.2748$. See Figure 3.8.9.

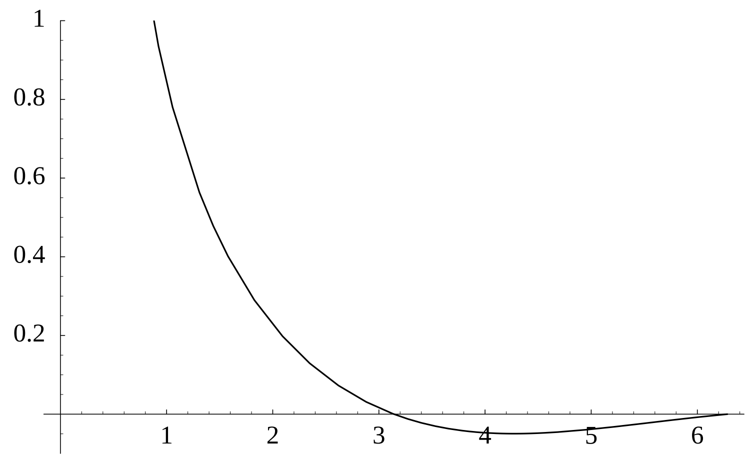


Figure 3.8.9 Graph of $f(t) = \dfrac{\sin(t)}{t^2}$

Example A company wishes to produce a metal can in the shape of a right circular cylinder which minimizes the amount of metal needed in its construction, yet will have a volume of V cubic centimeters. If we denote the radius of the base of the can by r, the height of the can by h, and the surface area of the can by S, then

$$S = 2\pi r^2 + 2\pi rh, \tag{3.8.2}$$

where the first term represents the combined area of the base and the top of the can and the second term represents the area of the side of the can, which, when flattened out, is a rectangle of length $2\pi r$ (the circumference of the base of the can) and width h. Our goal is to find the values of r and h which minimize S subject to the constraint that the can has to hold a volume V. This constraint translates into the condition

$$V = \pi r^2 h,$$

which means we must have

$$h = \frac{V}{\pi r^2}. \tag{3.8.3}$$

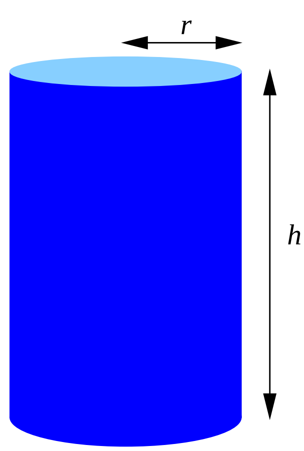


Figure 3.8.10 A cylindrical can

Substituting the value of h in (3.8.3) into (3.8.2), we have

$$S = 2\pi r^2 + \frac{2V}{r}, \tag{3.8.4}$$

giving S as a function of r which we want to minimize on the interval $(0, \infty)$. Now

$$\frac{dS}{dr} = 4\pi r - \frac{2V}{r^2},$$

so $\frac{dS}{dr} = 0$ when

$$4\pi r = \frac{2V}{r^2},$$

that is, when

$$r^3 = \frac{V}{2\pi}.$$

Hence

$$r = \sqrt[3]{\frac{V}{2\pi}}$$

is the only critical point of S in $(0, \infty)$. Moreover,

$$\frac{d^2S}{dr^2} = 4\pi + \frac{4V}{r^3},$$

so

$$\left.\frac{d^2S}{dr^2}\right|_{r=\sqrt[3]{\frac{V}{2\pi}}} = 4\pi + 8\pi = 12\pi > 0.$$

Hence S has a local minimum at

$$r = \sqrt[3]{\frac{V}{2\pi}}.$$

Since this is the only critical point in $(0, \infty)$, this is in fact the location of the minimum value of S. Now for

$$r = \sqrt[3]{\frac{V}{2\pi}},$$

we have, using (3.8.3),

$$h = \frac{V}{\pi r^2} = \frac{V}{\pi\left(\frac{V}{2\pi}\right)^{\frac{2}{3}}} = \frac{2^{\frac{2}{3}}\pi^{\frac{2}{3}}V}{\pi V^{\frac{2}{3}}} = 2\sqrt[3]{\frac{V}{2\pi}} = 2r.$$

That is, the surface area of the can, and hence the amount of metal used in the can, is minimized when the height of the can is equal to the diameter of the can. Figure 3.8.11

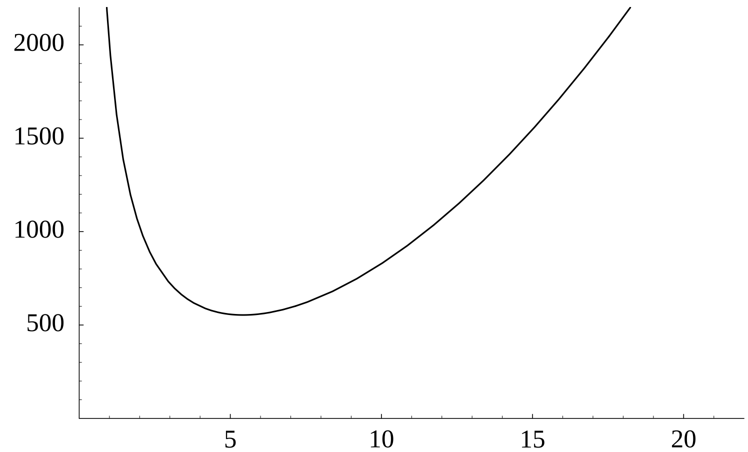


Figure 3.8.11 Graph of $S = 2\pi r^2 + \dfrac{2000}{r}$

shows the graph of S in the case $V = 1000$ cubic centimeters, in which case the surface area is minimized when r is approximately 5.42 centimeters.

Problems

1. Find the minimum and maximum values, and their locations, for each of the following functions on the given intervals.

   (a) $f(x) = x^2 - 4$ on $[-3, 4]$
   (b) $f(x) = x^3 - 3x$ on $[-2, 4]$
   (c) $g(t) = \cos(t) - \sin(t)$ on $[-\pi, \pi]$
   (d) $f(t) = 2t^3 + 3t^2 - 36t$ on $[-4, 3]$
   (e) $g(x) = x^2 \cos(x)$ on $[-2, 2]$
   (f) $f(t) = \cos(t) + \sin(2t)$ on $[0, \pi]$
   (g) $f(t) = t^2 \sin(t)$ on $[0, \pi]$

2. A farmer wishes to fence in a rectangular field with 600 yards of fencing. What should the dimensions of the field be in order to maximize the area of the field?

3. A farmer wishes to fence in a rectangular field, using a straight river for one side, with 500 yards of fencing. What should the dimensions of the field be in order to maximize the area of the field?

4. Suppose the farmer in Problem 2 wishes to divide his field into two equal rectangular fields using a fence parallel to two of the sides. What should the dimensions of the field be in order to maximize the combined areas of the fields?

5. When a potter sells his pots for p dollars apiece, he can sell $D(p) = 2500 - p^2$ of them. Suppose the pots cost him \$6.00 apiece to make. What price should the potter charge in order to maximize his profit?

6. A wire of unit length is to be cut into two pieces. One of the pieces will be used to form a square, the other a circle. Where should the wire be cut in order to maximum the total area enclosed by the square and the circle? Where should it be cut in order to minimize the total area enclosed by the square and the circle?

7. Find all local maximums and minimums, and their locations, for the following functions.

   (a) $f(x) = 3x^2 + 5$
   (b) $f(t) = t^4 + 3t^2$
   (c) $g(t) = t^3 + 3t^2$
   (d) $g(x) = \sin(x)\cos(x)$
   (e) $f(x) = \dfrac{1}{1+x^2}$
   (f) $h(x) = \dfrac{x^2}{1+x^2}$
   (g) $g(x) = x^5 - x^3$
   (h) $f(t) = t^4 - 2t^3$
   (i) $g(t) = 3t^5 - 5t^4$
   (j) $f(x) = \dfrac{x}{1+3x^2}$

8. Find the second derivative of each of the following functions.

   (a) $f(x) = 3x^2 + 2x - 3$
   (b) $g(t) = 13t^4 - 3t^3 + t^2 - 45$
   (c) $s = \dfrac{1}{2t-1}$
   (d) $g(x) = \sin^2(3x)$
   (e) $x = \sin(2t)\cos(4t)$
   (f) $y = x^2 \tan^2(3x)$

9. Find the maximum value of
$$f(x) = \frac{x-1}{x^2}$$
on the interval $(0, \infty)$. Does f have a minimum value on $(0, \infty)$?

10. We found the minimum value of
$$f(t) = \frac{\sin(t)}{t^2}$$
on $(0, 2\pi)$ in an example. Does f have a maximum value on $(0, 2\pi)$?

11. A farmer wishes to construct a rectangular storage bin with a volume of 1000 cubic feet. Both the top and the bottom of the bin are to be squares. Find the dimensions of the bin which will minimize its surface area.

12. Suppose the bin in Problem 11 does not require a bottom. Find the dimensions of the bin which minimize surface area in this case.

13. Suppose the material for the top and the bottom of the bin in Problem 11 costs \$2.00 per square foot while the material for the sides costs \$3.00 per square foot. Find the dimensions of the bin which minimize its cost.

14. A metal can in the shape of a right circular cylinder without a top is to be made so that it holds 100 cubic centimeters. Find the dimensions of the can which minimize its surface area.

15. Suppose the material for the top and bottom of a can in the shape of a right circular cylinder costs \$0.04 per square centimeter and the material for the side costs \$0.02 per square centimeter. If the can must hold 1000 cubic centimeters, for what dimensions is the cost of the can minimized?

16. A metal can in the shape of a right circular cylinder is to be made so that it holds 500 cubic centimeters. Suppose the top and bottom of the can are cut from square pieces of metal, with the scraps being discarded afterwards. Assuming there is no waste in making the side of the can, find the dimensions of the can which minimize the amount of material needed to make it.

17. Show that the rectangle with maximum area for a given perimeter P is a square.

18. Show that the rectangle with minimum perimeter for a given area A is a square.

19. A quality control engineer is studying the failure rate of a certain type of beam under stress. Test beams are put under a steady stress for 1000 hours. If p is the probability that the beam passes the test, then $1 - p$ is the probability that the beam fails the test. Suppose that in 50 trials, only 5 beams fail the test.

    (a) If $L(p)$ is the probability of the observed sequence of successes and failures, explain why
    $$L(p) = p^{45}(1-p)^5$$
    for $0 \leq p \leq 1$.

    (b) If $\hat{p}$ is the value of p which maximizes $L(p)$ on $[0, 1]$, show that $\hat{p} = 0.9$.

20. According to genetic theory, if a parent provides the gene A with probability θ and the gene a with probability $1 - \theta$, then the offspring is of genotype AA with probability θ^2, of genotype Aa with probability $2\theta(1-\theta)$, and of genotype aa with probability $(1-\theta)^2$. Suppose that in a sample of 100 people, 31 were observed to be of type AA, 48 of type Aa, and 21 of type aa. Let $L(\theta)$ be the probability of observing this specific sequence of genotypes.

    (a) Explain why
    $$L(\theta) = 2^{48}\theta^{110}(1-\theta)^{90}$$
    for $0 \leq \theta \leq 1$.

    (b) Find the value of θ which maximizes $L(\theta)$ on $[0, 1]$. What are the corresponding values for the probabilities of AA, Aa, and aa?

21. In the final example of this section, we showed that the surface area of a right circular cylindrical can is minimized when the height of the can is equal to the diameter of the can. Check out a local supermarket to see how many cans satisfy this condition. What other considerations might be important in the design of a can?

22. We have seen that if $x(t)$ is the position of an object moving on a straight line at time t, then the velocity of the object is given by $v(t) = \dot{x}(t)$ and the acceleration is given by $a(t) = \dot{v}(t)$. Hence a is the second derivative of x; that is, $a(t) = \ddot{x}(t)$. Suppose an object is oscillating at the end of a spring so that its position at time t is $x = 3\sin(\pi t)$.

    (a) Find $v(t)$.

    (b) Find $a(t)$.

    (c) Discuss the behavior of the object over the interval $[0, 2]$, taking into account the values of $x(t)$, $v(t)$, and $a(t)$.

Section 3.9

The Geometry of Graphs

In Section 2.1 we discussed the graph of a function $y = f(x)$ in terms of plotting points $(x, f(x))$ for many different values of x and connecting the resulting points with straight lines. This is a standard procedure when using a computer and, if the function is well behaved and sufficiently many points are plotted, will produce a reasonable picture of the graph. However, as we noted at that time, this method assumes that the behavior of the graph between any two successive points is approximated well by a straight line. With a sufficient number of points and a differentiable function, this assumption will be reasonable. Yet to understand a graph fully, it is important to have alternative techniques to verify the picture at least qualitatively. We have already developed several important aids for understanding the shape of a graph, including techniques for determining the location of local extreme values and techniques for finding intervals where the function is increasing and intervals where it is decreasing. In this section we will use this information, along with additional information contained in the second derivative, to piece together a picture of the graph of a given function.

To see the importance of the second derivative, consider the graphs of $f(x) = x^2$ and $g(x) = \sqrt{x}$ on the interval $(0, \infty)$. Now

$$f'(x) = 2x$$

and

$$g'(x) = \frac{1}{2\sqrt{x}},$$

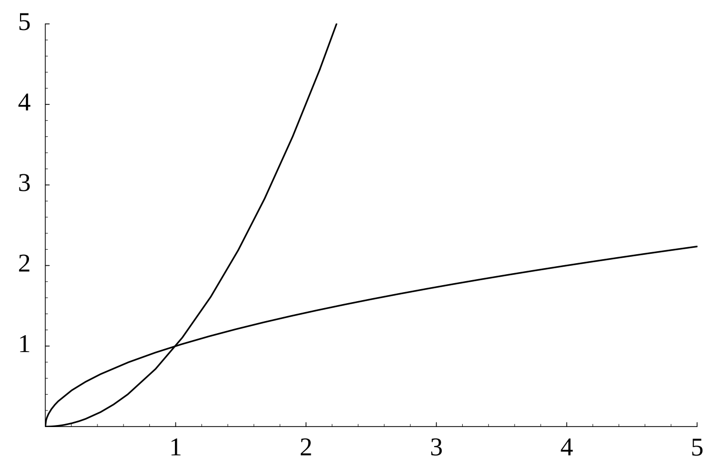


Figure 3.9.1 Graphs of $y = x^2$ and $y = \sqrt{x}$

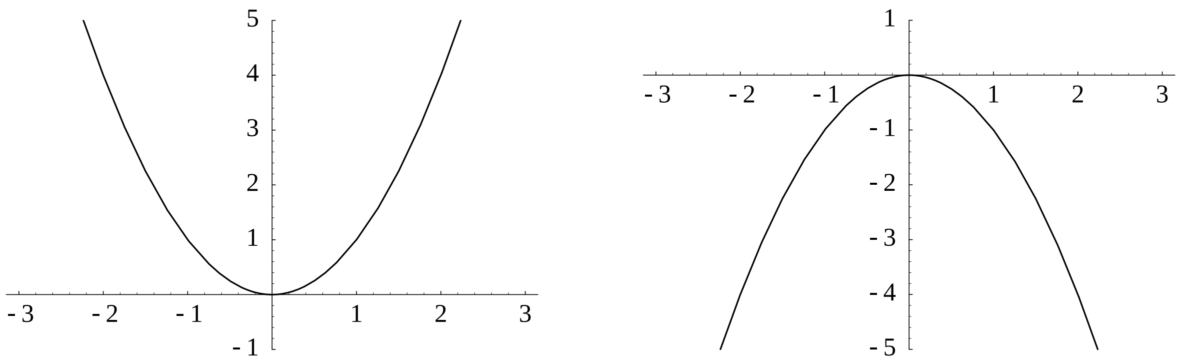


Figure 3.9.2 Graphs of $y = x^2$ and $y = -x^2$ on $(-\infty, \infty)$

so $f'(x) > 0$ and $g'(x) > 0$ for all x in $(0, \infty)$. Thus f and g are both increasing on $(0, \infty)$. However, the graphs of f and g, as shown in Figure 3.9.1, are dramatically different. The graph of f is not only increasing, but is becoming steeper and steeper as x increases, whereas the graph of g is increasing, but flattening out as x increases. In other words, f' is itself an increasing function, causing the rate of growth of the function to increase with x, while g' is a decreasing function, resulting in a decrease in the rate of growth of g and a flattening out of the graph. In the terminology of the next definition, we say that the graph of f is concave up on $(0, \infty)$ and the graph of g is concave down on $(0, \infty)$.

Definition Suppose f is differentiable on the open interval (a, b). If f' is an increasing function on (a, b), then we say the graph of f is *concave up* on (a, b). If f' is a decreasing function on (a, b), then we say the graph of f is *concave down* on (a, b).

Of course, to check for the intervals where f' is increasing and the intervals where f' is decreasing, we consider where f'', the derivative of f', is positive and where it is negative.

Proposition Suppose f is twice differentiable on the interval (a, b). If $f''(x) > 0$ for all x in (a, b), then the graph of f is concave up on (a, b); if $f''(x) < 0$ for all x in (a, b), then the graph of f is concave down on (a, b).

Example Two basic examples to keep in mind are $f(x) = x^2$ and $g(x) = -x^2$. Since $f''(x) = 2 > 0$ and $g''(x) = -2 < 0$ for all values of x, the graph of f is concave up on $(-\infty, \infty)$ and the graph of g is concave down on $(-\infty, \infty)$. See Figure 3.9.2.

Example Consider $g(t) = t^3$. Then $g''(t) = 6t$, so $g''(t) < 0$ when $t < 0$ and $g''(t) > 0$ when $t > 0$. Hence the graph of g is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$. Notice in Figure 3.9.3 how, even though g is increasing on $(-\infty, \infty)$, the change in concavity at $(0, 0)$ changes the shape of the graph.

Definition A point on the graph of a function f where the concavity changes from up to down or from down to up is called an *inflection point*.

Example In our previous example, $(0, 0)$ is an inflection point for the graph of $g(t) = t^3$.

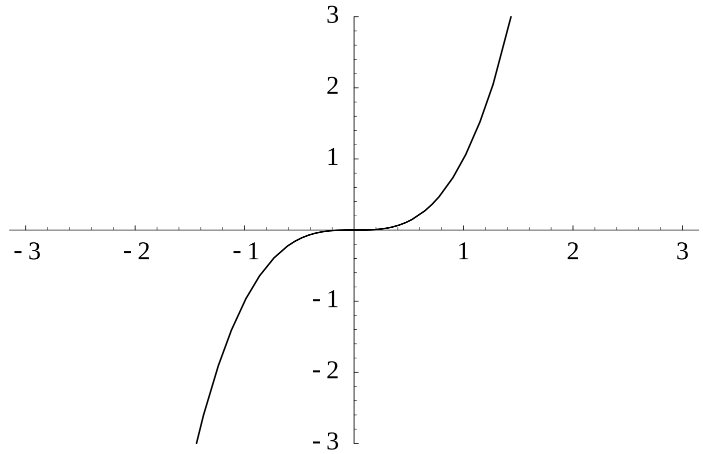


Figure 3.9.3 Graph of $g(t) = t^3$

Example Let $f(x) = \frac{1}{x}$. Then

$$f'(x) = -\frac{1}{x^2}$$

and

$$f''(x) = \frac{2}{x^3}.$$

Hence $f'(x) < 0$ on both $(-\infty, 0)$ and $(0, \infty)$, while $f''(x) < 0$ when $x < 0$ and $f''(x) > 0$ when $x > 0$. Thus f is decreasing on both $(-\infty, 0)$ and $(0, \infty)$, but the fact that the graph is concave down on $(-\infty, 0)$ shows up in the way the steepness of the graph increases as x approaches 0 from the right, while the fact that the graph is concave up on $(0, \infty)$ shows up in the way the graph flattens out as x increases toward ∞. See Figure 3.9.4. Also note that, although the concavity of the graph of f changes, the graph does not have an inflection point since f is not defined at 0.

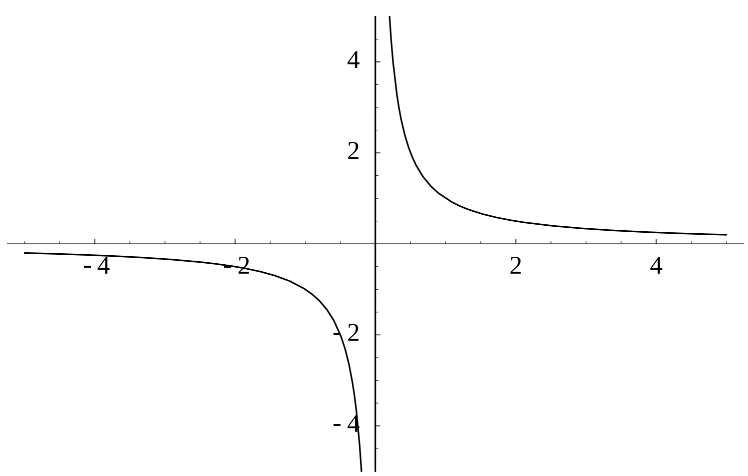


Figure 3.9.4 Graph of $f(x) = \frac{1}{x}$

Note that if $(c, f(c))$ is an inflection point on the graph of a function f, then either $f''(c) = 0$ or f'' is not defined at c. However, the converse does not hold. For example, if $f(x) = x^4$, then $f''(0) = 0$, even though $f''(x) = 12x^2$ is positive for all x in both $(-\infty, 0)$ and $(0, \infty)$.

From the foregoing, it is clear that f' and f'' provide enough information to obtain a good understanding of the shape of the graph of f. Specifically, to sketch the graph of f, we use the first derivative to find (1) intervals where f is increasing, (2) intervals where f is decreasing, and (3) locations of any local extreme values; we use the second derivative to find (1) intervals where the graph of f is concave up, (2) intervals where the graph f is concave down, and (3) any inflection points. Combining this information with a few values of the function, the location of any asymptotes, and information on the behavior of $f(x)$ as x goes to $-\infty$ and as x goes to ∞, we can piece together a qualitatively accurate picture of the graph of f.

Example Consider $f(x) = 3x^2 - x^3 + 2$. Then

$$f'(x) = 6x - 3x^2 = 3x(2 - x),$$

so the critical points of f are 0 and 2. Since $f'(-1) = -9 < 0$, $f'(1) = 3 > 0$, and $f'(3) = -9 < 0$, f is decreasing on the intervals $(-\infty, 0)$ and $(2, \infty)$ and increasing on $(0, 2)$. Moreover, this shows that f has a local minimum of 2 at $x = 0$ and a local maximum of 6 at $x = 2$.

Next, we have

$$f''(x) = 6 - 6x = 6(1 - x),$$

so $f''(x) = 0$ when $x = 1$. Now $1 - x > 0$ when $x < 1$ and $1 - x < 0$ when $x > 1$, so $f''(x) > 0$ on $(-\infty, 1)$ and $f''(x) < 0$ on $(1, \infty)$. Hence the graph of f is concave up on $(\infty, 1)$and concave down on $(1, \infty)$, and $(1, 4)$ is an inflection point.

Combining this information with the values $f(-1) = 6$, $f(3) = 2$,

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} (3x^2 - x^3 + 2) = \lim_{x\to-\infty} x^3 \left(\frac{3}{x} - 1 + \frac{2}{x^3}\right) = \infty,$$

and

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} (3x^2 - x^3 + 2) = \lim_{x\to\infty} x^3 \left(\frac{3}{x} - 1 + \frac{2}{x^3}\right) = -\infty,$$

we can easily draw a graph which, even though we are only plotting five points (the two local extreme values, the inflection point, and one point on each side of these points), captures the shape of the graph of f very well. See Figure 3.9.5.

Example Consider $g(x) = 12x^5 + 15x^4 - 40x^3 - 10$. Then

$$g'(x) = 60x^4 + 60x^3 - 120x^2 = 60x^2(x^2 + x - 2) = 60x^2(x + 2)(x - 1),$$

implying that g has three critical points, namely, $x = -2$, $x = 0$, and $x = 1$. Now $60x^2 \geq 0$ for all values of x; $x + 2 < 0$ when $x < -2$ and $x + 2 > 0$ when $x > -2$; and $x - 1 < 0$

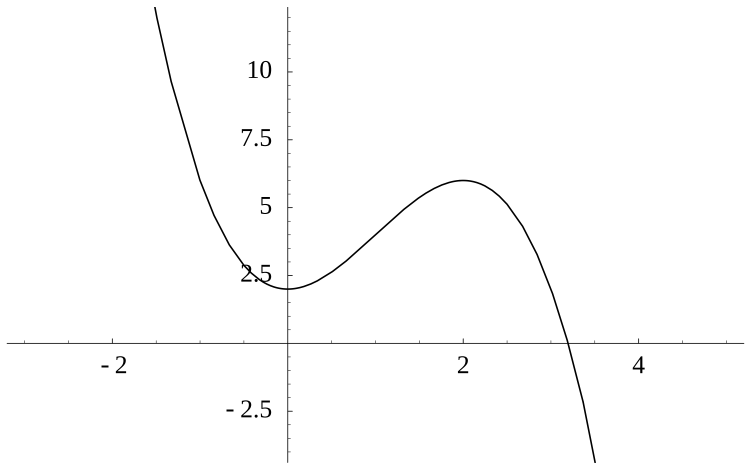


Figure 3.9.5 Graph of $f(x) = 3x^2 - x^3 + 2$

when $x < 1$ and $x - 1 > 0$ when $x > 1$. Thus $g'(x) > 0$ when $x < -2$ and when $x > 1$, and $g'(x) < 0$ when $-2 < x < 0$ and when $0 < x < 1$. So g is increasing on $(-\infty, -2)$ and $(1, \infty)$, and g is decreasing on $(-2, 0)$ and $(0, 1)$. In particular, g has a local maximum of 166 at $x = -2$ and a local minimum of -23 at $x = 1$. Although g has neither a local maximum nor a local minimum at the critical point 0, for drawing the graph of g it is important to note that the slope of the curve is 0 at $(0, -10)$.

Next,

$$g''(x) = 240x^3 + 180x^2 - 240x = 60x(4x^2 + 3x - 4),$$

so $g''(x) = 0$ when $x = 0$ and when $x^2 + 3x - 4 = 0$. Using the quadratic formula, the latter equation has solutions

$$x = \frac{-3 - \sqrt{73}}{8} = -1.4430$$

and

$$x = \frac{-3 + \sqrt{73}}{8} = 0.6930,$$

rounding to four decimal places. Now $4x^2 + 3x - 4 < 0$ only when x is between the two roots -1.4430 and 0.6930. Since $60x < 0$ when $x < 0$ and $60x > 0$ when $x > 0$, we may conclude that $g''(x) < 0$ for $x < -1.4430$ and $0 < x < 0.6930$, and $g''(x) > 0$ for $-1.4430 < x < 0$ and $x > 0.6930$. Hence the graph of g is concave down on $(-\infty, -1.4430)$ and $(0, 0.6930)$ and concave up on $(-1.4430, 0)$ and $(0.6930, \infty)$. In particular, g has three inflection points: $(-1.4430, 100.1459)$, $(0, -10)$, and $(0.6930, -17.9349)$.

Adding to this information the values $g(-3) = -631$, $g(2) = 294$,

$$\begin{aligned} \lim_{x \to -\infty} g(x) &= \lim_{x \to -\infty} (12x^5 + 15x^4 - 40x^3 - 10) \\ &= \lim_{x \to -\infty} x^5 \left(12 + \frac{15}{x} - \frac{40}{x^2} - \frac{10}{x^5}\right) \\ &= -\infty, \end{aligned}$$

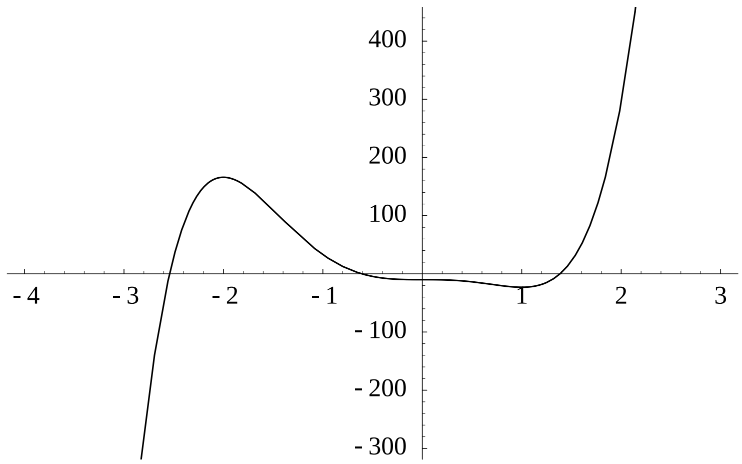


Figure 3.9.6 Graph of $g(x) = 12x^5 + 15x^4 - 40x^3 - 10$

and

$$\begin{aligned}\lim_{x\to\infty} g(x) &= \lim_{x\to\infty}(12x^5 + 15x^4 - 40x^3 - 10)\\ &= \lim_{x\to\infty} x^5\left(12 + \frac{15}{x} - \frac{40}{x^2} - \frac{10}{x^5}\right)\\ &= \infty,\end{aligned}$$

we can now sketch the graph of g. See Figure 3.9.6.

Example For our final example, consider

$$h(t) = \frac{t^2}{t^2 - 1}.$$

Then

$$h'(t) = \frac{(t^2-1)(2t) - (t^2)(2t)}{(t^2-1)^2} = -\frac{2t}{(t^2-1)^2},$$

so $h'(t) = 0$ when $2t = 0$. Thus h has one critical point, $t = 0$. However, we must also take into consideration the two points where h and h' are not defined, namely, $t = -1$ and $t = 1$. Now $(t^2 - 1)^2 \geq 0$ for all t, so the sign of h' is determined by the sign of $-2t$. Thus $h'(t) > 0$ when $t < -1$ and when $-1 < t < 0$, and $h'(t) < 0$ when $0 < t < 1$ and when $t > 1$. In other words, h is increasing on $(-\infty, -1)$ and $(-1, 0)$, and h is decreasing on $(0, 1)$ and $(1, \infty)$. From this we see that h has a local maximum of 0 at $t = 0$. For the second derivative, we have

$$h''(t) = \frac{(t^2-1)^2(-2) - (-2t)(2(t^2-1)(2t))}{(t^2-1)^4} = \frac{-2(t^2-1) + 8t^2}{(t^2-1)^3} = \frac{6t^2+2}{(t^2-1)^3}.$$

Since $6t^2 + 2 > 0$ for all values of t, it follows that $h''(t) \neq 0$ for all t. However, as with the first derivative, we need to consider the points $t = -1$ and $t = 1$ where h'' is not defined. Now $t^2 - 1 < 0$ only when $-1 < t < 1$, so $h''(t) < 0$ when $-1 < t < 1$ and $h''(t) > 0$ when

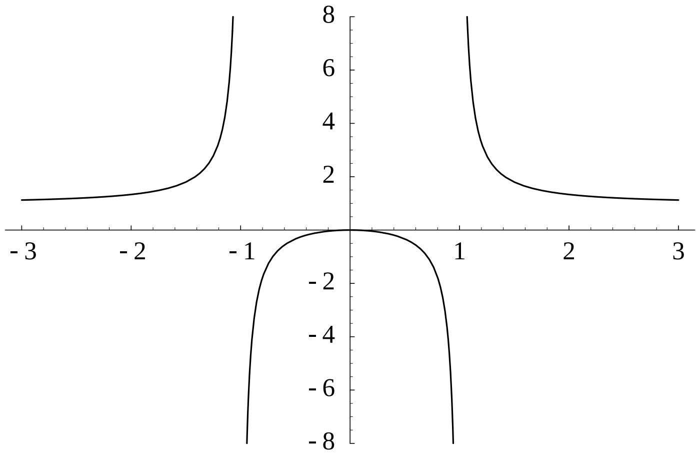


Figure 3.9.7 Graph of $h(t) = \dfrac{t^2}{t^2 - 1}$

$t < -1$ and when $t > 1$. Hence the graph of h is concave down on $(-1, 1)$ and concave up on $(-\infty, -1)$ and $(1, \infty)$. Note, however, that there are no points of inflection.

Since h is not defined at $t = -1$ and $t = 1$, we need to check for vertical asymptotes at these points. We have

$$\lim_{t \to -1^-} h(t) = \lim_{t \to -1^-} \frac{t^2}{t^2 - 1} = \infty,$$

$$\lim_{t \to -1^+} h(t) = \lim_{t \to -1^+} \frac{t^2}{t^2 - 1} = -\infty,$$

$$\lim_{t \to 1^-} h(t) = \lim_{t \to 1^-} \frac{t^2}{t^2 - 1} = -\infty,$$

and

$$\lim_{t \to 1^+} h(t) = \lim_{t \to 1^+} \frac{t^2}{t^2 - 1} = \infty,$$

showing that the graph of h has vertical asymptotes at $t = -1$ and $t = 1$. Finally,

$$\lim_{t \to -\infty} h(t) = \lim_{t \to -\infty} \frac{t^2}{t^2 - 1} = \lim_{t \to -\infty} \frac{1}{1 - \frac{1}{t^2}} = 1$$

and

$$\lim_{t \to \infty} h(t) = \lim_{t \to \infty} \frac{t^2}{t^2 - 1} = \lim_{t \to \infty} \frac{1}{1 - \frac{1}{t^2}} = 1$$

show that the graph of h has a horizontal asymptote at $y = 1$. With all of this geometric information, we may now draw the graph of h, as shown in Figure 3.9.7.

Problems

1. Discuss the geometry of the graphs of each of the following functions. That is, find the intervals where the function is increasing and where it is decreasing, find the intervals where the graph is concave up and where it is concave down, find all local extreme values and where they are located, find all inflection points, find any vertical or horizontal asymptotes, and use this information to sketch the graph.

   (a) $f(x) = x^2 - x$
   (b) $g(t) = 3t^2 + 2t - 6$
   (c) $g(x) = x^3 + 3x^2$
   (d) $f(t) = t^4 + 2t^2$
   (e) $f(x) = x^3 - 3x$
   (f) $g(x) = 3x^5 - 5x^3$
   (g) $h(x) = x^5 - x^3$
   (h) $f(x) = 3x^5 - 5x^4$
   (i) $g(z) = \dfrac{1}{z-1}$
   (j) $y(t) = \dfrac{1}{t^2+1}$
   (k) $f(x) = x^4 - 2x^3$
   (l) $h(t) = \dfrac{t}{1+t^2}$
   (m) $h(t) = \dfrac{t}{t^2-4}$
   (n) $g(x) = \dfrac{x}{1+3x^2}$
   (o) $f(x) = \dfrac{x^2}{1+x^2}$
   (p) $f(x) = \dfrac{1}{x^2-1}$
   (q) $x(t) = \dfrac{2t+1}{t-1}$
   (r) $f(z) = \dfrac{z^2}{z^2-4}$

2. Suppose the function f has the following properties:

$$
\begin{aligned}
&f(0) = 0\\
&f'(x) > 0 \text{ for } x \text{ in } (-\infty, 2)\\
&f'(x) < 0 \text{ for } x \text{ in } (2, \infty)\\
&f''(x) < 0 \text{ for } x \text{ in } (-2, 6)\\
&f''(x) > 0 \text{ for } x \text{ in } (-\infty, -2) \text{ and for } x\ (6, \infty)\\
&\lim_{x\to-\infty} f(x) = -2\\
&\lim_{x\to\infty} f(x) = 0
\end{aligned}
$$

   Sketch the graph of a function satisfying these conditions.

3. Suppose $f(0) = 0$ and $f'(x) = x^2 - 1$.

   (a) Sketch what the graph of f must look like.
   (b) Graph f' on the same axes with f.
   (c) Is there more than one function f which satisfies these conditions?

4. Suppose $f(0) = 0$ and $f'(x) = x^3 + x^2 - 6x$.
   (a) Sketch what the graph of f must look like.
   (b) Graph f' on the same axes with f.
   (c) Is there more than one function f which satisfies these conditions?
5. Suppose $g(1) = 0$ and $g'(t) = \frac{1}{t}$.
   (a) Sketch what the graph of g must look like on $(0, \infty)$.
   (b) Graph g' on the same axes with g.
   (c) Is there more than one function g which satisfies these conditions?
6. Suppose $f(0) = 1$ and $f'(x) = f(x)$. What must the graph of f look like? Is this enough information to determine the graph of f?

Difference Equations to Differential Equations

Section 4.1

The Definite Integral

As we discussed in Section 1.1, and mentioned again at the beginning of Section 3.1, there are two basic problems in calculus. In Chapter 3 we considered one of these, the problem of finding tangent lines to curves in the plane; we are now ready to turn to the second, quadrature, the problem of finding the area of a region in the plane. Although at first these problems would seem to have no connection, in Section 4.3 we shall see that Fundamental Theorem of Calculus relates them in an interesting and useful way. This theorem, first fully utilized by Newton and Leibniz, reveals that the problem of quadrature involves reversing the process of differentiation; as a consequence, the facility we developed in Chapter 3 for handling derivatives will be very helpful in many basic quadrature problems.

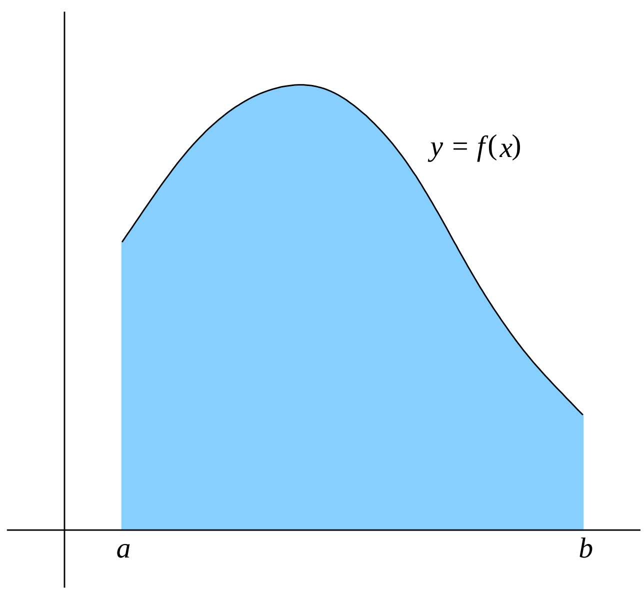


Figure 4.1.1 Region beneath the graph of $y = f(x)$ and over the interval $[a, b]$

As illustrated in Figure 4.1.1, our basic example for studying quadrature will be the problem of finding the area of a region R in the plane which is bounded above by the graph of a continuous function f and below by an interval $[a, b]$ on the x-axis. Later we will see how to extend our techniques to more complicated planar regions. Recall that in Section 1.1 we considered the problem of finding the area of the unit circle. In that case, we attacked the problem by approximating the area of the circle by the area of inscribed regular polygons, which were themselves divided into triangles. We used these to find the area of the circle by taking the limit of the areas of the inscribed polygons as the number

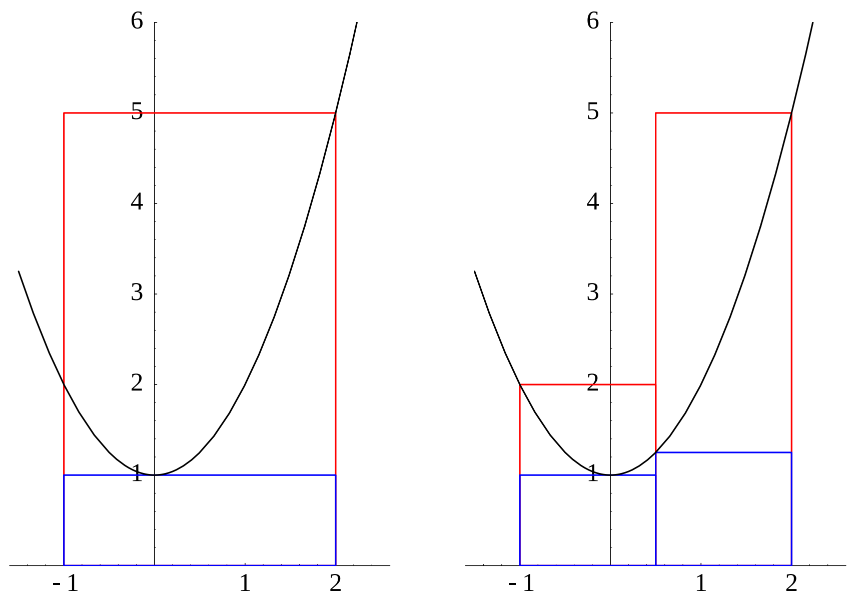


Figure 4.1.2 Inscribed and circumscribed rectangles for $f(x) = x^2 + 1$

of sides went to infinity. Here we will see that it is sufficient to use rectangles, rather than triangles, as our units of approximation. That is, we will approximate the area of the desired region by the area of rectangles and then ask about the limit as the number of rectangles used in the approximation goes to infinity. We begin with an example.

Example Consider the region R beneath the graph of the function $f(x) = x^2 + 1$ and above the interval $[-1, 2]$ on the x-axis. Let A be the area of R. If R_1 is the rectangle with base on the interval $[-1, 2]$ and height $f(2) = 5$, then, since 5 is the maximum value of f on $[-1, 2]$, R_1 contains R. We call R_1 a *circumscribed rectangle* for the region R. Hence the area of R is less than the area of R_1, showing that $A \leq 15$. Similarly, if R_2 is the rectangle with base on the interval $[-1, 2]$ and height $f(0) = 1$, then, since 1 is the minimum value of f on $[-1, 2]$, R contains R_2. We call R_2 an *inscribed rectangle* for the region R. Hence the area of R is greater than the area of R_2, showing that $A \geq 3$. See the figure on the left in Figure 4.1.2.

At this point we know that

$$3 \leq A \leq 15.$$

To improve our approximations for A, we begin by subdividing the interval $[-1, 2]$ into two equal intervals, namely $[-1, 0.5]$ and $[0.5, 2]$. If A_1 is the area of the region beneath the curve over the interval $[-1, 0.5]$, we can construct inscribed and circumscribed rectangles as we did in the last paragraph and obtain bounds for the area of A_1. Indeed, the rectangle with base on $[-1, 0.5]$ and height $f(-1) = 2$ circumscribes this region, while the rectangle with base on $[-1, 0.5]$ and height $f(0) = 1$ is inscribed in it. Hence we have

$$\frac{3}{2} \leq A_1 \leq 3.$$

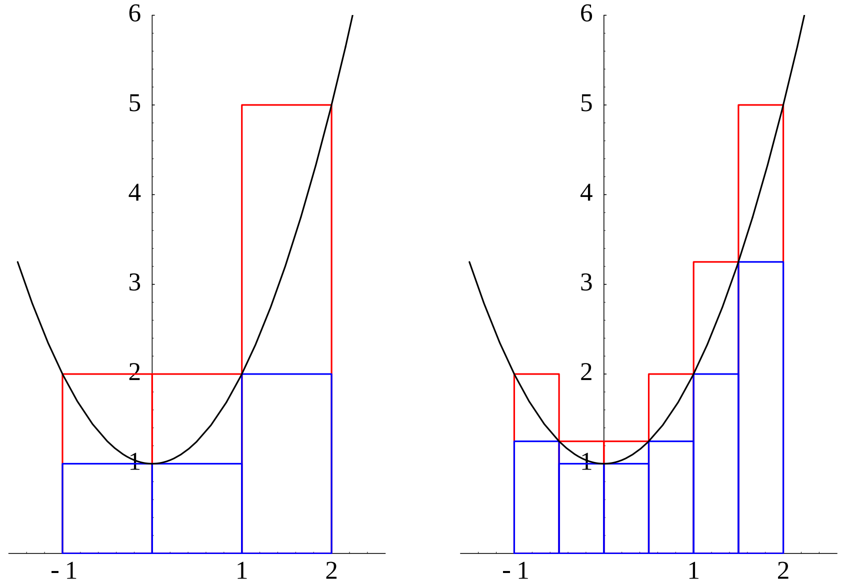


Figure 4.1.3 Inscribed and circumscribed rectangles for $f(x) = x^2 + 1$

Moreover, the region beneath the curve over the interval $[0.5, 2]$ is circumscribed by a rectangle of height $f(2) = 5$ and has inscribed within it a rectangle of height $f(0.5) = 1.25$. So if A_2 is the area of this region, we have

$$\frac{15}{8} \leq A_2 \leq \frac{15}{2}.$$

Since

$$A = A_1 + A_2,$$

putting these last two results together gives us

$$\frac{27}{8} \leq A \leq \frac{21}{2},$$

an improvement on our previous approximation. See the figure on the right in Figure 4.1.2.

To improve our approximation further, divide $[-1, 2]$ into three equal intervals: $[-1, 0]$, $[0, 1]$, and $[1, 2]$. You should check that the heights of the inscribed rectangles over these intervals are 1, 1, and 2, respectively. Since each rectangle has a base of length 1, we have

$$A \geq (1)(1) + (1)(1) + (2)(1) = 4.$$

Moreover, the heights of the circumscribed rectangles are 2, 2, and 5, respectively, and so

$$A \leq (2)(1) + (2)(1) + (5)(1) = 9.$$

Hence we now have

$$4 \leq A \leq 9.$$

See the figure on the left in Figure 4.1.3.

It is clear that we can approximate A using inscribed and circumscribed rectangles for any number of intervals. For example, you might check that if we use six intervals of equal length we would have

$$A \geq \left(\frac{5}{4}\right)\left(\frac{1}{2}\right) + (1)\left(\frac{1}{2}\right) + (1)\left(\frac{1}{2}\right) + \left(\frac{5}{4}\right)\left(\frac{1}{2}\right) + (2)\left(\frac{1}{2}\right) + \left(\frac{13}{4}\right)\left(\frac{1}{2}\right) = \frac{39}{8}$$

and

$$A \leq (2)\left(\frac{1}{2}\right) + \left(\frac{5}{4}\right)\left(\frac{1}{2}\right) + \left(\frac{5}{4}\right)\left(\frac{1}{2}\right) + (2)\left(\frac{1}{2}\right) + \left(\frac{13}{4}\right)\left(\frac{1}{2}\right) + (5)\left(\frac{1}{2}\right) = \frac{59}{8}$$

showing that

$$4.875 \leq A \leq 7.375,$$

(see the figure on the right in Figure 4.1.2). Continuing in this manner, subdividing the interval $[-1, 2]$ into smaller and smaller intervals, we would expect that we could approximate A to any desired level of accuracy. Moreover, we would expect that the area of the inscribed rectangles would increase toward A as the number of intervals increases, and that the area of the circumscribed rectangles would decrease toward A. Put another way, we might think of the area A as the unique number which is at once larger than the area of any set of inscribed rectangles and smaller than the area of any set of circumscribed rectangles. We will use this idea as the basis for our definition of the *definite integral.*

The definite integral

We now want to take the ideas of the previous example and develop a general procedure which, when applied to the appropriate function, will yield the area of certain types of regions in the plane. To do so, we require some preliminary terminology and notation.

Let f be a function defined on an interval $[a, b]$. We will not require that f be positive on $[a, b]$, although it will be necessary to require $f(x) \geq 0$ for all x in in order to talk about the area between the graph of f and the interval $[a, b]$, as in the previous example. However, we will assume that f is *bounded* on $[a, b]$; that is, we assume there exist numbers m and M such that $m \leq f(x) \leq M$ for all x in $[a, b]$. In particular, by the Extreme Value Theorem, f is bounded if f is continuous on $[a, b]$ We will return to the problem of unbounded functions in Section 4.7.

We call a set $P = \{x_0, x_1, \ldots, x_n\}$ a *partition* of the interval $[a, b]$ if

$$a = x_0 < x_1 < x_2 < \ldots < x_n = b.$$

Such a partition P divides $[a, b]$ into n intervals, $[x_{i-1}, x_i]$, of lengths

$$\Delta x_i = x_i - x_{i-1},$$

where $i = 1, 2, 3, \ldots, n$. For each such interval $[x_{i-1}, x_i]$, let M_i be the smallest number such that $f(x) \leq M_i$ for all x in $[x_{i-1}, x_i]$ and let m_i be the largest number such that $f(x) \geq m_i$ for all x in $[x_{i-1}, x_i]$. Note that if f is continuous on $[a, b]$, then M_i is the maximum value of f on $[x_{i-1}, x_i]$ and m_i is the minimum value of f on $[x_{i-1}, x_i]$, both of which are guaranteed to exist by the Extreme Value Theorem. If f is not continuous, properties of bounded sets of real numbers, alluded to in our discussion of bounded sequences in Section 1.2, nevertheless guarantee the existence of the values M_i and m_i. Also, note that if $f(x) \geq 0$ for all x in $[x_{i-1}, x_i]$, then, in the language of our previous example, the rectangle with base $[x_{i-1}, x_i]$ and height M_i is a circumscribed rectangle and the rectangle with base $[x_{i-1}, x_i]$ and height m_i is an inscribed rectangle.

Now let

$$U(f, P) = M_1 \Delta x_1 + M_2 \Delta x_2 + \cdots + M_n \Delta x_n = \sum_{i=1}^{n} M_i \Delta x_i, \tag{4.1.1}$$

the *upper sum* of f with respect to the partition P, and

$$L(f, P) = m_1 \Delta x_1 + m_2 \Delta x_2 + \cdots + m_n \Delta x_n = \sum_{i=1}^{n} m_i \Delta x_i, \tag{4.1.2}$$

the *lower sum* of f with respect to the partition P. Note that we always have

$$L(f, P) \leq U(f, P). \tag{4.1.3}$$

Also, if $f(x) \geq 0$ for all x in $[a, b]$, then $U(f, P)$ is the sum of the areas of the circumscribed rectangles for the partition P and $L(f, P)$ is the sum of the areas of the inscribed rectangles. In that case, if A is the area beneath the graph of f and above the interval , we would expect that we could make $U(f, P)$ and $L(f, P)$ arbitrarily close to A. This would imply that A is the only number with the property that

$$L(f, P) \leq A \leq U(f, P) \tag{4.1.4}$$

for all partitions P. This is the motivation for the following definition.

Definition Using the above notation, we say a function f is *integrable* on an interval $[a, b]$ if there exists a unique number I such that

$$L(f, P) \leq I \leq U(f, P) \tag{4.1.5}$$

for all partitions P of $[a, b]$. If f is integrable on $[a, b]$, we call I the *definite integral* of f on $[a, b]$, which we denote

$$I = \int_a^b f(x)dx. \tag{4.1.6}$$

Example Consider again our example of finding the area of the region beneath the graph of $f(x) = x^2 + 1$ and above the interval $[-1, 2]$ on the x-axis. Let P_n denote the partition using $n + 1$ equally spaced points (giving us n intervals of equal length). For examples,

$$P_2 = \{-1, 0.5, 2\}$$

and

$$P_6 = \{-1, -0.5, 0, 0.5, 1, 1.5, 2\}.$$

Our work above shows that, in our current notation,

$$U(f, P_6) = 7.375$$

and

$$L(f, P_6) = 4.875.$$

Using 100 intervals, and a computer to ease the computations, we find that

$$U(f, P_{100}) = 6.075$$

and

$$L(f, P_{100}) = 5.925,$$

where the results have been rounded to three decimal places. This shows that if f is integrable on $[-1, 2]$, then

$$5.925 \leq \int_{-1}^{2} (x^2 + 1)dx \leq 6.075.$$

Of course, we expect f to be integrable, and for the value of the definite integral to be the sought for area under the graph.

It is not easy to verify directly from the definition that a given function is integrable on some interval. However, it may be shown that any continuous function is integrable. The reasons for this are rather technical, but we can give some feeling for why this should be so. Suppose f is continuous on $[a, b]$ and let $P_n = \{x_0, x_1, x_2, \ldots, x_n\}$ denote the partition of $[a, b]$ using $n + 1$ equally spaced points. Let M_i and m_i be as defined above, and let

$$\Delta x = \Delta x_i = \frac{b-a}{n}$$

be the length of the intervals $[x_{i-1}, x_i]$, $i = 1, 2, 3, \ldots, n$. Given any number $\epsilon > 0$, we can choose n large enough (equivalently, Δx small enough) so that $M_i - m_i < \epsilon$ for $i = 1, 2, 3, \ldots, n$. This fact is a consequence of the continuity of f on $[a, b]$, although it requires a deeper property of continuous functions on closed intervals known as *uniform continuity.* We then have

$$\begin{aligned} 0 &\leq U(f, P_n) - L(f, P_n) \\ &= \sum_{i=1}^{n} M_i \Delta x - \sum_{i=1}^{n} m_i \Delta x \\ &= \sum_{i=1}^{n} (M_i - m_i)\Delta x < \sum_{i=1}^{n} \epsilon \Delta x \\ &= n\epsilon\Delta x = \epsilon(b-a). \end{aligned} \tag{4.1.7}$$

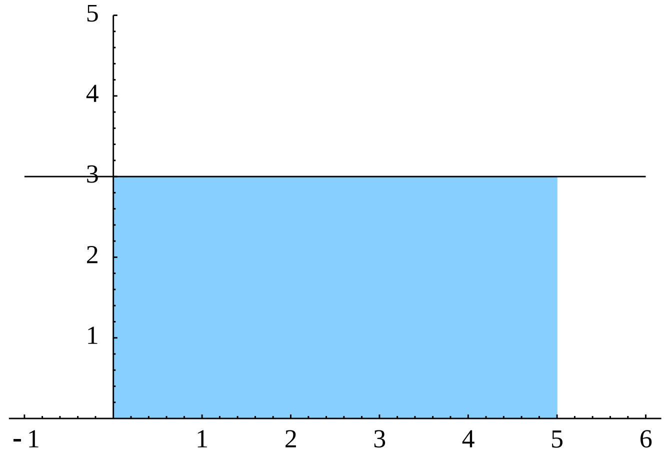


Figure 4.1.4 Region beneath $y = 3$ over the interval $[0, 5]$

Since ϵ may be made arbitrarily small, it follows that the difference between upper sums and lowers sums may be made arbitrarily small, and hence that there must be only one number which is between the upper and lower sums for all possible partitions.

Proposition If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

Example We now know that $f(x) = x^2 + 1$ is integrable on $[-1, 2]$.

Although our motivation for this section has been the computation of area, we have not actually defined the term. We do so now for the special case we have been considering.

Definition Given an integrable function f with $f(x) \geq 0$ for all x in an interval $[a, b]$, let R be the region in the plane bounded above by the curve $y = f(x)$, below by the interval $[a, b]$ on the x-axis, and on the sides by the vertical lines $x = a$ and $x = b$. Then we define the *area* A of R to be

$$A = \int_a^b f(x)dx. \tag{4.1.8}$$

Example Of course, we should verify that the above definition of area agrees with our previous notion of area. For example, if $f(x) = 3$ for all x in $[0, 5]$, then the region beneath the graph of f and above the x-axis is a rectangle with base of length 5 and height of 3, as shown in Figure 4.1.4. Hence we should have

$$\int_0^5 3dx = 15.$$

To verify this, let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be any partition of $[0, 5]$. Then on any interval $[x_{i-1}, x_i]$, $i = 1, 2, 3, \ldots, n$, the maximum value of f is $M_i = 3$ and the minimum value of f is $m_i = 3$. Hence

$$U(f, P) = L(f, P) = \sum_{i=1}^{n} 3\Delta x_i = 3\sum_{i=1}^{n} \Delta x_i = (3)(5) = 15,$$

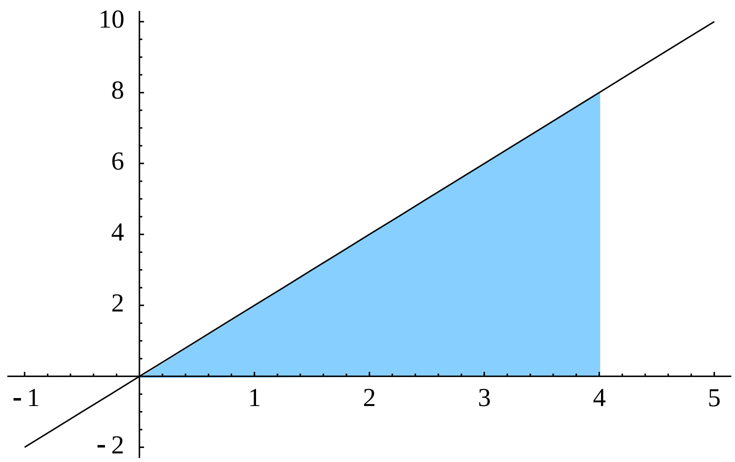


Figure 4.1.5 Region beneath $y = 2x$ over the interval $[0, 4]$

where we have used the fact that the sum of the lengths of the partition intervals must equal the length of the entire interval . Thus $I = 15$ is the only number satisfying

$$L(f, P) \leq I \leq U(f, P)$$

for all partitions P, and so

$$\int_0^5 3dx = 15,$$

as expected.

Note that the previous example could be generalized to show that for any constant c and any interval $[a, b]$,

$$\int_a^b cdx = c(b - a). \tag{4.1.9}$$

Example To verify another previously known area, consider the function $f(x) = 2x$ on the interval $[0, 4]$. Then the region beneath the graph of f and above the interval $[0, 4]$ on the x-axis is a triangle with base of length 4 and height 8, as shown in Figure 4.1.5. Thus it has area

$$\frac{1}{2}(4)(8) = 16,$$

and so we should have

$$\int_0^4 2xdx = 16.$$

To verify this, let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be a partition of $[0, 4]$ and let m_i and M_i be the minimum and maximum values, respectively, of f on $[x_{i-1}, x_i]$, $i = 1, 2, 3, \ldots, n$. Since f is an increasing function on $[0, 4]$, we have $m_i = f(x_{i-1})$ and $M_i = f(x_i)$. Thus

$$L(f, P) = \sum_{i=1}^{n} f(x_{i-1})\Delta x_i$$

and

$$U(f,P)=\sum_{i=1}^{n} f(x_i)\Delta x_i.$$

We will now use a technique which will be useful in the proof of the Fundamental Theorem of Calculus in Section 4.3. Let $F(x)=x^2$. Then $F'(x)=2x$, so F is an antiderivative of f. By the Mean Value Theorem, for every interval $[x_{i-1},x_i]$ there exists a point c_i in $[x_{i-1},x_i]$ such that

$$\frac{F(x_i)-F(x_{i-1})}{x_i-x_{i-1}}=F'(c_i)=f(c_i). \tag{4.1.10}$$

Now $x_i-x_{i-1}=\Delta x_i$, so from (4.1.10) we obtain

$$f(c_i)\Delta x_i=F(x_i)-F(x_{i-1}). \tag{4.1.11}$$

Moreover, $f(x_{i-1})\le f(c_i)\le f(x_i)$, so

$$L(f,P)=\sum_{i=1}^{n} f(x_{i-1})\Delta x_i\le\sum_{i=1}^{n} f(c_i)\Delta x_i\le\sum_{i=1}^{n} f(x_i)\Delta x_i=U(f,P). \tag{4.1.12}$$

But, using (4.1.11),

$$\begin{aligned}
\sum_{i=1}^{n} f(c_i)\Delta x_i &= \sum_{i=1}^{n}(F(x_i)-F(x_{i-1}))\\
&=(F(x_1)-F(x_0))+(F(x_2)-F(x_1))+(F(x_3)-F(x_2))+\cdots\\
&\qquad+(F(x_n)-F(x_{n-1}))\\
&=-F(x_0)+(F(x_1)-F(x_1))+(F(x_2)-F(x_2))+(F(x_3)-F(x_3))+\cdots\\
&\qquad+(F(x_{n-1})-F(x_{n-1}))+F(x_n)\\
&=F(x_n)-F(x_0).
\end{aligned}$$

Now $x_0=0$ and $x_n=4$, so

$$F(x_n)-F(x_0)=F(4)-F(0)=16-0=16.$$

It now follows from (4.1.11) that, for any partition P,

$$L(f,P)\le 16\le U(f,P). \tag{4.1.13}$$

Since we know that f is integrable on $[0,4]$ (it is continuous on $[0,4]$), the definite integral of f is the only number which satisfies the inequalities in (4.1.13) for any partition P. Hence we must have

$$\int_0^4 2x=16,$$

in agreement with our geometric argument above.

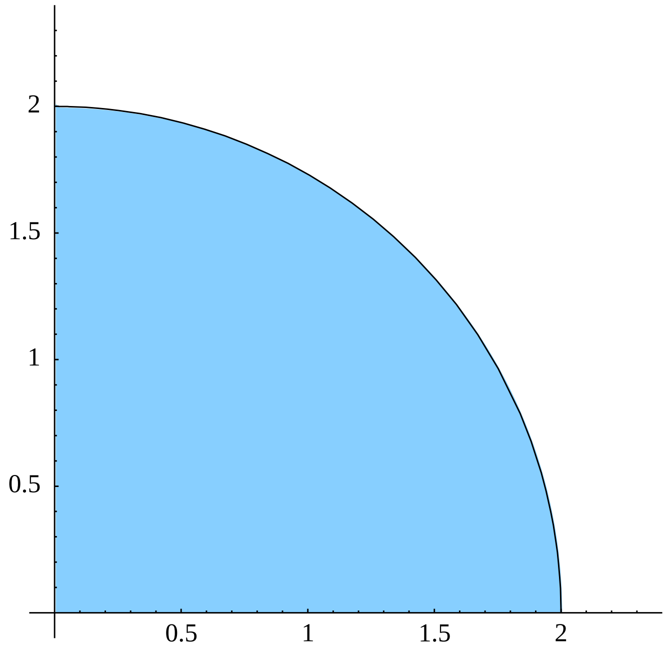


Figure 4.1.6 Region beneath $y = \sqrt{4 - x^2}$ over the interval $[0, 2]$

As we proceed with our study of integration we shall from time to time have occasion to verify that areas computed using a definite integral are consistent with areas computed by other geometric means. At the same time, we shall take this consistency as a given. For example, we shall accept that

$$\int_0^2 \sqrt{4 - x^2}\, dx = \pi,$$

since the region beneath the curve $y = \sqrt{4 - x^2}$ and above the interval $[0, 2]$ is one-quarter of a circle of radius 2 centered at the origin, as shown in Figure 4.1.6.

Example It is important to realize that not all bounded functions are integrable. As an example, consider the function

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is a rational number,} \\ 0, & \text{if } x \text{ is an irrational number.} \end{cases}$$

For example, $f(0.12345) = 1$ and $f\left(\frac{1}{\sqrt{2}}\right) = 0$. Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be a partition of $[0, 1]$. Since every interval $[x_{i-1}, x_i]$, $i = 1, 2, 3, \ldots, n$, contains both rational and irrational numbers, the minimum value of f on $[x_{i-1}, x_i]$ is $m_i = 0$ and the maximum value of f on $[x_{i-1}, x_i]$ is $M_i = 1$. Thus

$$L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i = 0$$

and

$$U(f,P) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} \Delta x_i = 1.$$

Hence any number between 0 and 1 lies between $L(f,P)$ and $U(f,P)$ for any partition P. Since there is not a unique such number, we conclude that f is not integrable on $[0,1]$.

Computing a definite integral directly from the definition is usually a daunting task. We shall take a first look at approximating definite integrals in this section, and then refine these techniques in Section 4.2. In Section 4.3 we will look at the Fundamental Theorem Calculus, a result which will, in certain cases, allow us to compute definite integrals exactly with relative ease.

Riemann sums

Again let f be a function defined on an interval $[a,b]$ and let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be a partition of $[a,b]$. Recall that in the definition of the upper and lower sums, M_i and m_i are chosen, in part, so that $m_i \leq f(x) \leq M_i$ for all x in $[x_{i-1}, x_i]$, $i = 1, 2, 3, \ldots, n$. It follows that if we choose values $c_1, c_2, c_3, \ldots, c_n$ so that c_i is in the ith interval of the partition (that is, $x_{i-1} \leq c_i \leq x_i$), then

$$m_i \leq f(c_i) \leq M_i \tag{4.1.14}$$

for $i = 1, 2, 3, \ldots, n$, and so

$$L(f,P) = \sum_{i=1}^{n} m_i \Delta x_i \leq \sum_{i=1}^{n} f(c_i)\Delta x_i \leq \sum_{i=1}^{n} M_i \Delta x_i = U(f,P). \tag{4.1.15}$$

If f is integrable, it may be shown that is always possible to choose a partition P so that

$$|U(f,P) - L(f,P)| < \epsilon \tag{4.1.16}$$

for any specified $\epsilon > 0$. It follows that, when f is integrable, it is always possible to find, for any given $\epsilon > 0$, partitions for which

$$\left| \int_a^b f(x)dx - \sum_{i=1}^{n} f(c_i)\Delta x_i \right| < \epsilon \tag{4.1.17}$$

for any choice of the points $c_1, c_2, c_3, \ldots, c_n$. In fact, it may be shown that if we let L be the maximum length of the intervals $[x_{i-1}, x_i]$, then it is possible to find a $\delta > 0$ such that (4.1.17) will hold for any partition with $L < \delta$.

The sum

$$\sum_{i=1}^{n} f(c_i)\Delta x_i \tag{4.1.18}$$

is called a *Riemann sum*, after the German mathematician G. B. F. Riemann (1826-1866). From what we have just seen, we may use Riemann sums to approximate definite integrals. We will consider two important special cases of Riemann sums here (we will look at another in Section 4.2). First, to make calculations simpler, we will restrict to partitions with intervals of equal length. As above, let $P_n = \{x_0, x_1, x_2, \ldots, x_n\}$ be the partition of $[a, b]$ using $n + 1$ equally spaced points and let

$$\Delta x = \frac{b-a}{n}$$

be the length of the intervals $[x_{i-1}, x_i]$, $i = 1, 2, 3, \ldots, n$. Note that

$$\lim_{n\to\infty} \Delta x = 0.$$

Hence, if we choose points $c_1, c_2, c_3, \ldots, c_n$ with $x_{i-1} \le c_i \le x_i$, then we have, for an integrable f,

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x. \tag{4.1.19}$$

In other words, we may approximate the definite integral $\int_a^b f(x)dx$ to any desired level of accuracy using Riemann sums

$$\sum_{i=1}^{n} f(c_i)\Delta x \tag{4.1.20}$$

with sufficiently large n. To do this efficiently requires specifying how the points c_1, c_2, c_3, $\ldots$, c_n are to be chosen. One method is to simply choose c_i to be the right-hand endpoint of the interval $[x_{i-1}, x_i]$. In that case, since the points in the partition are equally spaced, we have

$$\begin{aligned} c_1 &= x_1 = x_0 + \Delta x = a + \Delta x, \\ c_2 &= x_2 = x_1 + \Delta x = a + 2\Delta x, \\ c_3 &= x_3 = x_2 + \Delta x = a + 3\Delta x, \\ &\vdots \\ c_n &= x_n = x_{n-1} + \Delta x = a + n\Delta x. \end{aligned} \tag{4.1.21}$$

Using these points in (4.1.20), we have

$$\sum_{i=1}^{n} f(c_i)\Delta x = \Delta x \sum_{i=1}^{n} f(a + i\Delta x). \tag{4.1.22}$$

This approximation is known as the *right-hand rule approximation* for $\int_a^b f(x)dx$.

Definition If f is integrable on $[a,b]$, the *right-hand rule approximation* for the definite integral

$$\int_a^b f(x)dx$$

using n intervals is given by

$$A_R = \Delta x \sum_{i=1}^{n} f(a + i\Delta x), \tag{4.1.23}$$

where

$$\Delta x = \frac{b-a}{n}.$$

A similar rule is derived by using the left-hand endpoints of the intervals. In this case we choose

$$\begin{aligned} c_1 &= x_0 = x_0 = a, \\ c_2 &= x_1 = x_0 + \Delta x = a + \Delta x, \\ c_3 &= x_2 = x_1 + \Delta x = a + 2\Delta x, \\ &\vdots \\ c_n &= x_{n-1} = x_{n-2} + \Delta x = a + (n-1)\Delta x. \end{aligned} \tag{4.1.24}$$

Definition If f is integrable on $[a,b]$, the *left-hand rule approximation* for the definite integral

$$\int_a^b f(x)dx$$

using n intervals is given by

$$A_L = \Delta x \sum_{i=0}^{n-1} f(a + i\Delta x), \tag{4.1.25}$$

where

$$\Delta x = \frac{b-a}{n}.$$

Example Returning to our first example, suppose $f(x) = x^2 + 1$ and let A be the area of the region beneath the graph of f and above the interval $[-1,2]$. With $n = 6$, we have

$$\Delta x = \frac{2-(-1)}{6} = \frac{1}{2}$$

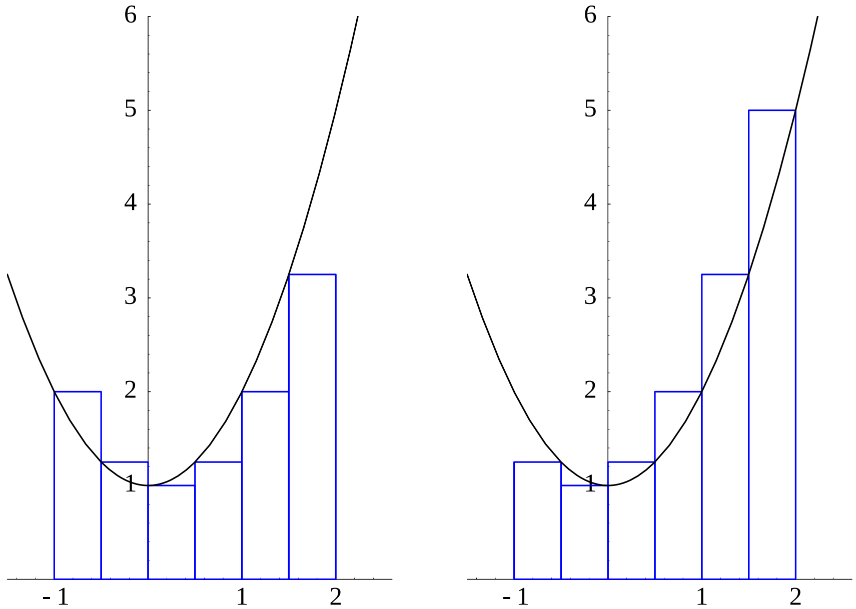


Figure 4.1.7 Left-hand and right-hand rule approximations for $\int_{-1}^{2}(x^2+1)dx$

and the left-hand rule approximation for A is

$$\begin{aligned}
A_L &= \frac{1}{2}\sum_{i=0}^{5} f\left(-1+\frac{1}{2}i\right)\\
&= \frac{1}{2}\left(f(-1)+f\left(-\frac{1}{2}\right)+f(0)+f\left(\frac{1}{2}\right)+f(1)+f\left(\frac{3}{2}\right)\right)\\
&= \frac{1}{2}\left(2+\frac{5}{4}+1+\frac{5}{4}+2+\frac{13}{4}\right)\\
&= \frac{1}{2}\left(\frac{43}{4}\right)\\
&= \frac{43}{8} = 5.375.
\end{aligned}$$

See the figure on the left in Figure 4.1.7. Similarly, the right-hand rule approximation is

$$\begin{aligned}
A_R &= \frac{1}{2}\sum_{i=1}^{6} f\left(-1+\frac{1}{2}i\right)\\
&= \frac{1}{2}\left(f\left(-\frac{1}{2}\right)+f(0)+f\left(\frac{1}{2}\right)+f(1)+f\left(\frac{3}{2}\right)+f(2)\right)\\
&= \frac{1}{2}\left(\frac{5}{4}+1+\frac{5}{4}+2+\frac{13}{4}+5\right)
\end{aligned}$$

$$= \frac{1}{2}\left(\frac{55}{4}\right)$$
$$= \frac{55}{8} = 6.875.$$

See the figure on the right in Figure 4.1.7. Recall that, for a partition of 6 intervals of equal length, we computed a lower sum of 4.875 and an upper sum 7.375. Hence, as we would expect for any Riemann sums, A_L and A_R lie between the lower and upper sums.

Using $n = 100$ and a computer, we find $A_L = 5.955$ and $A_R = 6.045$, which again lie between the lower sum of 5.925 and the upper sum of 6.075.

Example Now let A be the area of the region beneath the graph of

$$g(t) = \frac{1}{t}$$

over the interval $[1, 10]$, as shown in Figure 4.1.8. Then

$$A = \int_1^{10} \frac{1}{t}\, dt.$$

In Section 6.2 we will see that this integral is equal to the natural logarithm of 10, which, to 6 decimal places, is 2.302585. The following table summarizes the left-hand and right-hand rule approximations for A:

n	A_R	A_L	$\lvert A - A_R \rvert$	$\lvert A - A_L \rvert$
10	1.960214	2.770214	0.342371	0.467629
20	2.116477	2.521477	0.186108	0.218892
40	2.205491	2.407991	0.097094	0.105406
80	2.253003	2.354253	0.049582	0.051668
160	2.277534	2.328159	0.025052	0.025574
320	2.289994	2.315307	0.012591	0.012722

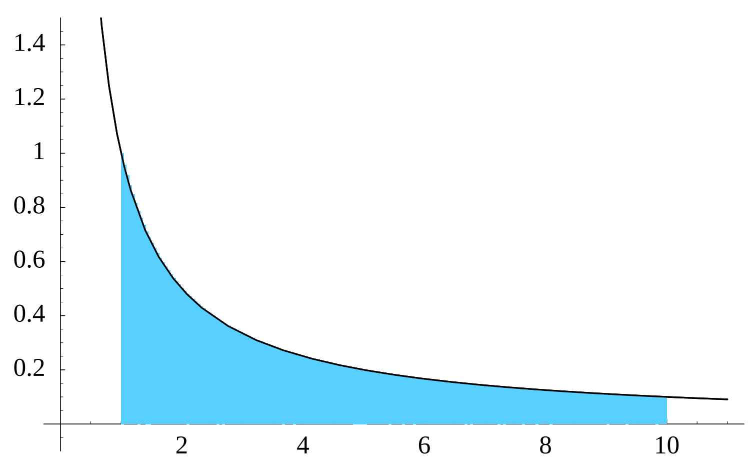


Figure 4.1.8 Region beneath $y = \dfrac{1}{t}$ over the interval $[1, 10]$

As we should expect, the error in our approximations decreases as the number of subdivisions increases. What is more interesting to note is that, in this particular case, when the number of subdivisions is doubled, the error committed by both the right-hand and the left-hand rules decreases by a factor of, roughly, $\frac{1}{2}$. For example, this might lead us to predict that the error in using 640 intervals would be about 0.0063; in fact, it turns out to be 0.006312 for the right-hand rule and 0.006344 for the left-hand rule. This type of behavior is typical for this method of approximation, a point we will come back to when we investigate other methods of approximation in Section 4.2.

Properties of the definite integral

Since the integral of an integrable function may be computed as the limit of Riemann sums, the basic properties of limits and sums hold true for integrals as well. In particular, if f and g are integrable on $[a, b]$ and k is any constant, then

$$\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx, \tag{4.1.26}$$

$$\int_a^b (f(x) - g(x))dx = \int_a^b f(x)dx - \int_a^b g(x)dx, \tag{4.1.27}$$

and

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx. \tag{4.1.28}$$

Example We know that

$$\int_0^3 xdx = \frac{9}{2}$$

(since the region under the graph of $y = x$ over the interval $[0, 3]$ is a triangle with base of length 3 and a height of 3) and

$$\int_0^3 4dx = 12$$

(either using (4.1.9) or the fact that the region under the graph of $y = 4$ is a rectangle with base of length 3 and a height of 4), so it follows from (4.1.24) that

$$\int_0^3 (x+4)dx = \int_0^3 xdx + \int_0^3 4dx = \frac{9}{2} + 12 = \frac{33}{2}.$$

Example The graph of $g(t) = \sqrt{1-t^2}$ over the interval $[-1, 1]$ is a semicircle of radius 1 centered at the origin, so

$$\int_{-1}^1 5\sqrt{1-t^2}\ dt = 5\int_{-1}^1 \sqrt{1-t^2}\ dt = \frac{5\pi}{2}.$$

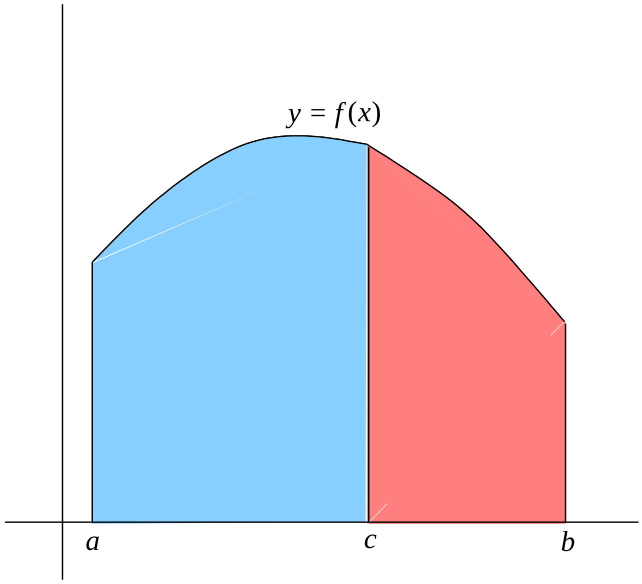


Figure 4.1.9 $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Now suppose f is integrable on $[a, b]$ and c is a point with $a < c < b$. It may be shown that f is integrable on both $[a, c]$ and $[c, b]$. Moveover, using partitions which include c, we may write a Riemann sum for f over $[a, b]$ as the sum of two Riemann sums, the first over the interval $[a, c]$ and the second over the interval $[c, b]$. After taking limits, it follows that

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx. \tag{4.1.29}$$

If $f(x) \geq 0$ for all x in $[a, b]$, we may think of (4.1.29) as saying that the area under the graph of f over the interval $[a, b]$ is equal to the area under the graph of f over the interval $[a, c]$ plus the area under the graph of f over the interval $[c, b]$. See Figure 4.1.9.

Example Suppose

$$f(x) = \begin{cases} x, & \text{if } 0 \leq x \leq 1, \\ 3, & \text{if } 1 < x \leq 2. \end{cases}$$

The region under the graph of f is shown in Figure 4.1.10. Now

$$\int_0^2 f(x)dx = \int_0^1 f(x)dx + \int_1^2 f(x)dx = \int_0^1 xdx + \int_1^2 3dx = \frac{1}{2} + 3 = \frac{7}{2}.$$

Technically, before applying (4.1.29) in the previous example we should have verified that f is integrable on $[0, 2]$. Since f is not continuous on $[0, 2]$, its integrability does not follow from our previous results. However, f is an example of what is known as a *piecewise continuous* function, which we will now define.

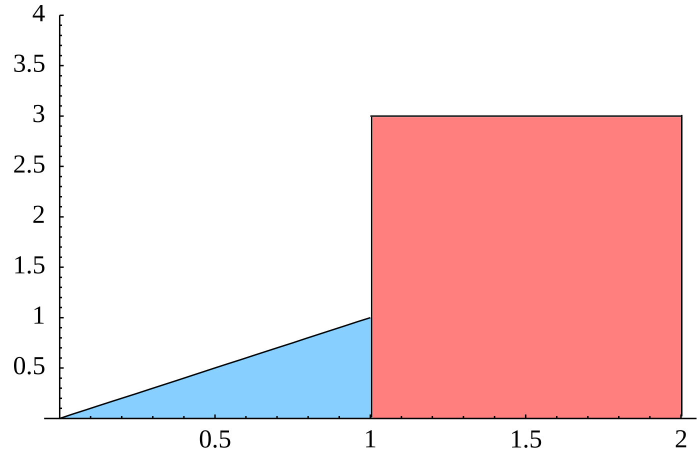


Figure 4.1.10 $\int_0^2 f(x)dx = \int_0^1 f(x)dx + \int_1^2 f(x)dx$

Definition A function is said to be *piecewise continuous* on an interval $[a,b]$ if there is a partition $P = \{x_0, x_1, x_2, \ldots, x_n\}$ of $[a,b]$ such that f is continuous on each open interval (x_{i-1}, x_i), $i = 1, 2, 3, \ldots, n$; has limits from both the right and the left at each partition point x_i, $i = 1, 2, 3, \ldots, x_{n-1}$; and has a right-hand limit at a and a left-hand limit at b.

Proposition If f is piecewise continuous on $[a,b]$, then f is integrable on $[a,b]$.

Example The function f in the previous example is piecewise continuous on $[0,2]$, and hence integrable on $[0,2]$ by the previous proposition.

Now suppose f and g are both integrable on $[a,b]$ and $f(x) \leq g(x)$ for all x in $[a,b]$. It follows that for any given partition P, the upper sum of g will be greater than or equal to the corresponding upper sum of f. Since the definite integral is the largest number less than or equal to the value of any upper sum, it follows that

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx. \tag{4.1.30}$$

Example Since $0 \leq x^2 \leq x$ for all x in $[0,1]$, we have

$$\int_0^1 0dx \leq \int_0^1 x^2 dx \leq \int_0^1 xdx.$$

Now

$$\int_0^1 0dx = 0(1-0) = 0$$

and

$$\int_0^1 xdx = \frac{1}{2},$$

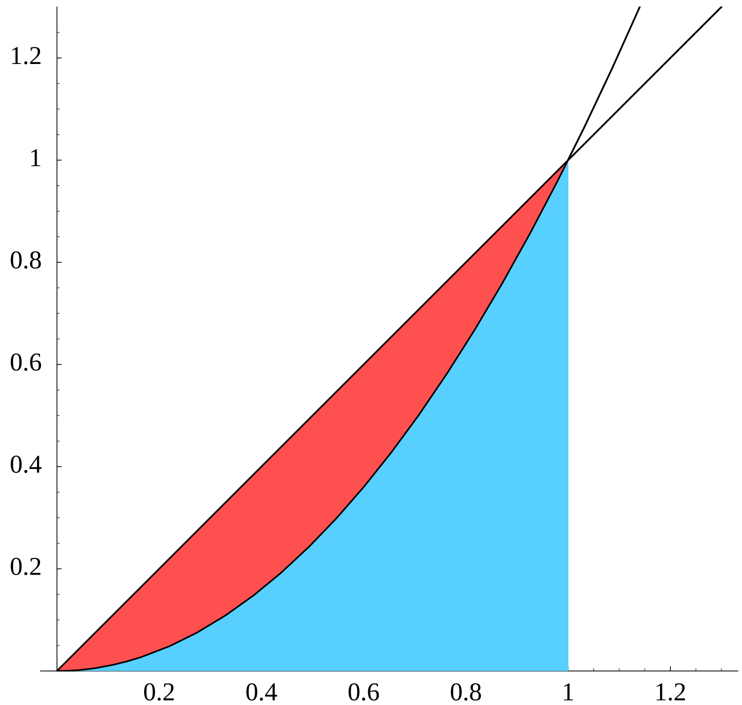


Figure 4.1.11 $0 \le \int_0^1 x^2 dx \le \int_0^1 x dx$

so it follows that

$$0 \le \int_0^1 x^2 \le \frac{1}{2}.$$

See Figure 4.1.11.

Geometric interpretations

The original motivation for this section was the problem of finding the area of a region in the plane. Given an integrable function f with $f(x) \ge 0$ for all x in an interval $[a, b]$, we eventually defined the area of the region beneath the graph of f and above the interval $[a, b]$ to be $\int_a^b f(x)dx$. Now suppose $f(x) \le 0$ for all x in $[a, b]$ and let R be the region between the graph of f and the interval $[a, b]$. If S is the region beneath the graph of $y = -f(x)$ and above $[a, b]$, then we have

$$\text{area of } R = \text{ area of } S = \int_a^b -f(x)dx = -\int_a^b f(x)dx. \tag{4.1.31}$$

Hence, in this case, $\int_a^b f(x)dx$ is not the area of the region R, but rather

$$\int_a^b f(x)dx = -(\text{area of } R). \tag{4.1.32}$$

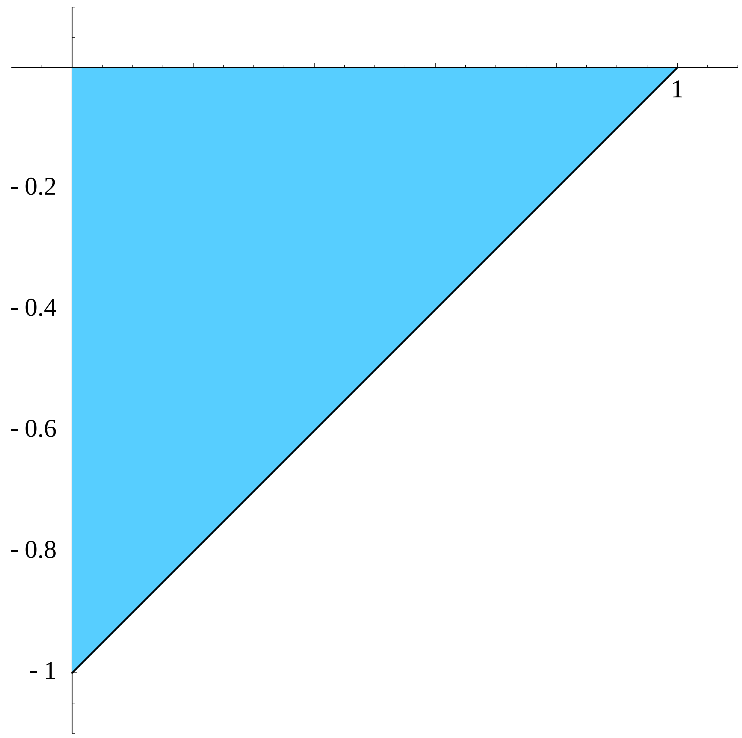


Figure 4.1.12 Region between the graph of $y = x - 1$ and the interval $[0, 1]$

Example If $f(x) = x - 1$, then $f(x) \leq 0$ for all x in $[0, 1]$. Since the region between the graph of f and the interval $[0, 1]$ on the x-axis is a triangle with area $\frac{1}{2}$, we must have

$$\int_0^1 (x-1)dx = -\frac{1}{2}.$$

See Figure 4.1.12.

More generally, we may think of $\int_a^b f(x)dx$ as representing the difference of the area of any regions between the graph of f and the x-axis which lie above the x-axis and the area of those regions which lie below the x-axis. For example, we have

$$\int_{-\pi}^{\pi} \sin(x)dx = 0$$

because the area of the region beneath the graph of $y = \sin(x)$ over the interval $[0, \pi]$ is negated by the area of the region between the graph $y = \sin(x)$ and the interval $[-\pi, 0]$, as shown in Figure 4.1.13.

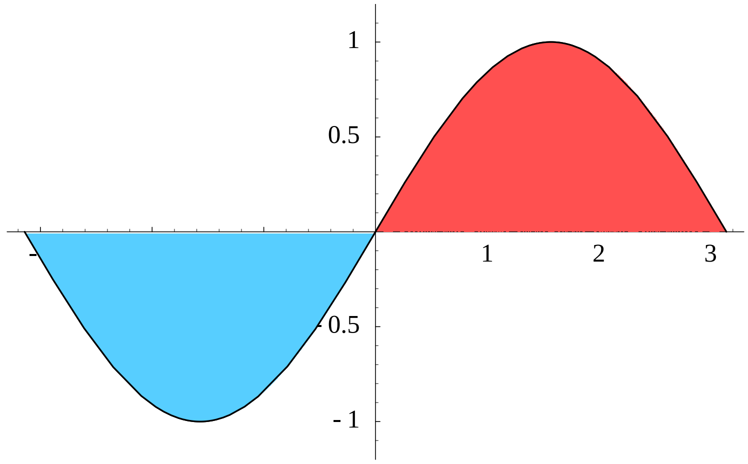


Figure 4.1.13 Area above the x-axis cancels area beneath the x-axis

Problems

1. For each of the following, find upper and lower bounds for the area of the region beneath the given curve over the given interval using four inscribed rectangles and four circumscribed rectangles.

   (a) $y = \dfrac{1}{x}$ on $[1, 5]$ (b) $y = x^2$ on $[0, 4]$

   (c) $y = x^2 + 1$ on $[-2, 2]$ (d) $y = \sin(x)$ on $[0, \pi]$

2. Find the upper and lower sums for the following integrals using a partition with six equal intervals.

   (a) $\displaystyle\int_1^4 3x\,dx$ (b) $\displaystyle\int_{-2}^4 x^2\,dx$

   (c) $\displaystyle\int_{-\pi}^{\pi} \cos(x)dx$ (d) $\displaystyle\int_{-2}^{2} (4 - x^2)dx$

   (e) $\displaystyle\int_0^1 (x^3 - x)dx$ (f) $\displaystyle\int_0^1 \sin(2\pi t)dt$

3. For each of the following, approximate the area beneath the graph of the function over the given interval using the right-hand and left-hand rules with four intervals.

   (a) $f(x) = x^2$ on $[0, 4]$ (b) $f(x) = x^2$ on $[-2, 2]$

   (c) $g(t) = \dfrac{1}{t}$ on $[1, 9]$ (d) $g(t) = \dfrac{1}{t}$ on $[1, 2]$

   (e) $h(x) = x^3$ on $[0, 1]$ (f) $f(t) = 1 - t^2$ on $[-1, 1]$

4. For each of the following, approximate the area beneath the graph of the function over the given interval using the right-hand and left-hand rules with 100 intervals.

   (a) $f(x) = x^2$ on $[0, 1]$ (b) $g(x) = \sin(x)$ on $[0, \pi]$

   (c) $f(t) = t^3$ on $[0, 2]$ (d) $g(z) = z^2$ on $[-2, 2]$

(e) $h(x) = \dfrac{1}{x}$ on $[1, 2]$

(f) $f(x) = \sqrt{1 - x^2}$ on $[-1, 1]$

(g) $h(\theta) = \sec(\theta)$ on $\left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$

(h) $g(t) = \sin(2t)$ on $\left[0, \dfrac{\pi}{2}\right]$

5. Use the right-hand and left-hand rules with four intervals to approximate the following definite integrals.

(a) $\displaystyle\int_0^2 x^2 dx$

(b) $\displaystyle\int_2^3 \frac{1}{x}\, dx$

(c) $\displaystyle\int_{-\pi}^{\pi} \cos(t) dt$

(d) $\displaystyle\int_{-3}^{1} s^3 ds$

(e) $\displaystyle\int_{-1}^{1} (x^2 - 1) dx$

(f) $\displaystyle\int_{-\pi}^{\pi} \sin(z) dz$

6. Use the right-hand and left-hand rules with 100 intervals to approximate the following definite integrals.

(a) $\displaystyle\int_0^3 x^2 dx$

(b) $\displaystyle\int_{-1}^{2} x^3 dx$

(c) $\displaystyle\int_{-2}^{2} \sqrt{4 - t^2}\, dt$

(d) $\displaystyle\int_0^{2\pi} \sin(x) dx$

(e) $\displaystyle\int_{-1}^{1} (x^2 - 1) dx$

(f) $\displaystyle\int_0^{\pi} \sin(3\theta) d\theta$

(g) $\displaystyle\int_{-1}^{0} \frac{x}{x^2 + 1}\, dx$

(h) $\displaystyle\int_{-4}^{-2} \frac{1}{t}\, dt$

7. Use geometric arguments to determine the value of each of the following definite integrals.

(a) $\displaystyle\int_0^4 x dx$

(b) $\displaystyle\int_0^3 (2x + 3) dx$

(c) $\displaystyle\int_0^3 \sqrt{9 - x^2}\, dx$

(d) $\displaystyle\int_{-2}^{2} 4\sqrt{4 - t^2}\, dt$

(e) $\displaystyle\int_{-2}^{2} x^3 dx$

(f) $\displaystyle\int_0^{2\pi} \sin(t) dt$

8. Suppose

$$f(x) = \begin{cases} x + 1, & \text{if } 0 \le x \le 1, \\ 4, & \text{if } 1 < x \le 3. \end{cases}$$

Combine geometric arguments with properties of definite integrals to determine the value of the following definite integrals.

(a) $\displaystyle\int_0^1 f(x) dx$

(b) $\displaystyle\int_1^3 f(x) dx$

(c) $\displaystyle\int_0^3 f(x)dx$ (d) $\displaystyle\int_0^2 f(x)dx$

9. The definition of $\displaystyle\int_a^b f(x)dx$ assumes $a < b$.

   (a) Explain why it would be reasonable to define

   $$\int_a^a f(x)dx = 0.$$

   (b) Explain why it would be reasonable to define

   $$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

   whenever $a > b$.

   (c) Using the definitions given in (a) and (b), and assuming that f is integrable on the appropriate intervals, show that

   $$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

   whether $a \le c \le b$, $a \le b \le c$, $b \le a \le c$, $b \le c \le a$, $c \le a \le b$, or $c \le b \le a$. Note that this generalizes (4.1.29).

10. Suppose f is integrable on $[a, b]$ and m and M are constants such that $m \le f(x) \le M$ for all x in $[a, b]$. Show that

    $$m(b-a) \le \int_a^b f(x)dx \le M(b-a).$$

11. Given that f is integrable on $[a, b]$, it may be shown that $g(x) = |f(x)|$ is also integrable on $[a, b]$. Show that

    $$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx.$$

    Hint: Use the fact that $-|f(x)| \le f(x) \le |f(x)|$ for all x in $[a, b]$.

12. In this section we showed that

    $$\int_0^4 2x dx = 16$$

    directly from the definition of the definite integral (with some help from the Mean Value Theorem).

(a) Use these ideas to show that
$$\int_0^1 x dx = \frac{1}{2}.$$

(b) More generally, show that
$$\int_0^b x dx = \frac{b^2}{2}.$$

(c) Let
$$F(x) = \int_0^x t dt.$$

What is the relationship between F and the function $f(x) = x$?

13. In this section we showed that
$$\int_0^4 2x = 16$$
directly from the definition of the definite integral (with some help from the Mean Value Theorem). Use these ideas to show that
$$\int_0^1 x^2 dx = \frac{1}{3}.$$

Section 4.2

Numerical Approximations of Definite Integrals

Computing a definite integral of a function f over an interval $[a, b]$ using upper and lower sums, or even as the limit of Riemann sums, is, for all but the simplest cases, a difficult task. As a result, definite integrals are almost never computed in that manner. For the most part, definite integrals are evaluated either using the Fundamental Theorem of Calculus or using numerical approximation techniques. We will take up the Fundamental Theorem of Calculus approach in the next section; in this section we consider several methods for numerical approximation.

The left-hand and right-hand rules

Recall that for an integrable function f on an interval $[a, b]$, the left-hand rule approximation for $\int_a^b f(x)dx$, using n intervals, is given by

$$A_L = h\sum_{i=0}^{n-1} f(a + ih) \tag{4.2.1}$$

and the right-hand rule approximation by

$$A_R = h\sum_{i=1}^{n} f(a + ih), \tag{4.2.2}$$

where

$$h = \frac{b-a}{n}.$$

We now look at the accuracy of these approximations. Let $x_i = a+ih$, $i = 0, 1, 2, \ldots, n$, the endpoints for a partition of $[a, b]$ using n intervals of equal length h. Assume f is continuous on $[a, b]$ and differentiable on (a, b) and that x is a point in the ith interval, that is, $x_{i-1} \le x \le x_i$. Then the Mean Value Theorem tells us that there exists a point c_i in the interval (x_{i-1}, x_i) such that

$$f'(c_i) = \frac{f(x) - f(x_{i-1})}{x - x_{i-1}}. \tag{4.2.3}$$

Solving for $f(x)$ in (4.2.3), we have

$$f(x) = f(x_{i-1}) + f'(c_i)(x - x_{i-1}). \tag{4.2.4}$$

Integrating both sides of (4.2.4) over the interval $[x_{i-1}, x_i]$ gives us

$$\begin{aligned}\int_{x_{i-1}}^{x_i} f(x)dx &= \int_{x_{i-1}}^{x_i} f(x_{i-1})dx + \int_{x_{i-1}}^{x_i} f'(c_i)(x - x_{i-1})dx \\ &= f(x_{i-1})(x_i - x_{i-1}) + \int_{x_{i-1}}^{x_i} f'(c_i)(x - x_{i-1})dx \\ &= f(x_{i-1})h + \int_{x_{i-1}}^{x_i} f'(c_i)(x - x_{i-1})dx,\end{aligned} \tag{4.2.5}$$

where we have used the fact that the integral of a constant equals the constant multiplied by the length of the interval. Hence

$$\begin{aligned}\int_a^b f(x)dx &= \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f(x)dx \\ &= \sum_{i=1}^n f(x_{i-1})h + \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f'(c_i)(x - x_{i-1})dx \\ &= A_L + \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f'(c_i)(x - x_{i-1})dx.\end{aligned} \tag{4.2.6}$$

Thus we have

$$\begin{aligned}\left|\int_a^b f(x)dx - A_L\right| &= \left|\sum_{i=1}^n \int_{x_{i-1}}^{x_i} f'(c_i)(x - x_{i-1})dx\right| \\ &\le \sum_{i=1}^n \left|\int_{x_{i-1}}^{x_i} f'(c_i)(x - x_{i-1})dx\right|.\end{aligned} \tag{4.2.7}$$

Now

$$\left|\int_{x_{i-1}}^{x_i} f'(c_i)(x - x_{i-1})dx\right| \le \int_{x_{i-1}}^{x_i} |f'(c_i)(x - x_{i-1})|dx, \tag{4.2.8}$$

(see Problem 11 in Section 4.1), so

$$\begin{aligned}\left|\int_a^b f(x)dx - A_L\right| &\le \sum_{i=1}^n \int_{x_{i-1}}^{x_i} |f'(c_i)(x - x_{i-1})|dx \\ &= \sum_{i=1}^n \int_{x_{i-1}}^{x_i} |f'(c_i)|(x - x_{i-1})dx,\end{aligned} \tag{4.2.9}$$

where the last equality follows from the fact that $x - x_{i-1} \ge 0$ for all x in $[x_{i-1}, x_i]$. Now suppose f' is defined and continuous on $[a, b]$ and let M be the maximum value of $|f'(x)|$ for x in $[a, b]$. Then

$$\begin{aligned}\int_{x_{i-1}}^{x_i} |f'(c_i)|(x - x_{i-1})|dx &\le \int_{x_{i-1}}^{x_i} M(x - x_{i-1})dx \\ &= M\int_{x_{i-1}}^{x_i} (x - x_{i-1})dx = \frac{Mh^2}{2},\end{aligned} \tag{4.2.10}$$

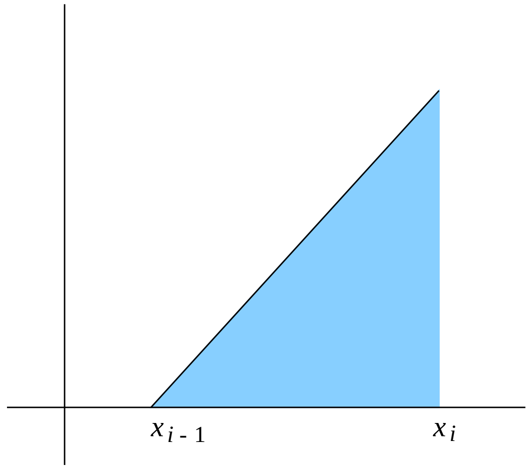


Figure 4.2.1 Graph of $y = x - x_{i-1}$ over the interval $[x_{i-1}, x_i]$

since the region beneath the graph of $y = x - x_{i-1}$ over the interval $[x_{i-1}, x_i]$ is a triangle with base and height both of length $h = x_i - x_{i-1}$ (see Figure 4.2.1). Substituting (4.2.10) into (4.2.9), and recalling that $h = \frac{b-a}{n}$, we have

$$\left| \int_a^b f(x)dx - A_L \right| \leq \sum_{i=1}^{n} \frac{Mh^2}{2} = \frac{nMh^2}{2} = \frac{nM(b-a)^2}{2n^2} = \frac{M(b-a)^2}{2n}. \tag{4.2.11}$$

In other words, the absolute value of the error of the left-hand rule approximation is bounded by a constant multiplied by $\frac{1}{n}$. This explains the behavior of the example in Section 4.1 where we saw that doubling the number of intervals would decrease the error by a factor of $\frac{1}{2}$. The same techniques yield a similar result for the right-hand rule.

The trapezoidal rule

For a decreasing function, the left-hand rule is an upper sum and the right-hand rule is a lower sum; for an increasing function, the left-hand rule is a lower sum and the right-hand rule is an upper sum. Hence, for such functions, it would seem that the average of the left-hand and right-hand rules, that is,

$$\frac{A_L + A_R}{2},$$

should provide a better approximation to $\int_a^b f(x)dx$ than either A_L or A_R. We will now show that this is true in general.

Suppose f, f', and f'' are all defined and continuous on $[a, b]$. From (4.2.4) we know that for any x in the interval $[x_{i-1}, x_i]$ there exists a point c_i in (x_{i-1}, x_i) such that

$$f(x) = f(x_{i-1}) + f'(c_i)(x - x_{i-1}). \tag{4.2.12}$$

Similarly, there exists a point d_i in (x_{i-1}, x_i) such that

$$f(x) = f(x_i) + f'(d_i)(x - x_i). \tag{4.2.13}$$

Using f' in place of f in (4.2.4), there exists a point p_i in (x_{i-1}, c_i) such that

$$f'(c_i) = f'(x_{i-1}) + f''(p_i)(c_i - x_{i-1}) \tag{4.2.14}$$

and a point q_i such that

$$f'(d_i) = f'(x_{i-1}) + f''(q_i)(d_i - x_{i-1}) \tag{4.2.15}$$

Substituting (4.2.14) into (4.2.12) and (4.2.15) into (4.2.13), we have

$$\begin{aligned} f(x) &= f(x_{i-1}) + (f'(x_{i-1}) + f''(p_i)(c_i - x_{i-1}))(x - x_{i-1}) \\ &= f(x_{i-1}) + f'(x_{i-1})(x - x_{i-1}) + f''(p_i)(c_i - x_{i-1})(x - x_{i-1}) \end{aligned} \tag{4.2.16}$$

and

$$\begin{aligned} f(x) &= f(x_i) + (f'(x_{i-1}) + f''(q_i)(d_i - x_{i-1}))(x - x_i) \\ &= f(x_i) + f'(x_{i-1})(x - x_i) + f''(q_i)(d_i - x_{i-1})(x - x_i). \end{aligned} \tag{4.2.17}$$

Taking the average of (4.2.16) and (4.2.17) gives us

$$\begin{aligned} f(x) &= \frac{f(x) + f(x)}{2} \\ &= \frac{f(x_{i-1}) + f(x_i)}{2} + \frac{f'(x_{i-1})((x - x_{i-1}) + (x - x_i))}{2} \\ &\quad + \frac{f''(p_i)(c_i - x_{i-1})(x - x_{i-1}) + f''(q_i)(d_i - x_{i-1})(x - x_i)}{2}. \end{aligned} \tag{4.2.18}$$

Now

$$\int_{x_{i-1}}^{x_i} \frac{f(x_{i-1}) + f(x_i)}{2}\, dx = \frac{f(x_{i-1}) + f(x_i)}{2}(x_i - x_{i-1}) = \frac{f(x_{i-1}) + f(x_i)}{2} h \tag{4.2.19}$$

and

$$\begin{aligned} \int_{x_{i-1}}^{x_i} \frac{f'(x_{i-1})((x - x_{i-1}) + (x - x_i))}{2}\, dx &= f'(x_{i-1}) \int_{x_{i-1}}^{x_i} \left(x - \frac{x_{i-1} + x_i}{2} \right) dx \\ &= 0, \end{aligned} \tag{4.2.20}$$

where the final equality follows from the fact that region between the graph of

$$y = x - \frac{x_{i-1} + x_i}{2}$$

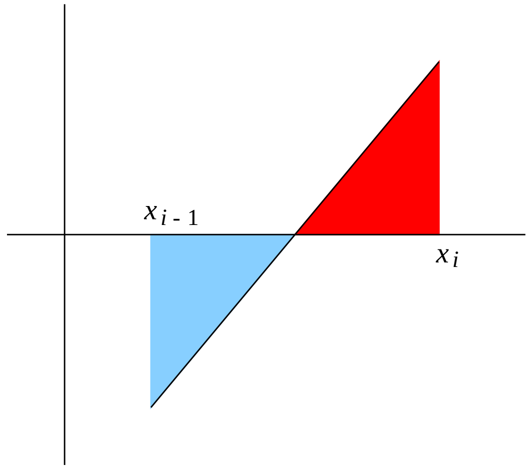


Figure 4.2.2 Graph of $y = x - \dfrac{x_{i-1} + x_i}{2}$ over the interval $[x_{i-1}, x_i]$

and the interval $[x_{i-1}, x_i]$ forms two triangles of equal area, one above the x-axis and one below (see Figure 4.2.2). Moreover, if K is the maximum value of $|f''(x)|$ for x in $[a, b]$, then

$$\left| \frac{f''(p_i)(c_i - x_{i-1})(x - x_{i-1}) + f''(q_i)(d_i - x_{i-1})(x - x_i)}{2} \right| \le \frac{K}{2}(|c_i - x_{i-1}||x - x_{i-1}| + |d_i - x_{i-1}||x - x_i|).$$

Since the points c_i, d_i, and x are all in $[x_{i-1}, x_i]$, it follows that

$$\left| \frac{f''(p_i)(c_i - x_{i-1})(x - x_{i-1}) + f''(q_i)(d_i - x_{i-1})(x - x_i)}{2} \right| \le Kh^2. \tag{4.2.21}$$

Hence

$$\int_{x_{i-1}}^{x_i} \left| \frac{f''(p_i)(c_i - x_{i-1})(x - x_{i-1}) + f''(q_i)(d_i - x_{i-1})(x - x_i)}{2} \right| dx \le \int_{x_{i-1}}^{x_i} Kh^2 dx.$$

Now

$$\int_{x_{i-1}}^{x_i} Kh^2 dx = Kh^2(x_i - x_{i-1}) = Kh^3, \tag{4.2.22}$$

so we have

$$\int_{x_{i-1}}^{x_i} \left| \frac{f''(p_i)(c_i - x_{i-1})(x - x_{i-1}) + f''(q_i)(d_i - x_{i-1})(x - x_i)}{2} \right| dx \le Kh^3. \tag{4.2.23}$$

Putting this all together, we have

$$\begin{aligned}
\int_a^b f(x)dx &= \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f(x)dx \\
&= \sum_{i=1}^n \int_{x_{i-1}}^{x_i} \frac{f(x_{i-1}) + f(x_i)}{2}\, dx \\
&\quad + \sum_{i=1}^n \int_{x_{i-1}}^{x_i} \frac{f'(x_{i-1})((x - x_{i-1}) + (x - x_i))}{2}\, dx \\
&\quad + \sum_{i=1}^n \int_{x_{i-1}}^{x_i} \frac{f''(p_i)(c_i - x_{i-1})(x - x_{i-1}) + f''(q_i)(d_i - x_{i-1})(x - x_i)}{2}\, dx \\
&= \sum_{i=1}^n \frac{f(x_{i-1}) + f(x_i)}{2} h + 0 \\
&\quad + \sum_{i=1}^n \int_{x_{i-1}}^{x_i} \frac{f''(p_i)(c_i - x_{i-1})(x - x_{i-1}) + f''(q_i)(d_i - x_{i-1})(x - x_i)}{2}\, dx \\
&= \frac{h\sum_{i=1}^n f(x_{i-1}) + h\sum_{i=1}^n f(x_i)}{2} \\
&\quad + \sum_{i=1}^n \int_{x_{i-1}}^{x_i} \frac{f''(p_i)(c_i - x_{i-1})(x - x_{i-1}) + f''(q_i)(d_i - x_{i-1})(x - x_i)}{2}\, dx \\
&= \frac{A_L + A_R}{2} \\
&\quad + \sum_{i=1}^n \int_{x_{i-1}}^{x_i} \frac{f''(p_i)(c_i - x_{i-1})(x - x_{i-1}) + f''(q_i)(d_i - x_{i-1})(x - x_i)}{2}\, dx,
\end{aligned}$$

from which it follows, using (4.2.23), that

$$\left| \int_a^b f(x)dx - \frac{A_L + A_R}{2} \right| \le \sum_{i=1}^n Kh^3 = nKh^3 = nK\left(\frac{b-a}{n}\right)^3 = \frac{K(b-a)^3}{n^2}. \qquad (4.2.24)$$

That is, if we approximate $\int_a^b f(x)dx$ by

$$\frac{A_L + A_R}{2},$$

then the absolute value of the error is bounded by a constant multiplied by $\frac{1}{n^2}$. In particular, if we double the number of intervals, we should expect the error to decrease by a factor of

$$\left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

We call

$$A_T = \frac{A_L + A_R}{2} \tag{4.2.25}$$

a *trapezoidal rule* approximation for $\int_a^b f(x)dx$. The name comes from the fact that (4.2.25) may also be derived by replacing the areas of rectangles by the areas of trapezoids in the Riemann sum approximations (see Problem 6).

Example In Section 4.1 we saw that the left-hand and right-hand rule approximations for $\int_{-1}^{2}(x^2 + 1)dx$ using $n = 6$ intervals are $A_L = 5.375$ and $A_R = 6.875$. Hence the corresponding trapezoidal rule approximation is

$$A_T = \frac{5.375 + 6.875}{2} = 6.125.$$

With $n = 100$, the left-hand and right-hand rules give us$A_L = 5.95545$ and $A_R = 6.04545$, yielding a trapezoidal rule approximation of

$$A_T = \frac{5.95545 + 6.04545}{2} = 6.00045.$$

Example In a Section 4.1 we approximated

$$A = \int_1^{10} \frac{1}{t}\, dt$$

using both left-hand and right-hand rules and, after noting that to 6 decimal places $A = 2.302585$, obtained the following table of values:

n	A_R	A_L	$\lvert A - A_R\rvert$	$\lvert A - A_L\rvert$
10	1.960214	2.770214	0.342371	0.467629
20	2.116477	2.521477	0.186108	0.218892
40	2.205491	2.407991	0.097094	0.105406
80	2.253003	2.354253	0.049582	0.051668
160	2.277534	2.328159	0.025052	0.025574
320	2.289994	2.315307	0.012591	0.012722

Using these results, we may compute the following trapezoidal rule approximations:

n	A_T	$\lvert A - A_T\rvert$
10	2.365214	0.062629
20	2.318977	0.016392
40	2.306741	0.004156
80	2.303628	0.001043
160	2.302846	0.000265
320	2.302650	0.000065

We see that the errors in the trapezoidal rule approximations are significantly smaller than the corresponding errors for the left-hand and right-hand rule approximations. Moreover, in agreement with our work above, the errors in the trapezoidal rule approximations decrease by a factor of, roughly, $\frac{1}{4}$ when we double the number of intervals. Hence we see the trapezoidal rule approximations converging to the value of the definite integral at a significantly faster rate than do the left-hand and right-hand rule approximations.

The midpoint rule

As above, let f be an integrable function on an interval $[a, b]$, n a positive integer, $h = \frac{b-a}{n}$, and, for $i = 0, 1, 2, \ldots, n$, $x_i = a + ih$, (the endpoints of a partition of $[a, b]$ with n intervals of equal length h). We may think of the trapezoidal rule as improving on the left-hand and right-hand rules by approximating the area of the region beneath the graph of f and above the interval $[x_{i-1}, x_i]$ using a rectangle with height equal to the average of $f(x_{i-1})$ and $f(x_i)$. Another approach is to average x_{i-1} and x_i before evaluating f. Since each interval is of length h, we may find the midpoint by adding $\frac{h}{2}$ to the left-hand endpoint. Namely, if we let

$$\begin{aligned}
c_1 &= a + \frac{h}{2}, \\
c_2 &= x_1 + \frac{h}{2} = a + h + \frac{h}{2} = a + \frac{3}{2}h, \\
c_3 &= x_2 + \frac{h}{2} = a + 2h + \frac{h}{2} = a + \frac{5}{2}h, \\
&\vdots \\
c_i &= x_{i-1} + \frac{h}{2} = a + (i-1)h + \frac{h}{2} = a + \left(i - \frac{1}{2}\right)h, \\
&\vdots \\
c_n &= x_{n-1} + \frac{h}{2} = a + (n-1)h + \frac{h}{2} = a + \left(n - \frac{1}{2}\right)h,
\end{aligned} \tag{4.2.26}$$

then c_i is the midpoint of the ith interval $[x_{i-1}, x_i]$. We call

$$A_M = \sum_{i=1}^{n} f(c_i)h = h\sum_{i=1}^{n} f\left(a + \left(i - \frac{1}{2}\right)h\right), \tag{4.2.27}$$

the Riemann sum formed by evaluating f at the points c_1, c_2, c_3, …, c_n, a *midpoint rule approximation* of the definite integral $\int_a^b f(x)dx$.

Example To find the midpoint rule approximation for $\int_{-1}^{2}(x^2 + 1)dx$ using $n = 6$ intervals, we would have

$$h = \frac{2 - (-1)}{6} = 0.5.$$

Then the interval endpoints are $x_0 = -1$, $x_1 = -0.5$, $x_2 = 0$, $x_3 = 0.5$, $x_4 = 1$, $x_5 = 1.5$, and $x_6 = 2$, from which we find the midpoints $c_1 = -0.75$, $c_2 = -0.25$, $c_3 = 0.25$, $c_4 = 0.75$,

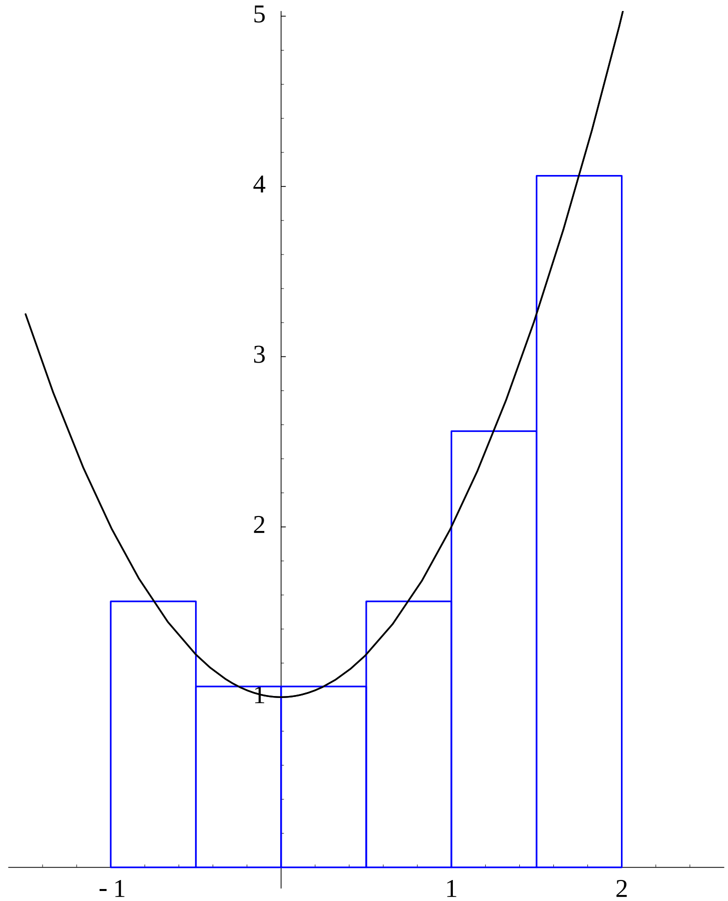


Figure 4.2.3 Midpoint rule approximation for $\int_{-1}^{2}(x^2+1)dx$

$c_5 = 1.25$, and $c_6 = 1.75$ (see Figure 4.2.3). Thus, letting $f(x) = x^2 + 1$, we have

$$\begin{aligned} A_M &= 0.5(f(-0.75) + f(-0.25) + f(0.25) + f(0.75) + f(0.125) + f(1.75)) \\ &= 0.5(1.5625 + 1.0625 + 1.0625 + 1.5625 + 2.5625 + 4.0625) \\ &= 5.9375. \end{aligned}$$

With $n = 100$ intervals, using (4.2.27) with a computer, we have $A_M = 5.999775$.

Example Applying the midpoint rule to the problem of approximating

$$A = \int_1^{10} \frac{1}{t}\ dt,$$

we obtain the following table (again rounded to 6 decimal places):

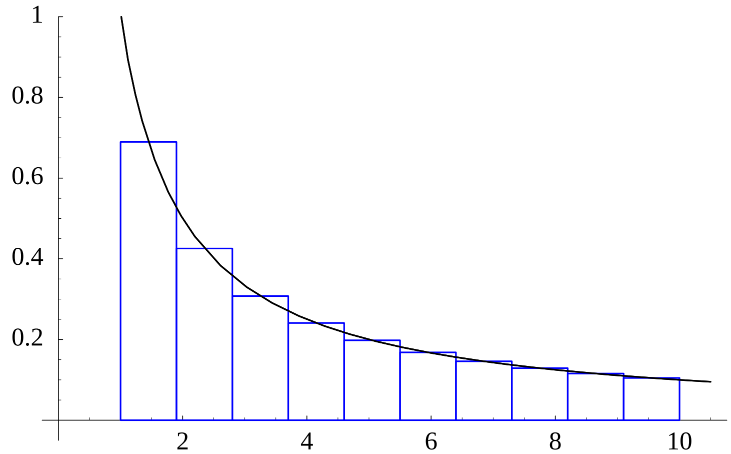


Figure 4.2.4 Midpoint rule approximation for $\displaystyle\int_1^{10} \frac{1}{t}\,dt$

n	A_M	$\lvert A - A_M \rvert$
10	2.272740	0.029845
20	2.294504	0.008081
40	2.300515	0.002070
80	2.302064	0.000521
160	2.302455	0.000130
320	2.302552	0.000033

Notice that, as with the trapezoidal rule, doubling the number of intervals decreases the error by a factor of about $\frac{1}{4}$. Moreover, note that the error in each approximation is approximately $\frac{1}{2}$ of the corresponding error for the trapezoidal rule.

An analysis of the error in the midpoint rule, similar to that which we did above for the left-hand, right-hand, and trapezoidal rules, would show that the absolute value of the error is bounded by a constant multiplied by $\frac{1}{n^2}$. Hence doubling the number of intervals will decrease the error by, roughly, a factor of $\frac{1}{4}$, as was evidenced in the previous example. Moreover, a more careful examination of the error (one requiring the use of Taylor polynomials, which we will discuss in Chapter 5) would show that there is a sense in which it is typically on the order of $\frac{1}{2}$ the size of the error of the trapezoidal rule.

Simpson's rule

We saw above that averaging the left-hand and right-hand rules, two approximation methods with errors bounded by a constant multiple of $\frac{1}{n}$, resulted in an approximation method, the trapezoidal rule, with an error bounded by a constant multiple of $\frac{1}{n^2}$. We might now think that we could improve on the trapezoidal and midpoint rules, two rules with errors bounded by a constant multiple of $\frac{1}{n^2}$, by computing their average. However, it turns out that the relationship between these two rules is not as simple as with the left-hand and right-hand rules; in fact, one really needs to use Taylor polynomials in order to understand the error terms fully. At the same time, there is a hint in our previous example. Given that the error from the midpoint rule was about $\frac{1}{2}$ of the error of the trapezoidal rule, it would be reasonable to guess that perhaps an average of the two which gives twice as much weight to the midpoint rule would be appropriate. This in fact turns out to be the right mixture, and we define

$$A_S = \frac{1}{3}A_T + \frac{2}{3}A_M = \frac{A_T + 2A_M}{3}. \tag{4.2.28}$$

We call A_S a *Simpson's rule approximation* for $\int_a^b f(x)dx$. This method of approximating definite integrals is named for Thomas Simpson (1710-1761). Simpson developed this rule in 1743 as a method for approximating the area under a curve after first approximating the curve with a number of parabolic arcs.

Example Using $n = 6$ intervals, we saw above that the trapezoidal rule approximation for $\int_{-1}^2 (x^2+1)dx$ is $A_T = 6.1250$ and the midpoint rule approximation is $A_M = 5.9375$. Thus the corresponding Simpson's rule approximation is

$$A_S = \frac{6.1250 + (2)(5.9375)}{3} = \frac{18}{3} = 6.$$

With $n = 100$ intervals, we have $A_T = 6.000450$ and $A_M = 5.999775$, giving us

$$A_S = \frac{6.000450 + (2)(5.999775)}{3} = \frac{18}{3} = 6.$$

It may seem surprising in this example that we get the same result with 100 intervals as we do with 6, but in fact this is the exact answer. It may be shown, either by careful examination of the error or by deriving the rule from parabolic approximations, that Simpson's rule will find the exact value for the definite integral of any quadratic polynomial. What is even more surprising, careful examination of the error using Taylor polynomials shows that Simpson's rule is exact for cubic polynomials as well.

Example Using the values for the trapezoidal and midpoint rule approximations obtained above, we have the following approximations for

$$A = \int_1^{10} \frac{1}{t}\, dt$$

using Simpson's rule:

n	A_S	$\lvert A - A_S \rvert$
10	2.303565	0.000980
20	2.302662	0.000077
40	2.302590	0.000005
80	2.302585	0.000000

We have stopped the table at 80 intervals because at that point the approximation is accurate to 6 decimal places.

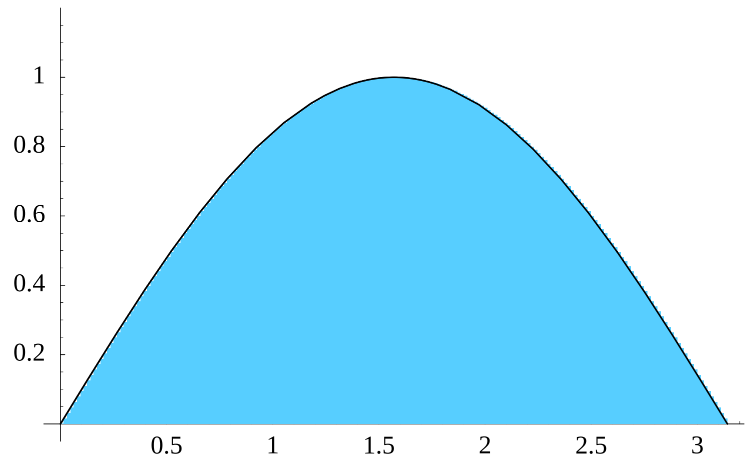


Figure 4.2.5 Region beneath $y = \sin(x)$ over the interval $[0, \pi]$

It may be shown that the absolute value of the error using Simpson's rule is bounded by a constant multiple of $\frac{1}{n^4}$, resulting in a dramatic improvement over both the trapezoidal and midpoint rules. For Simpson's rule, doubling the number of intervals typically decreases the error by a factor of

$$\left(\frac{1}{2}\right)^4 = \frac{1}{16},$$

a general fact for which we can see some evidence in the preceding example. To be fair, since Simpson's rule makes use of both the trapezoidal rule and the midpoint rule approximations, the function being integrated must be evaluated both at the endpoints and at the midpoint of each interval. This requires evaluating the function at $2n + 1$ points, whereas the trapezoidal rule evaluates the function at $n + 1$ points and the midpoint rule evaluates the function at n points. Thus, for direct comparison of errors, one should compare, for example, Simpson's rule with 10 subdivisions to the other rules using 20 subdivisions. Nevertheless, Simpson's rule converges to the value of the integral much faster than the other methods and, hence, is the method of preference among the ones we have discussed. Even faster methods exist, but we will leave them for a more advanced course.

When approximating the value of an integral, there is in general no practical way to know how many intervals are necessary in order to obtain an approximation to a desired level of accuracy. Analogous to the way in which we applied Newton's method, we normally compute a sequence of approximations, perhaps starting with only two intervals and then doubling the number of intervals from one approximation to the next, stopping when we obtain two successive approximations whose difference, in absolute value, is less than the desired level of accuracy. The next example illustrates this procedure.

Example Suppose we wish to approximate, with an error less than 0.0001, the area A of the region between one arch of the curve $y = \sin(x)$ and the x-axis, as shown in Figure 4.2.5. That is, we want to find

$$A = \int_0^\pi \sin(x)dx.$$

Starting with $n = 2$ intervals and using Simpson's rule, we generate the following table of approximations, rounding to 6 decimal places:

n	A_S
2	2.004560
4	2.000269
8	2.000017
16	2.000001

Since the absolute value of the difference between the last two approximations is less than 0.0001, we stop at this point and use 2.0000 as our approximation for A.

Problems

1. Approximate each of the following integrals using the trapezoidal and midpoint rules with $n = 4$ intervals.

   (a) $\int_0^1 x^2 dx$ (b) $\int_0^\pi \sin(x)dx$

   (c) $\int_1^5 \frac{1}{t}\, dt$ (d) $\int_{-1}^1 z^3 dz$

   (e) $\int_0^4 (t^2 + t)dt$ (f) $\int_0^2 \sqrt{4 - x^2}dx$

2. Use your results from Problem 1 to compute the corresponding Simpson's rule approximation for each integral.

3. Approximate each of the following integrals using the trapezoidal and midpoint rules with $n = 50$ intervals.

   (a) $\int_0^1 x^2 dx$ (b) $\int_0^\pi \sin(x)dx$

   (c) $\int_{-2}^5 (4x^2 + 3x - 6)dx$ (d) $\int_1^2 \frac{\sin(x)}{x}\, dx$

   (e) $\int_0^\pi x^2 \cos(x)dx$ (f) $\int_0^{2\pi} \sqrt{1 - \sin^2(t)}\, dt$

   (g) $\int_{-5}^5 \frac{1}{x^2 + 1}\, dx$ (h) $\int_{-\pi}^\pi \sin(3x)cos(x)dx$

4. Use your results from Problem 3 to compute the corresponding Simpson's rule approximation for each integral.

5. Approximate the following definite integrals using Simpson's rule. Starting with $n = 2$ intervals, compute a sequence of approximations by doubling the number of intervals from one approximation to the next. Stop when the absolute value of the difference between two successive approximations is less than 0.00001.

(a) $\displaystyle\int_1^4 x^2 dx$

(b) $\displaystyle\int_0^6 (3x^2 + 4x - 3)dx$

(c) $\displaystyle\int_0^\pi \sin^2(x)dx$

(d) $\displaystyle\int_0^{2\pi} \sin^2(x)\cos^2(x)dx$

(e) $\displaystyle\int_{-\pi}^{\pi} \sqrt{1+\cos^2(\theta)}\, d\theta$

(f) $\displaystyle\int_{0.1}^{2} \frac{\sin(x)}{x}\, dx$

6. Suppose f is integrable on the interval $[a,b]$. Divide $[a,b]$ into n equal intervals of length $h = \frac{b-a}{n}$ and let x_0, x_1, x_2, $\ldots$, x_n be the endpoints of these intervals. Let A_T be the trapezoidal rule approximation for $\int_a^b f(x)dx$.

   (a) Show that
   $$A_T = \frac{h}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)).$$

   (b) Assume $f(x) \geq 0$ for all x in $[a,b]$. For $i = 1, 2, 3, \ldots, n$, let A_i be the area of the trapezoid with vertices at $(x_{i-1}, 0)$, $(x_{i-1}, f(x_{i-1}))$, $(x_i, f(x_i))$, and $(x_i, 0)$ (that is, A_i is the area of a trapezoid with one side being the interval $[x_{i-1}, x_i]$ and parallel sides extending from the x-axis up to the graph of f). Then we could approximate $\int_a^b f(x)dx$ by $A_1 + A_2 + A_3 + \cdots + A_n$. Show that
   $$A_T = A_1 + A_2 + A_3 + \cdots + A_n.$$

7. Suppose f is integrable on the interval $[a,b]$. Divide $[a,b]$ into $2n$ equal intervals of length $h = \frac{b-a}{2n}$ and let x_0,, x_1, x_2, $\ldots$, x_{2n} be the endpoints of these intervals. Let A_T and A_M be the trapezoidal rule and midpoint rule approximations for $\int_a^b f(x)dx$ using the n intervals with endpoints x_0, x_2, x_4, $\ldots$, x_{2n}. Let A_S be the corresponding Simpson's rule approximation. Show that
   $$A_S = \frac{h}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{2n-2}) + 4f(x_{2n-1}) + f(x_{2n})).$$

8. Let $T(t)$ be the temperature at t hours after midnight at the Kalispell airport and suppose the following values for T were recorded on March 15, 1955:

Time (t)	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5
Temperature (T)	40	38	37	36	33	30	28	28	27	26
Time (t)	5.0	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5
Temperature (T)	24	26	30	30	32	35	37	38	40	45
Time (t)	10.0	10.5	11.0	11.5	12.0					
Temperature (T)	47	47	48	49	50					

   (a) Approximate $\int_0^{12} T(t)dt$ using Simpson's rule. You may wish to use the formula in Problem 7.

(b) What does

$$A = \frac{1}{12}\int_0^{12} T(t)dt$$

represent?

(c) How does A compare with $\frac{1}{25}\sum_{t=0}^{24} T\left(\frac{t}{2}\right)$?

9. Find the area beneath one arch of the curve $y = \sin^2(x)$.

10. Let R be the region in the plane bounded by the curves $y = x^2$ and $y = (x-2)^2$ and the x-axis. Find the area of R.

Difference Equations to Differential Equations

Section 4.3

The Fundamental Theorem of Calculus

We are now ready to make the long-promised connection between differentiation and integration, between areas and tangent lines. We will look at two closely related theorems, both of which are known as the Fundamental Theorem of Calculus. We will call the first of these the *Fundamental Theorem of Integral Calculus*.

Suppose f is integrable on $[a, b]$ and F is an antiderivative of f on (a, b) which is continuous on $[a, b]$. In particular, $F'(x) = f(x)$ for all x in (a, b). Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be a partition of $[a, b]$ and, as usual, let $\Delta x_i = x_i - x_{i-1}$, $i = 1, 2, 3, \ldots, n$. Now

$$\begin{aligned} F(b) - F(a) &= F(x_n) - F(x_0) \\ &= F(x_n) + (F(x_{n-1}) - F(x_{n-1})) + (F(x_{n-2}) - F(x_{n-2})) + \cdots \\ &\qquad + (F(x_1) - F(x_1)) - F(x_0) \\ &= (F(x_n) - F(x_{n-1})) + (F(x_{n-1}) - F(x_{n-2})) + \cdots \\ &\qquad + (F(x_1) - F(x_0)) \\ &= \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})). \end{aligned} \tag{4.3.1}$$

By the Mean Value Theorem, for every $i = 1, 2, 3, \ldots, n$, there exists a point c_i in the interval $[x_{i-1}, x_i]$ such that

$$F'(c_i) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}}. \tag{4.3.2}$$

Since $F'(c_i) = f(c_i)$ and $x_i - x_{i-1} = \Delta x_i$, it follows that

$$F(x_i) - F(x_{i-1}) = f(c_i)\Delta x_i. \tag{4.3.3}$$

Hence, putting (4.3.3) into (4.3.1),

$$F(b) - F(a) = \sum_{i=1}^{n} f(c_i)\Delta x_i. \tag{4.3.4}$$

Thus $F(b) - F(a)$ is equal to the value of a Riemann sum using the partition P, and so must lie between the upper and lower sums for P. That is, we have shown that for any partition P,

$$L(f, P) \le F(b) - F(a) \le U(f, P). \tag{4.3.5}$$

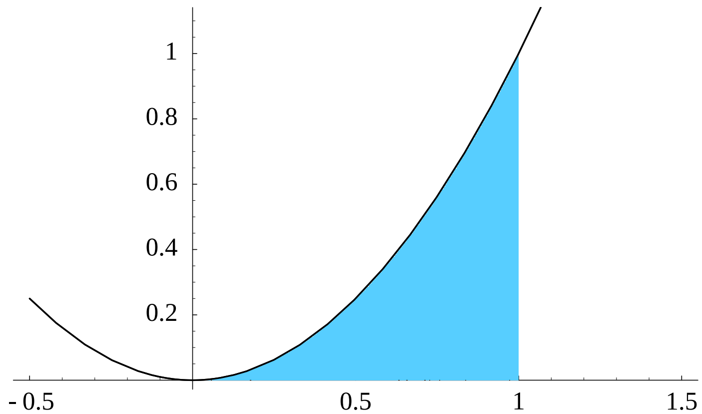


Figure 4.3.1 Region beneath the graph of $f(x) = x^2$ over the interval $[0, 1]$

But, since f is integrable, there is only one number that has this property, namely, $\int_a^b f(x)dx$. In other words, we have shown that

$$\int_a^b f(x)dx = F(b) - F(a). \tag{4.3.6}$$

Fundamental Theorem of Integral Calculus If f is integrable on $[a, b]$ and F is an antiderivative of f on (a, b) which is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a). \tag{4.3.7}$$

This result reveals a sense in which integration is the inverse of differentiation: The definite integral of a function f may be evaluated easily, using (4.3.7), provided we can find a function F whose derivative is f.

It is common to write

$$F(x)\Big|_a^b$$

for $F(b) - F(a)$. With this notation, (4.3.7) becomes

$$\int_a^b f(x)dx = F(x)\Big|_a^b. \tag{4.3.8}$$

Example Since

$$F(x) = \frac{1}{3}x^3$$

is an antiderivative of $f(x) = x^2$, we have

$$\int_0^1 x^2 dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3} - 0 = \frac{1}{3}.$$

Thus the area under the parabola $y = x^2$ and above the interval $[0, 1]$ on the x-axis is exactly $\frac{1}{3}$. See Figure 4.3.1.

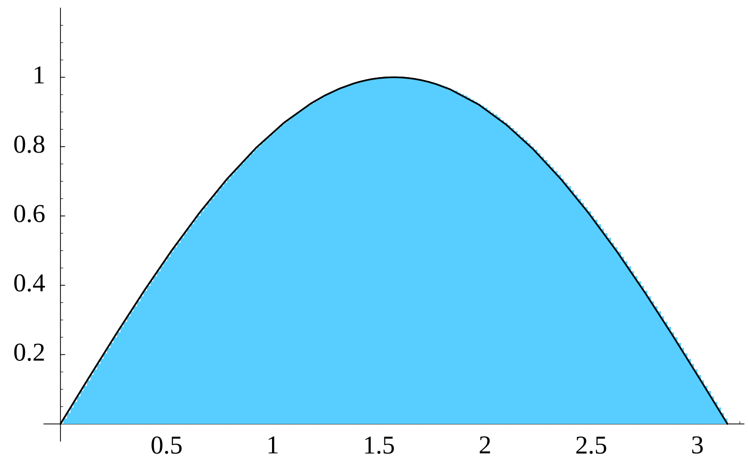


Figure 4.3.2 Region beneath the graph of $y = \sin(x)$ over the interval $[0, \pi]$

Note that F in the previous example is but one of an infinite number of antiderivatives of f. We can in fact use any antiderivative of f we want in applying (4.3.7), although we typically choose the simplest one we can find.

Example Since

$$G(x) = \frac{1}{3}x^3 + x$$

is an antiderivative of $g(x) = x^2 + 1$ (you may check by differentiating G), we have

$$\int_{-1}^{2} (x^2 + 1)dx = \left(\frac{1}{3}x^3 + x\right)\bigg|_{-1}^{2} = \left(\frac{8}{3} + 2\right) - \left(-\frac{1}{3} - 1\right) = 6,$$

as we claimed in Section 4.2.

Example If A is the area under one arch of the curve $y = \sin(x)$, then

$$A = \int_0^\pi \sin(x)dx.$$

Since $F(x) = -\cos(x)$ is an antiderivative of $f(x) = \sin(x)$, we have

$$A = \int_0^\pi \sin(x)dx = -\cos(x)\Big|_0^\pi = -\cos(\pi) - (-\cos(0)) = 1 + 1 = 2.$$

See Figure 4.3.2.

Example Since

$$F(x) = \frac{4}{3}x^3 - \frac{1}{2}x^2 + 2x$$

is an antiderivative of $f(x) = 4x^2 - x + 2$ (again, you may check this by differentiating F), we have

$$\begin{aligned}\int_{-2}^{3}(4x^2 - x + 2)dx &= \left(\frac{4}{3}x^3 - \frac{1}{2}x^2 + 2x\right)\Bigg|_{-2}^{3} \\ &= \left(\frac{108}{3} - \frac{9}{2} + 6\right) - \left(-\frac{32}{3} - 2 - 4\right) \\ &= \frac{325}{6}.\end{aligned}$$

Example Since

$$F(t) = \frac{2}{3}t^{\frac{3}{2}}$$

is an antiderivative of $f(t) = \sqrt{t}$, we have

$$\int_0^4 \sqrt{t}\ dt = \frac{2}{3}t^{\frac{3}{2}}\Big|_0^4 = \frac{16}{3} - 0 = \frac{16}{3}.$$

As can be seen from these examples, the Fundamental Theorem of Integral Calculus provides us with a powerful tool for evaluating definite integrals exactly. However, to utilize the theorem we must first find an antiderivative for the function we are integrating. This turns out to be a difficult problem in general, and we will devote the next two sections, as well as parts of Chapter 6, to developing techniques to aid in finding antiderivatives. For example,

$$F(x) = -\frac{1}{2}x^3\cos(2x) + \frac{3}{4}x^2\sin(2x) + \frac{3}{4}x\cos(2x) - \frac{3}{8}\sin(2x)$$

is an antiderivative of $f(x) = x^3\sin(2x)$, as may be checked by differentiation, but at this point it is not clear how to find such an antiderivative in the first place. Moreover, there are integrable functions, even relatively simple ones such as

$$f(x) = \frac{\sin(x)}{x},$$

which do not have antiderivatives expressible in terms of the elementary functions studied in calculus.

The Fundamental Theorem of Integral Calculus tells us that if a function f has an antiderivative, then we may use that antiderivative to evaluate a definite integral of f, but it does not tell us which functions have antiderivatives. The Fundamental Theorem of Differential Calculus will tell us, in part, that every continuous function has an antiderivative. Before beginning that discussion, we need to extend the definition of the definite integral slightly.

The definition of $\int_a^b f(x)dx$ in Section 4.1 implicitly assumes that $a < b$. For the work we are about to do, we need to extend the definition to include $a \geq b$, as we did in Problem 9 of Section 4.1. First of all, if $a = b$, it would seem reasonable for the value of the definite integral to be 0 since the region between the graph of the function and the x-axis has been reduced to a line segment. Hence we make the following definition.

Definition For any function f defined at a point a, we define

$$\int_a^a f(x)dx = 0. \tag{4.3.9}$$

Note that with this definition, the statement

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, \tag{4.3.10}$$

which we discussed in Section 4.1 in the case $a < c < b$, holds true even if $a = c$, $b = c$, or $a = b = c$. Now suppose we have $a < b < c$. Then

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx, \tag{4.3.11}$$

from which it follows that

$$\int_a^b f(x)dx = \int_a^c f(x)dx - \int_b^c f(x)dx. \tag{4.3.12}$$

If we define

$$\int_c^b f(x)dx = -\int_b^c f(x)dx, \tag{4.3.13}$$

then we may rewrite (4.3.12) in the form of (4.3.10). For this reason, we make the following definition.

Definition If $b < a$ and f is integrable on $[b, a]$, we define

$$\int_a^b f(x)dx = -\int_b^a f(x)dx. \tag{4.3.14}$$

You may check that with these two extensions to the definition of the definite integral, we may now state the following proposition.

Proposition If f is integrable on a closed interval containing the points a, b, and c, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx. \tag{4.3.15}$$

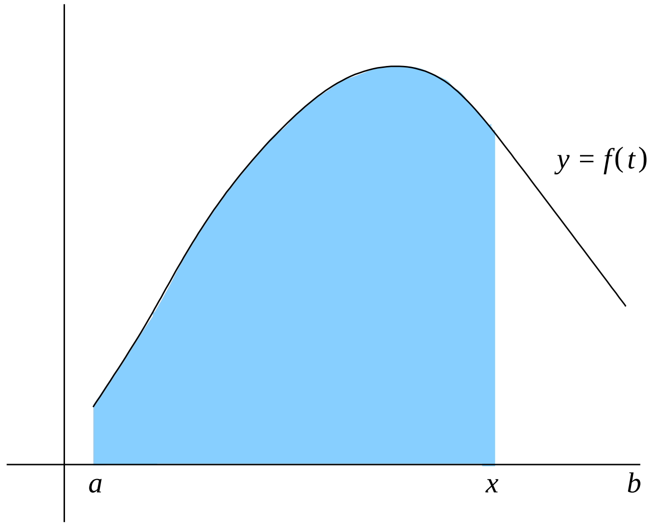


Figure 4.3.3 $F(x) = \int_a^x f(t)dt$ is the area from a to x

We may now return to our discussion of antiderivatives and the Fundamental Theorem of Differential Calculus. Suppose f is continuous on the interval $[a, b]$. We want to construct an antiderivative for f on (a, b). From the Fundamental Theorem of Integral Calculus, we know that if F is an antiderivative of f on (a, b) which is continuous on $[a, b]$, then for any x in (a, b) we would have

$$\int_a^x f(t)dt = F(x) - F(a), \tag{4.3.16}$$

that is,

$$F(x) = F(a) + \int_a^x f(t)dt. \tag{4.3.17}$$

Hence, if we are seeking an antiderivative for f, it makes sense to define

$$F(x) = \int_a^x f(t)dt \tag{4.3.18}$$

and verify that $F'(x) = f(x)$ for all x in (a, b). Note that $F(x)$, geometrically, is the cumulative area between the graph of f and the x-axis from a to x, as shown in Figure 4.3.3. We need to compute

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{1}{h} \left(\int_a^{x+h} f(t)dt - \int_a^x f(t)dt \right) \tag{4.3.19}$$

for x in (a, b). Now

$$\int_a^{x+h} f(t)dt = \int_a^x f(t)dt + \int_x^{x+h} f(t)dt, \tag{4.3.20}$$

so

$$\int_a^{x+h} f(t)dt - \int_a^x f(t)dt = \int_x^{x+h} f(t)dt. \tag{4.3.21}$$

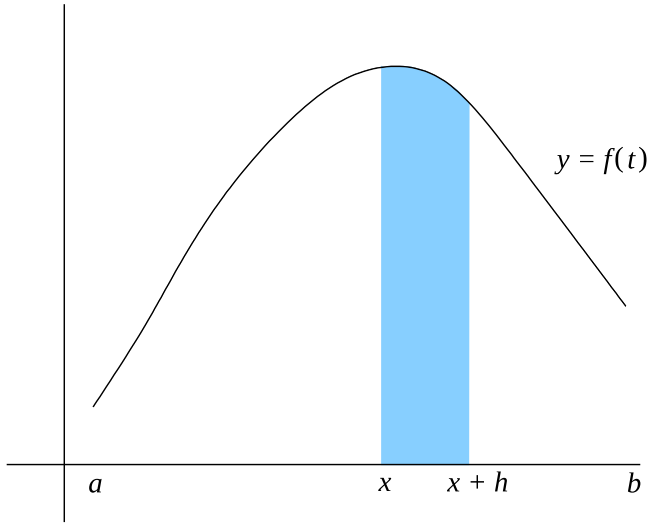


Figure 4.3.4 $\int_a^{x+h} f(t)dt - \int_x^a f(t)dt = \int_x^{x+h} f(t)dt$

See Figure 4.3.4. Thus

$$F'(x) = \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)dt. \tag{4.3.22}$$

Suppose $h > 0$. Since f is continuous, f has a minimum value $m(h)$ and a maximum value $M(h)$ on the interval $[x, x+h]$. Hence $m(h) \le f(x) \le M(h)$ for all x in $[x, x+h]$, from which it follows that

$$\int_x^{x+h} m(h)dt \le \int_x^{x+h} f(t)dt \le \int_x^{x+h} M(h)dt. \tag{4.3.23}$$

Since $m(h)$ and $M(h)$ are constants, (4.3.23) implies

$$m(h)h \le \int_x^{x+h} f(t)dt \le M(h)h. \tag{4.3.24}$$

Thus

$$m(h) \le \frac{1}{h} \int_x^{x+h} f(t)dt \le M(h). \tag{4.3.25}$$

Now $m(h) = f(c)$ for some c in $[x, x+h]$. Moreover, as h approaches 0, $x+h$ approaches x, and so c must also approach x. Hence, since f is continuous,

$$\lim_{h\to 0^+} m(h) = \lim_{h\to 0^+} f(c) = f(x). \tag{4.3.26}$$

Similarly,

$$\lim_{h\to 0^+} M(h) = f(x). \tag{4.3.27}$$

It now follows from (4.3.25) that

$$\lim_{h \to 0^+} \frac{1}{h} \int_x^{x+h} f(t)dt = f(x). \tag{4.3.28}$$

A similar argument shows that

$$\lim_{h \to 0^-} \frac{1}{h} \int_x^{x+h} f(t)dt = f(x), \tag{4.3.29}$$

and so we may conclude that

$$F'(x) = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)dt = f(x). \tag{4.3.30}$$

That is,

$$F(x) = \int_a^x f(t)dt$$

is an antiderivative of f on (a, b).

Fundamental Theorem of Differential Calculus If f is continuous on the closed interval $[a, b]$ and F is the function on (a, b) defined by

$$F(x) = \int_a^x f(t)dt, \tag{4.3.31}$$

then F is differentiable on (a, b) with $F'(x) = f(x)$ for all x in (a, b). In other words,

$$\frac{d}{dx} \int_a^x f(t)dt = f(x) \tag{4.3.32}$$

for all x in (a, b).

It is worth noting that (4.3.32) holds for $x < a$ as well, as long as f is continuous on a closed interval which contains both x and a.

Example Let

$$f(x) = \begin{cases} \dfrac{\sin(x)}{x}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$$

Then f is continuous on $(-\infty, \infty)$, so

$$F(x) = \int_0^x f(t)dt$$

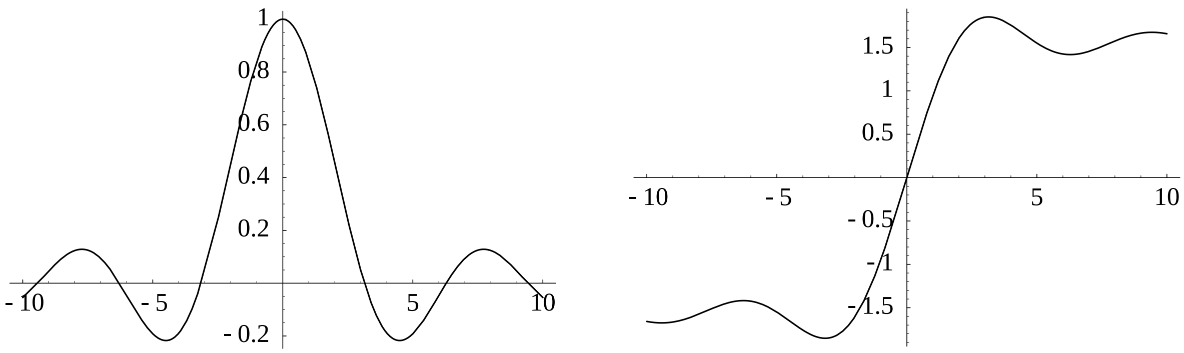


Figure 4.3.5 Graphs of $f(x) = \dfrac{\sin(x)}{x}$ and $F(x) = \displaystyle\int_0^x \frac{\sin(t)}{t}\,dt$

is an antiderivative of f on $(-\infty, \infty)$. In particular,

$$F'(x) = \frac{\sin(x)}{x}$$

for all $x \neq 0$. The graphs of F and f are shown in Figure 4.3.5. Geometrically, $F(x)$ is the cumulative area between the graph of f and the x-axis from 0 to x and $F'(x)$ is the rate at which area is accumulating as x increases. Since the rate at which area is accumulating depends on the height of the curve, it is natural to expect, and the Fundamental Theorem of Differential Calculus confirms, that $F'(x) = f(x)$. The function F is known as the *sine integral function*. It may be shown that it is not representable in closed form in terms of the elementary functions of calculus.

Example Using Leibniz notation,

$$\frac{d}{dx}\int_0^x \sin(t^2)dt = \sin(x^2).$$

Example Suppose

$$G(x) = \int_0^{3x} \sin(t^2)dt.$$

Then $G(x) = F(h(x))$, where $h(x) = 3x$ and

$$F(x) = \int_0^x \sin(t^2)dt.$$

Hence, using the chain rule,

$$G'(x) = F'(h(x))h'(x) = \sin((3x)^2)(3) = 3\sin(9x^2).$$

Example Suppose

$$H(x) = \int_x^0 \frac{1}{1+t^4}\,dt.$$

Then, using (4.3.14),

$$H(x) = -\int_0^x \frac{1}{1+t^4}\,dt,$$

so

$$H'(x) = -\frac{d}{dx}\int_0^x \frac{1}{1+t^4}\,dt = -\frac{1}{1+x^4}.$$

Example Suppose

$$F(x) = \int_{2x}^{x^2} \sqrt{1+t^4}\,dt.$$

Then, using (4.3.15) and (4.3.14),

$$F(x) = \int_{2x}^{0} \sqrt{1+t^4}\,dt + \int_0^{x^2} \sqrt{1+t^4}\,dt = -\int_0^{2x} \sqrt{1+t^4}\,dt + \int_0^{x^2} \sqrt{1+t^4}\,dt.$$

Note that there is nothing special about using 0 in this decomposition, other than the requirement that the function $f(t) = \sqrt{1+t^4}$ be integrable on all of the relevant intervals. Now we have

$$\begin{aligned} F'(x) &= -\frac{d}{dx}\int_0^{2x} \sqrt{1+t^4}\,dt + \frac{d}{dx}\int_0^{x^2} \sqrt{1+t^4}\,dt \\ &= -\sqrt{1+(2x)^4}\,(2) + \sqrt{1+(x^2)^4}\,(2x) \\ &= 2x\sqrt{1+x^8} - 2\sqrt{1+16x^4}. \end{aligned}$$

To summarize this section, the Fundamental Theorem of Integral Calculus provides us with an elegant method for evaluating definite integrals, but is useful only when we can find an antiderivative for the function being integrated. The Fundamental Theorem of Differential Calculus tells us that every continuous function has an antiderivative and shows how to construct one using the definite integral. Unfortunately, this brings us in circle and does not provide us with an effective means for finding antiderivatives to use in applying the Fundamental Theorem of Integral Calculus. For example, we know that

$$F(x) = \int_1^x \frac{\sin(t)}{t}\,dt$$

is an antiderivative of

$$f(x) = \frac{\sin(x)}{x},$$

but this is of no help in evaluating, say,

$$\int_1^4 \frac{\sin(x)}{x}\,dx.$$

Hence in order to fully utilize the Fundamental Theorem of Integral Calculus in the evaluation of definite integrals, we must develop some procedures to aid in finding antiderivatives. We will turn to this problem in the next section.

Problems

1. Evaluate the following definite integrals using the Fundamental Theorem of Integral Calculus.

   (a) $\int_0^1 x dx$ (b) $\int_0^3 (x^2 + 2x) dx$

   (c) $\int_0^2 x^3 dx$ (d) $\int_{-1}^1 x^3 dx$

   (e) $\int_0^2 (2x^3 + 3x^2 + x - 4) dx$ (f) $\int_1^4 \frac{1}{x^2}\, dx$

   (g) $\int_1^4 \frac{1}{\sqrt{t}}\, dt$ (h) $\int_0^8 \sqrt{t+1}\, dt$

   (i) $\int_0^{\frac{\pi}{2}} \sin(x) dx$ (j) $\int_{-\pi}^{\pi} \cos(z) dz$

2. Evaluate the following definite integrals using the Fundamental Theorem of Integral Calculus.

   (a) $\int_0^2 (x+1)^2 dx$ (b) $\int_0^2 (2x+1)^2 dx$

   (c) $\int_0^4 \sqrt{1+2t}\, dt$ (d) $\int_0^{\frac{\pi}{4}} \sec^2(x) dx$

   (e) $\int_0^{\pi} \sin(2x) dx$ (f) $\int_0^{\pi} 5\cos(3x) dx$

   (g) $\int_0^{\frac{\pi}{3}} 4\sin(3x) dx$ (h) $\int_{-\pi}^{\pi} 8\cos(5\theta) d\theta$

   (i) $\int_0^{\sqrt{\pi}} 2x\sin(x^2) dx$ (j) $\int_{-1}^2 2x(1+x^2)^5 dx$

   (k) $\int_0^{\sqrt{\pi}} x\sin(x^2) dx$ (l) $\int_{-1}^2 x(1+x^2)^5 dx$

3. For each of the following functions, graph both f and

$$F(x) = \int_0^x f(t) dt$$

   together over the given interval.

   (a) $f(x) = \sin(x)$ on $[-2\pi, 2\pi]$ (b) $f(x) = \sin(x^2)$ on $[0, 10]$

   (c) $f(x) = \frac{1}{1+x^4}$ on $[-3, 3]$ (d) $f(x) = \frac{1}{x+1}$ on $[0, 10]$

4. Find the derivatives of each of the following functions.

   (a) $F(x) = \int_0^x \sin^2(4t)dt$

   (b) $g(x) = \int_2^x \frac{3}{t+2}\, dt$

   (c) $F(x) = \int_x^\pi \cos^3(t)dt$

   (d) $G(t) = \int_t^0 \sqrt{4-z^2}\, dz$

   (e) $f(x) = \int_0^{x^2} \frac{1}{1+s^2}\, ds$

   (f) $h(z) = \int_z^{3z} \sqrt{1+t^2}\, dt$

5. Evaluate the following derivatives.

   (a) $\frac{d}{dx}\int_1^x \frac{1}{1+t^2}\, dt$

   (b) $\frac{d}{dx}\int_1^{3x} \frac{\sin(3t)}{t}\, dt$

   (c) $\frac{d}{dt}\int_{t^2}^5 \sin^2(3x)dx$

   (d) $\frac{d}{dx}\int_{3x}^{x^2} \frac{1}{\sqrt{1+t^2}}\, dt$

6. Find the area of the region beneath one arch of the curve $y = 3\sin(2x)$.

7. Let R be the region bounded by the curves $y = x^2$ and $y = (x-2)^2$ and the x-axis. Find the area of R.

8. Explain why the integral
$$\int_0^1 (x - x^2)dx$$
is the area of the region bounded by the curves $y = x^2$ and $y = x$. Find this area.

9. Explain why the integral
$$\int_{-1}^1 (2 - 2x^2)dx$$
is the area of the region bounded by the curves $y = 1 - x^2$ and $y = x^2 - 1$. Find this area.

Difference Equations to Differential Equations

Section 4.4

Using the Fundamental Theorem

As we saw in Section 4.3, using the Fundamental Theorem of Integral Calculus reduces the problem of evaluating a definite integral to the problem of finding an antiderivative. Unfortunately, finding antiderivatives, even for relatively simple functions, cannot be done as routinely as the computation of derivatives. For example, suppose we let $f(x) = \sin(x)$, $g(x) = x$, and

$$h(x) = \frac{f(x)}{g(x)} = \frac{\sin(x)}{x}.$$

Then, since we know the derivative of f and we know the derivative of g, it is a simple matter to find the derivative of h using the quotient rule. However, knowing the antiderivatives of f and g in no way helps us find the antiderivative of h. In fact, it has been shown that the antiderivative of h is not expressible in terms of any finite combination of algebraic and elementary transcendental functions. Because of results like this, many of the definite integrals that are encountered in applications cannot be evaluated using the Fundamental Theorem of Integral Calculus; instead, they must be approximated using numerical techniques such as those we studied in Section 4.2. Of course, when antiderivatives are available, the Fundamental Theorem is the best way to evaluate an integral. To this end, we will investigate, in this section and in the next, techniques for evaluating definite integrals by finding antiderivatives and applying the Fundamental Theorem.

Before we begin, we need to introduce some additional notation and terminology. First of all, we will call the collection of all antiderivatives of a given function f the *general antiderivative* of f. For example, if $f(x) = 3x^2$, then the general antiderivative of f is given by $F(x) = x^3 + c$, where c is an arbitrary constant.

Second, since the Fundamental Theorem of Calculus draws a close connection between antiderivatives and definite integrals, it is customary to borrow the notation for the general antiderivative from the notation for the definite integral. Hence the general antiderivative of a function f with respect to the variable x is denoted by

$$\int f(x)dx. \tag{4.4.1}$$

This is usually referred to as the *indefinite integral* of f with respect to x. Thus the terms indefinite integral and general antiderivative are synonymous, and from this point on we will prefer the former to the latter.

Example In this notation, we write

$$\int 3x^2 dx = x^3 + c,$$

where c is assumed to be an arbitrary constant.

Since finding an indefinite integral involves reversing the process of differentiation, we can rewrite our basic results about derivatives in terms of indefinite integrals. Hence we have the following list of integration formulas:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c \text{ (where } n \neq -1 \text{ is a rational number)}, \tag{4.4.2}$$

$$\int \sin(x)dx = -\cos(x) + c, \tag{4.4.3}$$

$$\int \cos(x)dx = \sin(x) + c, \tag{4.4.4}$$

$$\int \sec^2(x)dx = \tan(x) + c, \tag{4.4.5}$$

$$\int \csc^2(x)dx = -\cot(x) + c, \tag{4.4.6}$$

$$\int \sec(x)\tan(x)dx = \sec(x) + c, \tag{4.4.7}$$

$$\int \csc(x)\cot(x)dx = -\csc(x) + c. \tag{4.4.8}$$

Note that each one of these formulas may be verified by checking that the derivative of the right-hand side is equal to the function inside the integral sign on the left-hand side. Also notice that we have not used any special techniques to find these results; rather, we know these formulas only because they are the inverses of differentiation formulas that we learned in Chapter 3. Thus, for example, we know that

$$\int \sec^2(x)dx = \tan(x) + c,$$

but we do not know, nor do we even know how to begin to find, $\int \sec(x)dx$, which would at first seem to be an easier problem.

The following proposition is a consequence of the corresponding basic properties of differentiation.

Proposition If the indefinite integrals of f and g exist, then

$$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx \tag{4.4.9}$$

and

$$\int (f(x) - g(x))dx = \int f(x)dx - \int g(x)dx. \tag{4.4.10}$$

Moreover, for any constant k,

$$\int kf(x)dx = k\int f(x)dx. \tag{4.4.11}$$

Example Using (4.4.2) with the results of the previous proposition, we have

$$\int (5x^3 - 3x + 2)dx = 5\int x^3 dx - 3\int x dx + 2\int 1 dx = \frac{5}{4}x^4 - \frac{3}{2}x^2 + 2x + c.$$

It is worth noting that $\int 1dx$ is typically denoted simply by $\int dx$.

Example Using (4.4.2),

$$\int \frac{1}{\sqrt{t}}\, dt = \int t^{-\frac{1}{2}} dt = \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + c = 2\sqrt{t} + c.$$

Example Using (4.4.2), (4.4.4), and (4.4.9), we have

$$\begin{aligned} \int \left(\cos(x) + \frac{4}{x^2}\right) dx &= \int \cos(x)dx + 4\int x^{-2}dx \\ &= \sin(x) - 4x^{-1} + c \\ &= \sin(x) - \frac{4}{x} + c. \end{aligned}$$

Sometimes the indefinite integral of a function, although not itself in the list (4.4.2) through (4.4.8), may be found with the use of some intelligent guessing. For example, $F(x) = \sin(2x)$ is not an antiderivative of $f(x) = \cos(2x)$ since $F'(x) = 2\cos(2x)$. However, since F' and f differ only by a factor of 2, we can correct for this by dividing F by 2. That is,

$$\int \cos(2x)dx = \frac{1}{2}\sin(2x) + c.$$

Again, as with all indefinite integrals, you may verify this result by differentiation.

Example To find $\int 3\sin(4x)dx$, we might begin with a guess using $F(x) = -3\cos(4x)$. However, $F'(x) = 12\sin(3x)$, which differs from the function we are integrating by a factor of 4. Thus, dividing our initial guess by 4, we have

$$\int 3\sin(4x)dx = -\frac{3}{4}\cos(4x) + c.$$

Example To find $\int \sqrt{2t+3}\; dt$, we might begin with a guess using

$$F(t) = \frac{(2t+3)^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3}(2t+3)^{\frac{3}{2}}.$$

However,

$$F'(t) = \sqrt{2t+3}\frac{d}{dt}(2t+3) = 2\sqrt{2t+3},$$

so we need to divide our guess by 2. Hence

$$\int \sqrt{2t+3}\, dt = \frac{1}{3}(2t+3)^{\frac{3}{2}} + c.$$

Example To find

$$\int \frac{1}{\sqrt{3z+1}}\, dz,$$

we might start with an initial guess of

$$F(z) = \frac{(3z+1)^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{3z+1}.$$

Since

$$F'(z) = \frac{3}{\sqrt{3z+1}},$$

we find that

$$\int \frac{1}{\sqrt{3z+1}}\, dz = \frac{2}{3}\sqrt{3z+1} + c.$$

Thus, for example,

$$\int_0^5 \frac{1}{\sqrt{3z+1}}\, dz = \frac{2}{3}\sqrt{3z+1}\,\Big|_0^5 = \frac{8}{3} - \frac{2}{3} = 2.$$

The common thread in the previous examples is the need to modify an initial guess because of the chain rule. For example, $F(x) = \sin(2x)$ is not an antiderivative of $f(x) = \cos(2x)$ because the chain rule comes into play when differentiating F, resulting in an extra factor of 2. This process of reversing the chain rule can be taken a step further to help evaluate integrals in even more complicated situations. For example, consider the indefinite integral

$$\int 2x\sqrt{1+x^2}\, dx.$$

The key to evaluating this integral is recognizing that the factor $2x$ is the derivative of the function inside the square root. That is, if we let

$$f(u) = \sqrt{u}$$

and

$$g(x) = 1 + x^2,$$

then

$$\int 2x\sqrt{1+x^2}\, dx = \int f(g(x))g'(x)dx.$$

Thus we are trying to find an antiderivative of a function which is in the form of the result from a chain rule differentiation. Now if F is an antiderivative of f, then, using the chain rule,

$$\frac{d}{dx}F(g(x)) = F'(g(x))g'(x) = f(g(x))g'(x).$$

Hence, thinking of u as $1 + x^2$, we really only need to find the antiderivative of f with respect to u. Now

$$\int f(u)du = \int \sqrt{u}\ du = \frac{2}{3}u^{\frac{3}{2}} + c,$$

so, substituting $1 + x^2$ back in for u, we should have

$$\int 2x\sqrt{1+x^2}\ dx = \frac{2}{3}(1+x^2)^{\frac{3}{2}} + c.$$

You should check this result by differentiation, noting in particular that the factor of $2x$ comes from the use of the chain rule.

In general, if F is an antiderivative of f and $u = g(x)$ is some differentiable function of x, then, by the chain rule,

$$\frac{d}{dx}F(u) = F'(u)\frac{du}{dx} = f(u)\frac{du}{dx}. \tag{4.4.12}$$

Writing this as an integration formula, we have

$$\int f(u)\frac{du}{dx}\ dx = F(u) + c = \int f(u)du. \tag{4.4.13}$$

This technique to help find indefinite integrals is called *integration by substitution.*

Example To find

$$\int 2x\sin(x^2)dx,$$

we should let $u = x^2$. Then

$$\frac{du}{dx} = 2x,$$

so, using (4.4.13) with $f(u) = \sin(u)$,

$$\begin{aligned}\int 2x\sin(x^2)dx &= \int \sin(u)\frac{du}{dx}\ dx \\ &= \int \sin(u)du \\ &= -\cos(u) + c \\ &= -\cos(x^2) + c.\end{aligned}$$

We summarize this technique as follows.

Integration by substitution To evaluate an indefinite integral of the form

$$\int f(g(x))g'(x)dx, \tag{4.4.14}$$

we may make the substitution $u = g(x)$. We then have

$$\int f(g(x))g'(x)dx = \int f(u)\frac{du}{dx}\,dx = \int f(u)du. \tag{4.4.15}$$

Of course, this technique will work only if we know an antiderivative for f. Indeed, all we have done is replace one indefinite integral with another, with the hope that the new integral will be simpler than the original. In our notation, we can think of the transition from

$$\int f(g(x))g'(x)dx$$

to

$$\int f(u)du$$

as replacing $g(x)$ by u and $g'(x)dx$ by du. Thus in practice we often denote the process of substitution by writing

$$\begin{aligned} u &= g(x) \\ du &= g'(x)dx \end{aligned} \tag{4.4.16}$$

and directly substituting into the integral

$$\int f(g(x))g'(x)dx$$

to obtain the integral

$$\int f(u)du.$$

We will illustrate this in the next examples.

Example To evaluate the indefinite integral

$$\int \frac{2x}{\sqrt{2+x^2}}\,dx,$$

we may let $u = 2 + x^2$. Then

$$\frac{du}{dx} = 2x,$$

which we write in the form

$$du = 2xdx.$$

Substituting, we have

$$\int \frac{2x}{\sqrt{2+x^2}}\,dx = \int \frac{1}{\sqrt{u}}\,du = 2\sqrt{u} + c = 2\sqrt{2+x^2} + c. \tag{4.4.17}$$

Example From (4.4.17), it is easy to see, after dividing through by 2, that

$$\int \frac{x}{\sqrt{2+x^2}}\,dx = \sqrt{2+x^2} + c.$$

We could also see this directly when making the substitution. Namely, if we let $u = 2+x^2$, then $du = 2xdx$ may be written as

$$\frac{1}{2}\,du = xdx.$$

Hence, if we substitute $2+x^2$ for u and $\frac{1}{2}du$ for xdx, we obtain

$$\int \frac{2x}{\sqrt{2+x^2}}\,dx = \int \frac{\frac{1}{2}}{\sqrt{u}}\,du = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \frac{1}{2}(2\sqrt{u}) + c = \sqrt{2+x^2} + c.$$

Example To evaluate the indefinite integral

$$\int 5x\cos(x^2)dx,$$

we may make the substitution

$$\begin{aligned} u &= x^2 \\ du &= 2xdx. \end{aligned}$$

Then

$$\frac{1}{2}\,du = xdx,$$

so we have

$$\int 5x\cos(x^2)dx = \frac{5}{2}\int \cos(u)du = \frac{5}{2}\sin(u) + c = \frac{5}{2}\sin(x^2) + c.$$

Example To evaluate the indefinite integral

$$\int \tan(3x)\sec^2(3x)dx,$$

we may make the substitution

$$\begin{aligned} u &= \tan(3x) \\ du &= 3\sec^2(3x)dx. \end{aligned}$$

Then

$$\frac{1}{3}\,du = \sec^2(3x)dx,$$

so we have

$$\int \tan(3x)\sec^2(3x)dx = \frac{1}{3}\int udu = \frac{1}{6}u^2 + c = \frac{1}{6}\tan^2(3x) + c.$$

Now suppose we want to evaluate the definite integral

$$\int_a^b f(g(x))g'(x)dx.$$

If F is an antiderivative of f, then we know that

$$\int f(g(x))g'(x)dx = F(g(x)) + c. \tag{4.4.18}$$

Hence

$$\int_a^b f(g(x))g'(x)dx = F(g(x))\Big|_a^b = F(g(b)) - F(g(a)). \tag{4.4.19}$$

Now we also have

$$\int_{g(a)}^{g(b)} f(u)du = F(u)\Big|_{g(a)}^{g(b)} = F(g(b)) - F(g(b)). \tag{4.4.20}$$

Putting (4.4.19) and (4.4.20) together, we see that

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du. \tag{4.4.21}$$

That is, similar to our work with indefinite integrals, we may evaluate the definite integral

$$\int_a^b f(g(x))g'(x)dx$$

by making a substitution

$$\begin{aligned} u &= g(x) \\ du &= g'(x)dx, \end{aligned} \tag{4.4.22}$$

the only difference being that in the definite integral we must also change the limits of integration. Note that the new limits of integration correspond to the range of values for u given that x is ranging from a to b.

Example To evaluate

$$\int_0^1 \frac{x^2}{(1+x^3)^2}\,dx,$$

we may make the substitution

$$\begin{aligned} u &= 1+x^3 \\ du &= 3x^2. \end{aligned}$$

Then

$$\frac{1}{3}\,du = x^2dx$$

and u varies from

$$1+0^3 = 1$$

to

$$1+1^3 = 2$$

as x varies from 0 to 1, so

$$\int_0^1 \frac{x^2}{(1+x^3)^2}\,dx = \frac{1}{3}\int_1^2 \frac{1}{u^2}\,du = -\frac{1}{3}\frac{1}{u}\bigg|_1^2 = -\frac{1}{6}+\frac{1}{3} = \frac{1}{6}.$$

Example To evaluate

$$\int_0^{\frac{\pi}{2}} \sin^2(x)\cos(x)dx,$$

we may make the substitution

$$\begin{aligned} u &= \sin(x) \\ du &= \cos(x)dx. \end{aligned}$$

Then u varies from 0 to 1 as x varies from 0 to $\frac{\pi}{2}$, so

$$\int_0^{\frac{\pi}{2}} \sin^2(x)\cos(x)dx = \int_0^1 u^2du = \frac{1}{2}u^3\bigg|_0^1 = \frac{1}{3}.$$

Example To evaluate

$$\int_0^{\frac{\pi}{2}} \cos^3(x)\sin(x)dx,$$

we may make the substitution

$$\begin{aligned} u &= \cos(x) \\ du &= -\sin(x)dx. \end{aligned}$$

Then $-du = -\sin(x)dx$ and u varies from 1 to 0 as x varies from 0 to $\frac{\pi}{2}$, so

$$\int_0^{\frac{\pi}{2}} \cos^3(x)\sin(x)dx = -\int_1^0 u^3du = \int_0^1 u^3du = \frac{1}{4}u^4\bigg|_0^1 = \frac{1}{4}.$$

Example So far all of our examples of substitution have involved reversing the results of the chain rule. However, substitutions can be useful in other situations as well. For example, to evaluate

$$\int_0^3 x\sqrt{1+x}\,dx,$$

the substitution

$$\begin{aligned} u &= 1+x \\ du &= dx \end{aligned}$$

turns out to be useful for rearranging the integral into a form which can be evaluated. Namely, since $u = 1+x$ implies that $x = u-1$, we may substitute to obtain

$$\begin{aligned} \int_0^3 x\sqrt{1+x}\,dx &= \int_1^4 (u-1)\sqrt{u}\,du \\ &= \int_1^4 (u^{\frac{3}{2}} - u^{\frac{1}{2}})du \\ &= \frac{2}{5}u^{\frac{5}{2}}\Big|_1^4 - \frac{2}{3}u^{\frac{3}{2}}\Big|_1^4 \\ &= \left(\frac{64}{5} - \frac{2}{5}\right) - \left(\frac{16}{3} - \frac{2}{3}\right) \\ &= \frac{116}{15}. \end{aligned}$$

We will continue the discussion of techniques for using the Fundamental Theorem of Integral Calculus in Section 4.5.

Problems

1. Evaluate the following indefinite integrals.

   (a) $\int (x^3 + 3x - 6)dx$ (b) $\int (3t^2 - 4t + 5)dt$

   (c) $\int \frac{1}{x^4}\,dx$ (d) $\int \left(3z - \frac{4}{z^2}\right)dz$

   (e) $\int \frac{12}{\sqrt{t}}\,dt$ (f) $\int \frac{3}{\sqrt{2+x}}\,dx$

   (g) $\int 7\sqrt{x+5}\,dx$ (h) $\int (\sin(\theta) - 2\cos(\theta))d\theta$

2. Evaluate the following indefinite integrals.

   (a) $\int \sin(3x)dx$ (b) $\int \cos(4x)dx$

   (c) $\int \sqrt{3t-1}\,dt$ (d) $\int \frac{4}{\sqrt{1+5z}}\,dz$

(e) $\int 7\sec^2(2x)dx$

(f) $\int 3\sec(4x)\tan(4x)dx$

(g) $\int 2\csc^2(7x)dx$

(h) $\int \sin(4x+1)dx$

3. Evaluate the following indefinite integrals.

(a) $\int 6x\sqrt{1+3x^2}\,dx$

(b) $\int 4x^3\cos(x^4)dx$

(c) $\int x^2(3+x^3)^{10}dx$

(d) $\int \frac{7x}{\sqrt{4+3x^2}}\,dx$

(e) $\int 4t\sin(t^2)dt$

(f) $\int 7z\cos(3z^2+1)dz$

(g) $\int \sin^3(t)\cos(t)dt$

(h) $\int 4\cos^4(3t)\sin(3t)dt$

4. Evaluate the following indefinite integrals.

(a) $\int \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx$

(b) $\int \sec^2(x)\tan^2(x)dx$

(c) $\int \sec^3(4x)\tan(4x)dx$

(d) $\int \sin(\theta)\cos(\theta)d\theta$

(e) $\int \frac{\sin(x)}{\cos^2(x)}\,dx$

(f) $\int \frac{\cos(3t)}{\sqrt{1+\sin(3t)}}\,dt$

(g) $\int t\sqrt{t-2}\,dt$

(h) $\int \frac{z}{\sqrt{z+1}}\,dz$

5. Evaluate the following definite integrals.

(a) $\int_0^1 (4x^2-3x-5)dx$

(b) $\int_0^1 \frac{1}{\sqrt{3x+1}}\,dx$

(c) $\int_0^{\frac{\pi}{4}} 3\sin(2x)dx$

(d) $\int_1^5 \sqrt{2t-1}\,dt$

(e) $\int_{-\frac{\pi}{12}}^0 6\sec(3t)\tan(3t)dt$

(f) $\int_0^2 \frac{1}{(7z+6)^2}\,dz$

(g) $\int_0^2 x\sqrt{x^2+1}\,dx$

(h) $\int_0^{\pi} \sin^4(t)\cos(t)dt$

6. Evaluate the following definite integrals.

(a) $\int_{-1}^1 \frac{5x^2}{(x^3+2)^2}\,dx$

(b) $\int_0^2 \frac{3x}{\sqrt{x^2+1}}\,dx$

(c) $\int_0^{\sqrt{\pi}} 3x\sin(x^2)dx$

(d) $\int_{-\frac{\pi}{2}}^0 \cos^2(t)\sin(t)dt$

(e) $\int_0^{\frac{\pi}{2}} \sin^3(2t)\cos(2t)dt$

(f) $\int_0^1 5x(2+x^2)^{10}dx$

(g) $\int_{-1}^{1} x(1+x^2)^{25}dx$ (h) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2(u)\tan(u)du$

7. Evaluate the following definite integrals.

(a) $\int_0^3 \frac{x}{\sqrt{x+1}}\, dx$ (b) $\int_0^1 x^3\sqrt{x^2+1}\, dx$

(c) $\int_{-\frac{\pi}{12}}^{\frac{\pi}{24}} \tan^4(4x)\sec^2(4x)dx$ (d) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot(t)\csc^2(t)dt$

(e) $\int_0^1 4x(1+x)^{25}dx$ (f) $\int_0^{\frac{\pi}{3}} \frac{\sin(2\theta)}{\cos^3(2\theta)}\, d\theta$

(g) $\int_1^5 5u\sqrt{2u-1}\, du$ (h) $\int_0^{\pi} \sin^5(w)\cos(w)dw$

8. Find the area beneath one arch of the curve $y = 4\sin(6t)$.

9. (a) Plot the graph of $y = \sin^2(x)\cos(x)$ over the interval $[0, \pi]$.

(b) Find the area of the region beneath the graph of $y = \sin^2(x)\cos(x)$ over the interval $\left[0, \frac{\pi}{2}\right]$.

(c) Verify that

$$\int_0^{\pi} \sin^2(x)\cos(x)dx = 0$$

and justify your result geometrically.

Difference Equations to Differential Equations

Section 4.5

More Techniques of Integration

In the last section we saw how we could exploit our knowledge of the chain rule to develop a technique for simplifying integrals using suitably chosen substitutions. In this section we shall see how we can develop a second technique, called *integration by parts*, using the product rule. Outside of algebraic manipulation and the use of various functional identities, like the trigonometric identities, substitution and parts are the only basic techniques we have available to us for simplifying the process of evaluating an integral.

Example Suppose we wish to find $\int x\cos(x)dx$. Since

$$\int \cos(x)dx = \sin(x) + c,$$

we might make an initial guess of $F(x) = x\sin(x)$ for an antiderivative of $f(x) = x\cos(x)$. But, of course, differentiation of F, using the product rule, yields

$$F'(x) = x\cos(x) + \sin(x),$$

which differs from the desired result, $f(x)$, by the term $\sin(x)$. However, since

$$\int \sin(x)dx = -\cos(x) + c,$$

we can obtain an antiderivative of $f(x)$ by adding on the term $\cos(x)$ to $F(x)$. That is,

$$G(x) = x\sin(x) + \cos(x)$$

is an antiderivative of $f(x)$ since the derivative of $\cos(x)$ will cancel the $\sin(x)$ term in $F'(x)$. Explicitly,

$$G'(x) = x\cos(x) + \sin(x) - \sin(x) = x\cos(x).$$

Thus

$$\int x\cos(x)dx = x\sin(x) + \cos(x) + c.$$

In general, suppose f and g are differentiable functions and we want to evaluate

$$\int f(x)g'(x)dx. \tag{4.5.1}$$

For example, in our previous example we would have $f(x) = x$ and $g(x) = \sin(x)$. From the product rule we know that

$$\frac{d}{dx}f(x)g(x) = f(x)g'(x) + g(x)f'(x). \tag{4.5.2}$$

Thus, integrating both sides of (4.5.2), we have

$$\int \frac{d}{dx}f(x)g(x) = \int f(x)g'(x)dx + \int g(x)f'(x)dx, \tag{4.5.3}$$

from which it follows that

$$f(x)g(x) = \int f(x)g'(x)dx + \int g(x)f'(x)dx. \tag{4.5.4}$$

Rearranging (4.5.4) gives us

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx. \tag{4.5.5}$$

Applying (4.5.5) to our example, with $f(x) = x$ and $g(x) = \sin(x)$, we have

$$\int x\cos(x)dx = x\sin(x) - \int \sin(x)dx = x\sin(x) + \cos(x) + c.$$

In effect, using (4.5.5), we have replaced the problem of evaluating $\int x\cos(x)dx$ with the simpler problem of evaluating $\int \sin(x)dx$. In general, the success of this method always depends on the integral

$$\int g(x)f'(x)dx \tag{4.5.6}$$

being easier to evaluate than the integral

$$\int f(x)g'(x)dx. \tag{4.5.7}$$

It is common with this technique to let $u = f(x)$ and $v = g(x)$ along with the notation, as we did with substitution,

$$dv = g'(x)dx \tag{4.5.8}$$

and

$$du = f'(x)dx. \tag{4.5.9}$$

With this notation, (4.5.5) becomes

$$\int u dv = uv - \int v du, \tag{4.5.10}$$

the standard form for what is known as integration by parts.

Example To evaluate the integral $\int x \sin(x)$ by parts, we must first make a choice for u and dv. Here we might choose

$$u = x \quad dv = \sin(x)dx.$$

It follows then that $du = dx$. However, there are many possible choices for v; all that we require is that the derivative of v must be $\sin(x)$. The simplest choice is to take $v = -\cos(x)$. Then we have, applying (4.5.10),

$$\int x \sin(x)dx = -x\cos(x) + \int \cos(x)dx = -x\cos(x) + \sin(x) + c.$$

Example To evaluate the integral $\int x^2 \cos(2x)dx$, we might choose

$$u = x^2 \quad dv = \cos(2x)dx,$$

from which we obtain

$$du = 2xdx \quad v = \frac{1}{2}\sin(2x).$$

Thus

$$\int x^2 \cos(2x)dx = \frac{1}{2}x^2\sin(2x) - \int x\sin(2x)dx.$$

This time we do not immediately know the value of the integral on the right, but we know we can find it using integration by parts. Namely, to evaluate $\int x\sin(2x)dx$, we let

$$\begin{aligned} u &= x & dv &= \sin(2x)dx \\ du &= dx & v &= -\frac{1}{2}\cos(2x). \end{aligned}$$

Then

$$\int x\sin(2x)dx = -\frac{1}{2}x\cos(2x) + \frac{1}{2}\int \cos(2x)dx = -\frac{1}{2}x\cos(2x) + \frac{1}{4}\sin(2x) + c.$$

Hence

$$\int x^2\cos(2x)dx = \frac{1}{2}x^2\sin(2x) + \frac{1}{2}x\cos(2x) - \frac{1}{4}\sin(2x) + c.$$

The key to success with integration by parts is in the choice of the parts, u and dv. For example, we saw in an example that the choices

$$\begin{aligned} u &= x & dv &= \sin(x)dx \\ du &= dx & v &= -\cos(x) \end{aligned}$$

work well for evaluating $\int x\sin(x)dx$. Alternatively, we could have chosen

$$\begin{aligned} u &= \sin(x) & dv &= xdx \\ du &= \cos(x)dx & v &= \frac{1}{2}x^2, \end{aligned}$$

which would yield

$$\int x\sin(x)dx = \frac{1}{2}x^2\sin(x) - \frac{1}{2}\int x^2\cos(x)dx.$$

All of this is correct, but useless (at least for our present purpose) since the resulting integral on the right is more complicated than the integral with which we started. If we had started to work the problem this way, we would probably stop at this point and rethink our strategy.

Example In using integration by parts to evaluate a definite integral, we must remember to evaluate all the pieces of the resulting antiderivative. For example, to evaluate

$$\int_0^{\frac{\pi}{3}} 4x\cos(3x)dx,$$

we might choose

$$\begin{aligned} u &= 4x & dv &= \cos(3x)dx \\ du &= 4dx & v &= \frac{1}{3}\sin(3x). \end{aligned}$$

Then

$$\begin{aligned} \int_0^{\frac{\pi}{3}} 4x\cos(3x)dx &= \frac{4}{3}x\sin(3x)\Big|_0^{\frac{\pi}{3}} - \frac{4}{3}\int_0^{\frac{\pi}{3}}\sin(3x)dx \\ &= (0-0) + \frac{4}{9}\cos(3x)\Big|_0^{\frac{\pi}{3}} \\ &= -\frac{4}{9} - \frac{4}{9} \\ &= -\frac{8}{9}. \end{aligned}$$

Example Although integration by parts is most frequently of use when integrating functions involving transcendental functions, such as the trigonometric functions, there are other times when the technique may be used. For example, to compute

$$\int_0^1 x(1+x)^{10}dx,$$

we could use

$$\begin{aligned} u &= x & dv &= (1+x)^{10}dx \\ du &= dx & v &= \frac{1}{11}(1+x)^{11}. \end{aligned}$$

Then

$$\begin{aligned}\int_0^1 x(1+x)^{10}dx &= \frac{1}{11}x(1+x)^{11}\Big|_0^1 - \frac{1}{11}\int_0^1 (1+x)^{11}dx \\ &= \frac{2048}{11} - \frac{1}{132}(1+x)^{12}\Big|_0^1 \\ &= \frac{2048}{11} - \left(\frac{4096}{132} - \frac{1}{132}\right) \\ &= \frac{6827}{44}.\end{aligned}$$

Notice that we could also evaluate this integral using the substitution $u = 1 + x$.

Miscellaneous examples

The techniques of substitution and parts are often useful for putting an integral into a form that can be readily evaluated by the Fundamental Theorem of Integral Calculus. The next several examples illustrate how basic trigonometric identities are also useful for rewriting integrals in more easily evaluated forms.

Example To evaluate $\int \sin^2(x)dx$, we may use the identity

$$\sin^2(x) = \frac{1-\cos(2x)}{2} \tag{4.5.11}$$

(see Problem 5, Section 2.2). Then

$$\int \sin^2(x)dx = \frac{1}{2}\int (1-\cos(2x))dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + c.$$

Example Similarly, to evaluate

$$\int_0^{\frac{\pi}{4}} \cos^2(2t)dt,$$

we use the identity

$$\cos^2(x) = \frac{1+\cos(2x)}{2}. \tag{4.5.12}$$

Then

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \cos^2(2t)dt &= \frac{1}{2}\int_0^{\frac{\pi}{4}} (1+\cos(4t))dt \\ &= \frac{1}{2}t\Big|_0^{\frac{\pi}{4}} + \frac{1}{8}\sin(4t)\Big|_0^{\frac{\pi}{4}} \\ &= \frac{\pi}{8}.\end{aligned}$$

Example To evaluate $\int \sin^2(x)\cos^2(x)dx$, the identity

$$\sin(x)\cos(x) = \frac{1}{2}\sin(2x) \tag{4.5.13}$$

is useful (see Problem 4, Section 2.2.). From it we obtain

$$\begin{aligned}\int \sin^2(x)\cos^2(x)dx &= \int (\sin(x)\cos(x))^2 dx \\ &= \int \left(\frac{1}{2}\sin(2x)\right)^2 dx \\ &= \frac{1}{4}\int \sin^2(2x)dx \\ &= \frac{1}{8}\int (1-\cos(4x))dx \\ &= \frac{1}{8}x - \frac{1}{32}\sin(4x) + c.\end{aligned}$$

Example To evaluate $\int \sin^3(x)dx$ we may use the identity

$$\sin^2(x) = 1 - \cos^2(x)$$

to write

$$\sin^3(x) = \sin^2(x)\sin(x) = (1-\cos^2(x))\sin(x).$$

Then the substitution

$$\begin{aligned}u &= \cos(x) \\ du &= -\sin(x)dx\end{aligned}$$

gives us

$$\begin{aligned}\int \sin^3(x)dx &= \int (1-\cos^2(x))\sin(x)dx \\ &= -\int (1-u^2)du \\ &= -u + \frac{1}{3}u^3 + c \\ &= -\cos(x) + \frac{1}{3}\cos^3(x) + c.\end{aligned}$$

This manipulation is useful in evaluating any integral of the form $\int \sin^n(x)dx$ or, in a similar fashion, $\int \cos^n(x)dx$, provided n is a positive odd integer.

Example As a final example, note that the identity

$$\tan^2(x) = \sec^2(x) - 1$$

(see Problem 3, Section 2.2) is useful in evaluating

$$\int_0^{\frac{\pi}{4}} \tan^2(x)dx.$$

Namely,

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \tan^2(x)dx &= \int_0^{\frac{\pi}{4}} (\sec^2(x) - 1)dx \\ &= \tan(x)\Big|_0^{\frac{\pi}{4}} - x\Big|_0^{\frac{\pi}{4}} \\ &= 1 - \frac{\pi}{4}.\end{aligned}$$

This concludes our discussion of techniques of integration. As we noted above, there are basically only two techniques for evaluating indefinite integrals, substitution and parts, and even these rely on an ability to reduce a given integral to a form where an antiderivative is recognizable. Hence the situation is not nearly as straightforward as it was for finding derivatives and best affine approximations. For this reason, in the past tables of indefinite integrals were compiled to aid in the evaluation of integrals; when faced with an integral more involved than the basic ones we have investigated in these last two sections, one could hope to find it, or one related to it through a substitution or an integration by parts, in a table. For the most part, tables of integrals have been replaced by computer programs, such as computer algebra systems, which are capable of finding antiderivatives symbolically. Such programs are then able to evaluate definite integrals exactly using the Fundamental Theorem. Although these programs are immensely useful and are an everyday tool for those working with applications of mathematics, one must use them with care. In particular, whenever possible, you should check your answer for reasonableness. Moreover, there are integrals which the system will not be able to evaluate symbolically, either because the given integral is beyond the capabilities of the system, or because a symbolic answer does not even exist. In such cases, one must, of necessity, fall back on numerical approximation techniques.

Problems

1. Evaluate the following indefinite integrals.

   (a) $\int 3x\sin(x)dx$

   (b) $\int 2x\cos(5x)dx$

   (c) $\int 4x\sin(3x)dx$

   (d) $\int x^2\cos(3x)dx$

   (e) $\int 2x^2\sin(4x)dx$

   (f) $\int x^3\cos(x)dx$

   (g) $\int 3x^3\sin(2x)dx$

   (h) $\int x\sqrt{1+x}\ dx$

2. Evaluate the following definite integrals.

(a) $\displaystyle\int_0^{\pi} 4x\sin(x)dx$ (b) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 3x\cos(2x)dx$

(c) $\displaystyle\int_0^{\frac{\pi}{3}} 2t\sin(3t)dt$ (d) $\displaystyle\int_0^{\frac{\pi}{2}} x^2\cos(x)dx$

(e) $\displaystyle\int_0^{\frac{\pi}{4}} 2x^2\sin(2x)dx$ (f) $\displaystyle\int_0^{\frac{\pi}{4}} z^3\cos(4z)dz$

3. Evaluate the following indefinite integrals.

(a) $\displaystyle\int \sin^2(2x)dx$ (b) $\displaystyle\int \cos^2(3t)dt$

(c) $\displaystyle\int 5\sin^2(2t)\cos^2(2t)dt$ (d) $\displaystyle\int \sin^3(3x)dx$

(e) $\displaystyle\int 6\cos^3(2z)dz$ (f) $\displaystyle\int \sin^5(t)dt$

(g) $\displaystyle\int \cos^5(2x)dx$ (h) $\displaystyle\int \tan^2(3\theta)d\theta$

4. Evaluate the following definite integrals.

(a) $\displaystyle\int_0^{\pi} \sin^2(x)dx$ (b) $\displaystyle\int_0^{\frac{\pi}{4}} \cos^2(2t)dt$

(c) $\displaystyle\int_0^{\frac{\pi}{2}} 3\sin^2(z)\cos^2(z)dz$ (d) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3(t)dt$

(e) $\displaystyle\int_0^{\pi} \sin^3(3t)dt$ (f) $\displaystyle\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \tan^2(2t)dt$

5. Evaluate the following integrals using a computer algebra system.

(a) $\displaystyle\int \cos^6(x)dx$ (b) $\displaystyle\int \sin^2(2t)\cos^4(2t)dt$

(c) $\displaystyle\int \sin^4(2t)\cos^4(3t)dt$ (d) $\displaystyle\int \sec^4(3x)dx$

(e) $\displaystyle\int_{-1}^{1} \sqrt{1-x^2}\ dx$ (f) $\displaystyle\int_0^{1} x^2\sqrt{1-x^2}\ dx$

(g) $\displaystyle\int_0^{2\pi} \sin^8(2t)dt$ (h) $\displaystyle\int_0^{\frac{\pi}{4}} \tan^6(t)dt$

6. Evaluate the following integrals with any method at your disposal.

(a) $\displaystyle\int_{-\pi}^{\pi} \sin^4(x)dx$ (b) $\displaystyle\int_1^{\pi} \frac{\sin(x)}{x}\ dx$

(c) $\displaystyle\int_0^{5} \sin(3x^2)dx$ (d) $\displaystyle\int_0^{1} \sqrt{1+x^2}\ dx$

(e) $\displaystyle\int_1^2 \frac{1}{x}\,dx$

(f) $\displaystyle\int_{-2}^{-1} \frac{1}{t}\,dt$

(g) $\displaystyle\int_0^{\pi} \sqrt{5-3\sin^2(t)}\,dt$

(h) $\displaystyle\int_0^{2\pi} \frac{1}{\sqrt{1+\sin^2(t)}}\,dt$

7. If a pendulum of length b is held, at rest, at an angle α from the perpendicular, $0<\alpha<\pi$, and then released, its period T, the time required for one complete oscillation, is given by

$$T = 4\sqrt{\frac{b}{g}}\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-k^2\sin^2(\varphi)}}\,d\varphi,$$

where $g = 980\ \text{cm/sec}^2$ (the acceleration due to gravity) and $k=\sin\left(\frac{\alpha}{2}\right)$.

(a) Find the period of a pendulum of length 50 centimeters which is released initially from an angle of $\alpha = \frac{\pi}{3}$.

(b) Repeat (a) for $\alpha = \frac{\pi}{4}, \frac{\pi}{6}, \frac{\pi}{50}$, and $\frac{\pi}{100}$.

(c) In Section 2.2 we noted that for small values of α, if $x(t)$ represents the angle the pendulum makes with the perpendicular at time t, then, to a good approximation,

$$x(t) = \alpha\cos\left(\sqrt{\frac{g}{b}}t\right).$$

Thus, in this approximation, x has a period of $2\pi\sqrt{\frac{b}{g}}$. For a pendulum of length 50 centimeters, compare this result with your results in parts (a) and (b).

(d) For a pendulum of length 50 centimeters, graph T as a function of α for $-\frac{\pi}{4}\le\alpha\le\frac{\pi}{4}$.

Difference Equations to Differential Equations

Section 4.6

Improper Integrals

In this section we will make two extensions to our definition of the definite integral. The first will cover integrals of functions over intervals of the form $[a, \infty]$ and $(-\infty, b]$, where a and b are fixed real numbers, as well as the interval $(-\infty, \infty)$, while the second will cover integrals of functions which have infinite discontinuities. An integral of either one of these two types is called an *improper integral*.

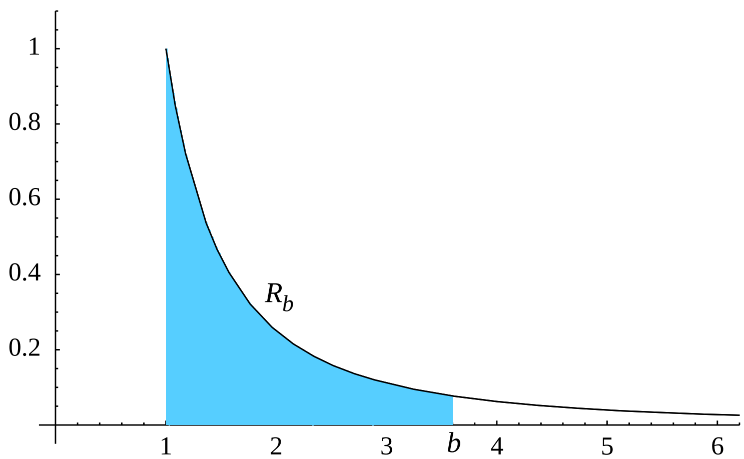


Figure 4.6.1 Area of R_b approaches area under $y = \dfrac{1}{x^2}$ as b increases

First, consider a function f defined on an interval $[a, \infty)$ with the property that f is integrable on every interval $[a, b]$ with $a < b < \infty$. For example, the function

$$f(x) = \frac{1}{x^2}$$

is defined for all x in $[1, \infty)$ and, since it is continuous on $[1, \infty)$, is integrable on any interval $[1, b]$ with $1 < b < \infty$. If we let R_b be the region beneath the graph of f over the interval $[1, b]$ and we let R be the region beneath the graph of f over the interval $[1, \infty)$, then we would expect that the area of R_b would approach the area of R in the limit as b goes to infinity (see Figure 4.6.1). In terms of integrals, this is saying that it would seem reasonable to define

$$\int_1^\infty \frac{1}{x^2}\, dx = \lim_{b \to \infty} \int_1^b \frac{1}{x^2}\, dx.$$

That is, we should have

$$\begin{aligned}\int_1^\infty \frac{1}{x^2}\,dx &= \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx\\ &= \lim_{b\to\infty} -\frac{1}{x}\Big|_1^b\\ &= \lim_{b\to\infty}\left(-\frac{1}{b}+1\right)\\ &= 1.\end{aligned}$$

Geometrically, this result says that R has finite area, namely, 1, even though it has infinite length.

We now state a general definition for this type of integral.

Definition If f is defined on $[a,\infty)$ and integrable on $[a,b]$ for all $a<b<\infty$, then we define

$$\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx, \tag{4.6.1}$$

provided the limit exists. Similarly, if f is defined on $(-\infty,b]$ and integrable on $[a,b]$ for all $-\infty<a<b$, then we define

$$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx, \tag{4.6.2}$$

provided the limit exists. Finally, if f is defined on $(-\infty,\infty)$ and integrable on any finite interval $[a,b]$, then we define

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^0 f(x)dx + \int_0^\infty f(x)dx, \tag{4.6.3}$$

provided both of the integrals on the right exist. In each case where the appropriate limit exists, we say the integral *converges*; otherwise, the integral is said to *diverge*.

Note that the use of 0 in (4.6.3) is not crucial; all that is important is that the integral is broken into two pieces, the meaning of each of the pieces already having been covered in the earlier parts of the definition.

Example The integral

$$\int_3^\infty \frac{1}{x^3}\,dx$$

converges, since

$$\begin{aligned}\int_3^\infty \frac{1}{x^3}\,dx &= \lim_{b\to\infty}\int_3^b \frac{1}{x^3}\,dx\\ &= \lim_{b\to\infty} -\frac{1}{2x^2}\Big|_3^b\end{aligned}$$

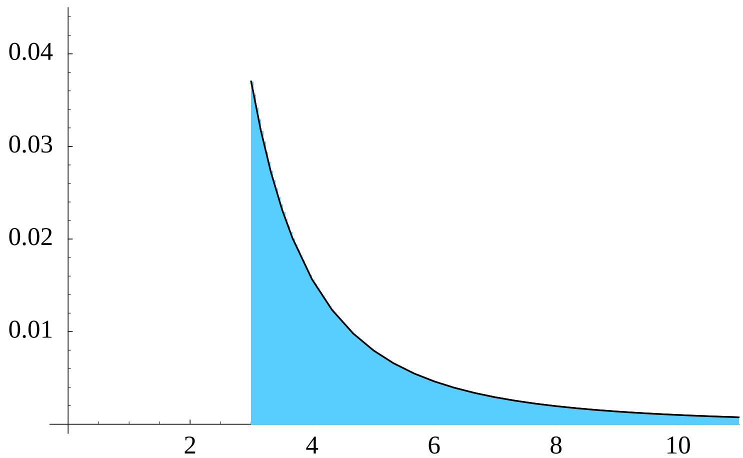


Figure 4.6.2 Region beneath $y = \dfrac{1}{x^3}$ beginning at 3

$$= \lim_{b\to\infty}\left(-\frac{1}{2b^2} + \frac{1}{18}\right)$$
$$= \frac{1}{18}.$$

See Figure 4.6.2.

Example The integral

$$\int_2^\infty \frac{1}{\sqrt{x}}\,dx$$

diverges, since

$$\lim_{b\to\infty}\int_2^b \frac{1}{\sqrt{x}}\,dx = \lim_{b\to\infty} 2\sqrt{x}\,\Big|_2^b = \lim_{b\to\infty}(2\sqrt{b} - 2\sqrt{2}) = \infty.$$

See Figure 4.6.3.

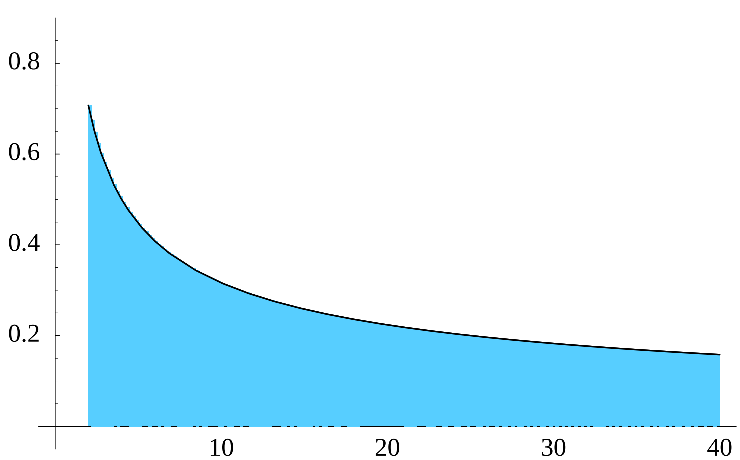


Figure 4.6.3 Region beneath $y = \dfrac{1}{\sqrt{x}}$ beginning at 2

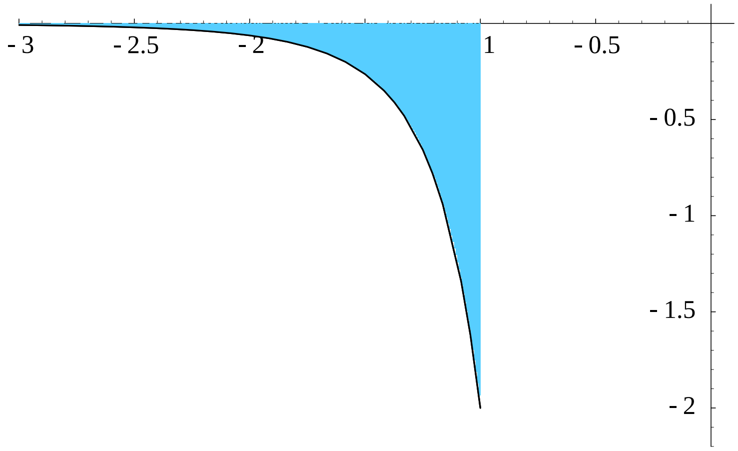


Figure 4.6.4 Region above $y = \dfrac{2}{x^5}$ ending at -1

Example The integral

$$\int_{-\infty}^{-1} \frac{2}{x^5}\, dx$$

converges, since

$$\begin{aligned}\int_{-\infty}^{-1} \frac{2}{x^5}\, dx &= \lim_{a\to-\infty} \int_a^{-1} \frac{2}{x^5}\, dx \\ &= \lim_{a\to-\infty} -\frac{2}{4x^4}\bigg|_a^{-1} \\ &= \lim_{a\to-\infty} \left(-\frac{1}{2} + \frac{1}{2a^4}\right) \\ &= -\frac{1}{2}.\end{aligned}$$

See Figure 4.6.4.

Example The integral

$$\int_{-\infty}^{\infty} \frac{x}{(1+x^2)^2}\, dx$$

converges, since

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{x}{(1+x^2)^2}\, dx &= \int_{-\infty}^{0} \frac{x}{(1+x^2)^2}\, dx + \int_0^{\infty} \frac{x}{(1+x^2)^2}\, dx \\ &= \lim_{a\to-\infty} \int_a^0 \frac{x}{(1+x^2)^2}\, dx + \lim_{b\to\infty} \int_0^b \frac{x}{(1+x^2)^2}\, dx \\ &= \lim_{a\to-\infty} -\frac{1}{2(1+x^2)}\bigg|_a^0 + \lim_{b\to\infty} -\frac{1}{2(1+x^2)}\bigg|_0^b \\ &= \lim_{a\to-\infty} \left(-\frac{1}{2} + \frac{1}{2(1+a^2)}\right) + \lim_{b\to\infty} \left(-\frac{1}{2(1+b^2)} + \frac{1}{2}\right)\end{aligned}$$

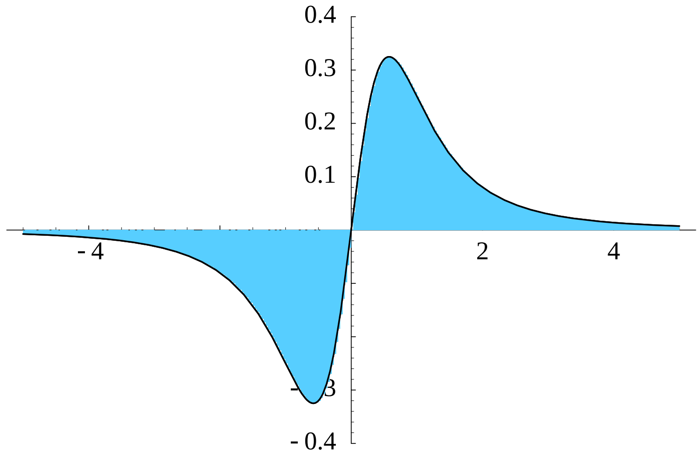


Figure 4.6.5 Region between $y = \dfrac{x}{(1+x^2)^2}$ and the x-axis

$$= -\frac{1}{2} + \frac{1}{2}$$
$$= 0.$$

Note that you could use the substitution $u = 1 + x^2$ to help evaluate the integral in this example. See Figure 4.6.5.

It is frequently important to know that an integral $\int_a^\infty f(x)dx$ converges even if we cannot compute its value exactly. For example, before trying to find numerical approximations for such an integral one should first check that it converges. We will first consider the following situation: Suppose f and g are defined on $[a,\infty)$, integrable on $[a,b]$ for all $a < b < \infty$, and $0 \le f(x) \le g(x)$ for all x in $[a,\infty)$. Moreover, suppose we know that $\int_a^\infty g(x)dx$ converges. Let

$$M = \int_a^\infty g(x)dx, \tag{4.6.4}$$

$$G(b) = \int_a^b g(x)dx, \tag{4.6.5}$$

and

$$F(b) = \int_a^b f(x)dx \tag{4.6.6}$$

for all $b \ge a$. Now for any $b \ge a$,

$$M = \int_a^\infty g(x)dx = \int_a^b g(x)dx + \int_b^\infty g(x)dx = G(b) + \int_b^\infty g(x)dx. \tag{4.6.7}$$

Since $g(x) \ge 0$ for all $x \ge a$,

$$\int_b^\infty g(x)dx \ge 0. \tag{4.6.8}$$

Thus (4.6.7) implies that

$$G(b) = M - \int_b^\infty g(x)dx \le M \tag{4.6.9}$$

for all $b \ge a$. Moreover, $f(x) \le g(x)$ for all $x \ge a$, so

$$F(b) = \int_a^b f(x)dx \le \int_a^b g(x)dx = G(b) \tag{4.6.10}$$

for all $b \ge a$. Putting (4.6.9) and (4.6.10) together, we have $F(b) \le M$ for all $b \ge a$. Furthermore, for any $c \ge b \ge a$,

$$F(c) = \int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx \ge \int_a^b f(x)dx = F(b), \tag{4.6.11}$$

where we know

$$\int_b^c f(x)dx \ge 0 \tag{4.6.12}$$

because $f(x) \ge 0$ for all $x \ge a$. From (4.6.11) we conclude that F is a nondecreasing function. Since we already know that F is bounded by M, it follows from our result about bounded nondecreasing sequences in Section 1.2 that

$$\lim_{b\to\infty} F(b) = \lim_{b\to\infty} \int_a^b f(x)dx \tag{4.6.13}$$

exists. That is,

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx \tag{4.6.14}$$

converges. Moreover, since $F(b) \le M$ for all $b \ge a$,

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} F(b) \le M = \int_a^\infty g(x)dx. \tag{4.6.15}$$

On the other hand, suppose f and g are defined on $[a, \infty)$, integrable on $[a, b]$ for all $a < b < \infty$, $0 \le f(x) \le g(x)$ for all x in $[a, \infty)$, and $\int_a^\infty f(x)dx$ diverges. If we define F and G as above, then $F(b)$ is nondecreasing and without a limit as b increases toward ∞. Hence it follows, again from our results in Section 1.2, that we must have

$$\lim_{b\to\infty} F(b) = \infty. \tag{4.6.16}$$

Since, as above, $G(b) \ge F(b)$ for all $b \ge a$, (4.6.16) implies that

$$\lim_{b\to\infty} \int_a^b g(x)dx = \lim_{b\to\infty} G(b) = \infty. \tag{4.6.17}$$

In particular, $\int_a^\infty g(x)dx$ diverges.

We summarize the previous results in the next proposition.

Proposition Suppose f and g are defined on $[a, \infty)$, integrable on $[a, b]$ for all $a < b < \infty$, and $0 \leq f(x) \leq g(x)$ for all x in $[a, \infty)$. If $\int_a^\infty g(x)dx$ converges, then $\int_a^\infty f(x)dx$ converges and

$$0 \leq \int_a^\infty f(x)dx \leq \int_a^\infty g(x)dx. \tag{4.6.18}$$

If $\int_a^\infty f(x)dx$ diverges, then $\int_a^\infty g(x)dx$ diverges.

Similar results hold for integrals on intervals of the form $(-\infty, b]$ and $(-\infty, \infty)$.

Example At present we cannot use the Fundamental Theorem to evaluate

$$\int_0^\infty \frac{1}{1+x^2}\,dx$$

because we do not know an antiderivative for

$$f(x) = \frac{1}{1+x^2}$$

(although we will find one in Section 6.5). However, since $x^2 < 1 + x^2$ for all values of x, we know that

$$0 < \frac{1}{1+x^2} < \frac{1}{x^2}$$

for all $x > 0$. Now we saw above that

$$\int_1^\infty \frac{1}{x^2}\,dx = 1,$$

so we know, by the previous proposition, that

$$\int_1^\infty \frac{1}{1+x^2}\,dx$$

converges with

$$\int_1^\infty \frac{1}{1+x^2}\,dx \leq 1. \tag{4.6.19}$$

Moreover, $1 + x^2 \geq 1$ for all x, so

$$\frac{1}{1+x^2} \leq 1$$

for all x. Hence

$$\int_0^1 \frac{1}{1+x^2}\,dx \leq \int_0^1 dx = 1. \tag{4.6.20}$$

Putting (4.6.19) and (4.6.20) together, we have

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \int_0^1 \frac{1}{1+x^2}\,dx + \int_1^\infty \frac{1}{1+x^2}\,dx \leq 1 + 1 = 2.$$

In the problems for Section 6.5, you will be asked to show that

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \frac{\pi}{2}.$$

Example Since $\sqrt{x}-1<\sqrt{x}$ for all $x \geq 0$, we see that

$$0 < \frac{1}{\sqrt{x}} < \frac{1}{\sqrt{x}-1}$$

for all $x \geq 1$. Thus, by the previous proposition,

$$\int_2^\infty \frac{1}{\sqrt{x}-1}\,dx$$

diverges since we saw above that

$$\int_2^\infty \frac{1}{\sqrt{x}}\,dx$$

diverges.

Although we will not go into the details, the previous proposition may be generalized as follows.

Proposition Suppose $h(x) \leq f(x) \leq g(x)$ for all x in an interval $[a,\infty)$ and f, g, and h are integrable on $[a,b]$ for all $a<b<\infty$. If both $\int_a^\infty h(x)dx$ and $\int_a^\infty g(x)dx$ converge, then $\int_a^\infty f(x)dx$ converges as well. Moreover, in that case,

$$\int_a^\infty h(x)dx \leq \int_a^\infty f(x)dx \leq \int_a^\infty g(x)dx. \tag{4.6.21}$$

Note that our previous proposition is a special case of this proposition with $h(x)=0$ for all $x \geq a$. As before, similar results hold for integrals on intervals of the form $(-\infty,b]$ and $(-\infty,\infty)$.

Example Since $-1 \leq \sin(x) \leq 1$ for all x, it follows that

$$-\frac{1}{x^2} \leq \frac{\sin(x)}{x^2} \leq \frac{1}{x^2}$$

for all $x \geq 1$. Moreover,

$$\int_1^\infty \frac{1}{x^2}\,dx = 1$$

and

$$\int_1^\infty -\frac{1}{x^2}\,dx = -\int_1^\infty \frac{1}{x^2}\,dx = -1.$$

Hence it follows that

$$\int_1^\infty \frac{\sin(x)}{x^2}\, dx$$

converges and

$$-1 \le \int_1^\infty \frac{\sin(x)}{x^2}\, dx \le 1.$$

After noticing that for any function f, $-|f(x)| \le f(x) \le |f(x)|$ for all values of x, the following proposition is a special case of the previous proposition.

Proposition If f is defined on $[a, \infty)$ and $\int_a^\infty |f(x)|dx$ converges, then $\int_a^\infty f(x)dx$ converges.

Example Another way to see that

$$\int_1^\infty \frac{\sin(x)}{x^2}\, dx$$

converges is to note that

$$\int_1^\infty \left|\frac{\sin(x)}{x^2}\right| dx$$

converges since

$$0 \le \left|\frac{\sin(x)}{x^2}\right| \le \frac{|\sin(x)|}{x^2} \le \frac{1}{x^2}$$

for all $x \ge 1$.

Once again, similar results hold for integrals on intervals of the form $(-\infty, b]$ and $(-\infty, \infty)$.

We now consider another extension to our definition of the definite integral. Suppose the function f is defined on the interval $(a, b]$ with

$$\lim_{x \to a^+} |f(x)| = \infty.$$

If f is integrable on every interval of the form $[c, b]$ with $a < c < b$, then we may, analogous to our earlier definitions, define

$$\int_a^b f(x)dx = \lim_{c \to a^+} \int_c^b f(x)dx, \tag{4.6.22}$$

provided the limit exists. See Figure 4.6.6.

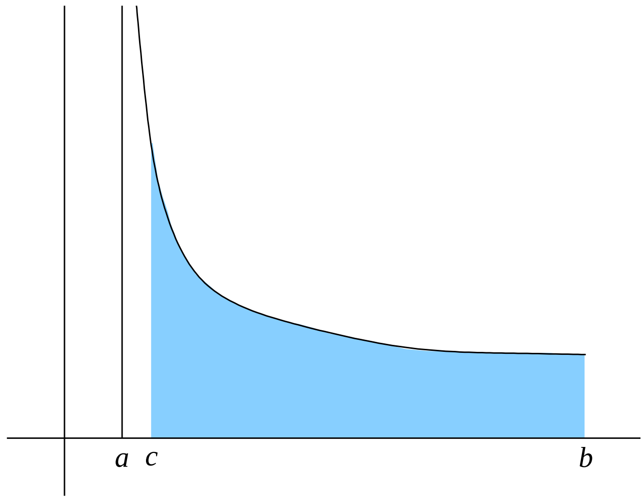


Figure 4.6.6 Area over $(a, b]$ is the area over $[c, b]$ as c approaches a

Definition If f is defined on the interval $(a, b]$ with

$$\lim_{x \to a^+} |f(x)| = \infty,$$

and is integrable on every interval of the form $[c, b]$ with $a < c < b$, then we define

$$\int_a^b f(x)dx = \lim_{c \to a^+} \int_c^b f(x)dx, \tag{4.6.23}$$

provided the limit exists. Similarly, if f is defined on the interval $[a, b)$ with

$$\lim_{x \to b^-} |f(x)| = \infty,$$

and is integrable on every interval of the form $[a, c]$ with $a < c < b$, then we define

$$\int_a^b f(x)dx = \lim_{c \to b^-} \int_a^c f(x)dx, \tag{4.6.24}$$

provided the limit exists. Finally, if f is defined on $[a, d)$ and $(d, b]$ with either

$$\lim_{x \to d^-} |f(x)| = \infty,$$

or

$$\lim_{x \to d^+} |f(x)| = \infty,$$

or both, and f is integrable on all intervals of the form $[a, c]$ with $a < c < d$ and of the form $[c, b]$ with $d < c < b$, then we define

$$\int_a^b f(x)dx = \int_a^d f(x)dx + \int_d^b f(x)dx, \tag{4.6.25}$$

provided both the integrals on the right exist. In each case where the appropriate limit exists, we say the integral converges; otherwise, the integral is said to diverge.

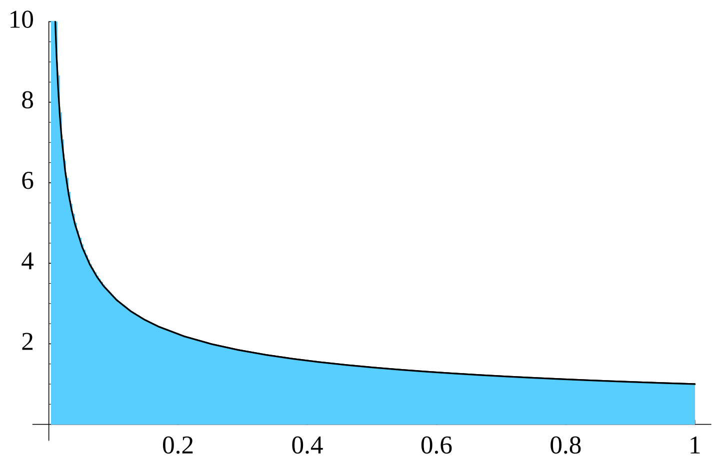


Figure 4.6.7 Region beneath $y = \frac{1}{\sqrt{x}}$ over $[0, 1]$

Example The integral

$$\int_0^1 \frac{1}{\sqrt{x}}\, dx$$

converges since

$$\int_0^1 \frac{1}{\sqrt{x}} = \lim_{c \to 0^+} \int_c^1 \frac{1}{\sqrt{x}}\, dx = \lim_{c \to 0^+} 2\sqrt{x}\Big|_c^1 = \lim_{c \to 0^+} (2 - 2\sqrt{c}) = 2.$$

See Figure 4.6.7.

Example The integral

$$\int_0^1 \frac{1}{x^2}\, dx$$

diverges since

$$\int_0^1 \frac{1}{x^2}\, dx = \lim_{c \to 0^+} \int_c^1 \frac{1}{x^2}\, dx = \lim_{c \to 0^+} -\frac{1}{x}\Big|_c^1 = \lim_{c \to 0^+} \left(-1 + \frac{1}{c}\right) = \infty.$$

See Figure 4.6.8.

Example The integral

$$\int_0^2 \frac{1}{(x-1)^{\frac{2}{3}}}\, dx$$

is improper since

$$\lim_{x \to 1^-} \frac{1}{(x-1)^{\frac{2}{3}}} = \infty$$

and

$$\lim_{x \to 1^+} \frac{1}{(x-1)^{\frac{2}{3}}} = \infty.$$

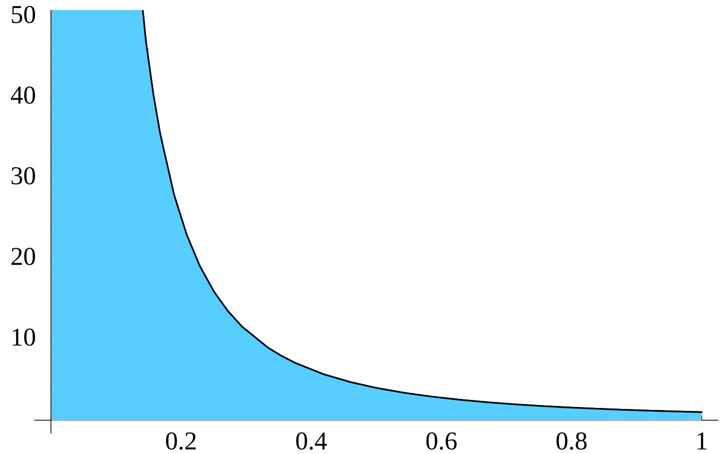


Figure 4.6.8 Region beneath $y = \dfrac{1}{x^2}$ over $[0, 1]$

Moreover, the integral converges since

$$\begin{aligned}\int_0^1 \frac{1}{(x-1)^{\frac{2}{3}}}\,dx &= \lim_{c\to 1^-}\int_0^c \frac{1}{(x-1)^{\frac{2}{3}}}\,dx\\ &= \lim_{c\to 1^-} 3(x-1)^{\frac{1}{3}}\Big|_0^c\\ &= \lim_{c\to 1^-}\left(3(c-1)^{\frac{1}{3}}+3\right)\\ &= 3\end{aligned}$$

and

$$\begin{aligned}\int_1^2 \frac{1}{(x-1)^{\frac{2}{3}}}\,dx &= \lim_{c\to 1^+}\int_c^2 \frac{1}{(x-1)^{\frac{2}{3}}}\,dx\\ &= \lim_{c\to 1^+} 3(x-1)^{\frac{1}{3}}\Big|_c^2\\ &= \lim_{c\to 1^+}\left(3-3(c-1)^{\frac{1}{3}}\right)\\ &= 3,\end{aligned}$$

which together imply that

$$\int_0^2 \frac{1}{(x-1)^{\frac{2}{3}}}\,dx = \int_0^1 \frac{1}{(x-1)^{\frac{2}{3}}}\,dx + \int_1^2 \frac{1}{(x-1)^{\frac{2}{3}}}\,dx.$$

See Figure 4.6.9.

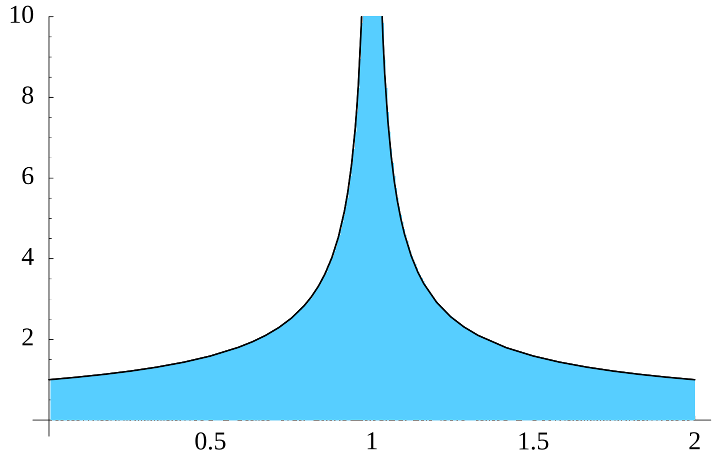


Figure 4.6.9 Region beneath $y = \dfrac{1}{(x-1)^{\frac{2}{3}}}$ over $[0, 2]$

Problems

1. Evaluate the following integrals.

(a) $\displaystyle\int_1^{\infty} \frac{1}{x^3}\,dx$

(b) $\displaystyle\int_4^{\infty} \frac{3}{x^7}\,dx$

(c) $\displaystyle\int_{10}^{\infty} \frac{1}{5x^4}\,dx$

(d) $\displaystyle\int_0^{\infty} \frac{1}{\sqrt{x+1}}\,dx$

(e) $\displaystyle\int_0^{\infty} \frac{3}{(2x+3)^2}\,dx$

(f) $\displaystyle\int_0^{\infty} \sin(x)dx$

2. Evaluate the following integrals.

(a) $\displaystyle\int_{-\infty}^{-2} \frac{3}{x^2}\,dx$

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(x^2+4)^4}\,dx$

(c) $\displaystyle\int_{-\infty}^{0} \frac{3}{\sqrt{1-x}}\,dx$

(d) $\displaystyle\int_{-\infty}^{\infty} \frac{5t}{\sqrt{t^2+1}}\,dt$

3. For each of the following, decide, without evaluating, whether the integral converges or diverges.

(a) $\displaystyle\int_1^{\infty} \frac{1}{x^3+2}\,dx$

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+5}\,dx$

(c) $\displaystyle\int_2^{\infty} \frac{1}{(z^2-2)^{1/3}}\,dz$

(d) $\displaystyle\int_0^{\infty} \frac{1}{\sqrt{t^4+1}}\,dt$

(e) $\displaystyle\int_1^{\infty} \frac{\sin^3(t)}{t^2}\,dt$

(f) $\displaystyle\int_{\pi}^{\infty} \frac{\cos(z)}{z^5}\,dz$

4. Evaluate the following integrals.

(a) $\displaystyle\int_0^8 \frac{1}{x^{\frac{1}{3}}}\,dx$

(b) $\displaystyle\int_0^1 \frac{3}{x^4}\,dx$

(c) $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x}}\,dx$

(d) $\displaystyle\int_0^5 \frac{5}{(t-2)^{\frac{2}{5}}}\,dt$

(e) $\displaystyle\int_{-2}^0 \frac{6}{(z+2)^2}\,dz$

(f) $\displaystyle\int_{-1}^2 \frac{3}{x^{\frac{1}{3}}}\,dx$

5. (a) Show that
$$\int_1^\infty \frac{1}{x^p}\,dx$$
converges for $p > 1$. Find its value.

(b) Show that
$$\int_1^\infty \frac{1}{x^p}\,dx$$
diverges for $p < 1$.

6. (a) Show that
$$\int_0^1 \frac{1}{x^p}\,dx$$
converges for $p < 1$. Find its value.

(b) Show that
$$\int_0^1 \frac{1}{x^p}\,dx$$
diverges for $p > 1$.

7. Let
$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$
for $n = 1, 2, 3, \ldots$. That is, s_n is the nth partial sum of the harmonic series (see Section 1.3).

(a) Show that
$$s_n \leq 1 + \int_1^n \frac{1}{x}\,dx$$
for $n = 1, 2, 3, \ldots$. (Hint: Use the right-hand rule to approximate the integral.)

(b) Show that
$$\int_1^\infty \frac{1}{x}\,dx$$
diverges.

(c) Use a geometric argument to conclude that

$$\int_0^1 \frac{1}{x}\,dx$$

also diverges.

8. For constants $\sigma > 0$ and $\alpha > 0$, the function

$$p(x) = \frac{\alpha\sigma^\alpha}{x^{\alpha+1}},$$

where $x \geq \sigma$, is called a *Pareto distribution*. It is often used in modeling the distribution of incomes or wealth in a population. In the income interpretation, the function

$$P(x) = \int_x^\infty p(t)dt,$$

$x \geq \sigma$, gives the proportion of the population whose income exceeds x. Here σ represents the minimum income of any person in the population and α controls how rapidly the income distribution diminishes as x increases.

(a) Find $P(x)$.

(b) If $\alpha > 1$, the average income of a population described by this model is

$$A = \int_\sigma^\infty xp(x)dx.$$

Find A.

(c) Why is the condition $\alpha > 1$ needed in (b)?

(d) Suppose $\sigma = 10,000$ and $\alpha = 1.2$. Find A, $P(A)$, and $P(2A)$. Interpret the meaning of these values.

(e) Find the general expression for $P(A)$ as a function of α and graph it. Use this graph to interpret the fairness of the income distribution for different values of α.

9. If f is integrable on $[-b, b]$ for all $b > 0$ and

$$\lim_{b\to\infty} \int_{-b}^b f(x)dx$$

exists, then we call

$$I(f) = \lim_{b\to\infty} \int_{-b}^b f(x)dx$$

the *Cauchy integral* of f.

(a) Show that if $\int_{-\infty}^\infty f(x)dx$ converges, then

$$I(f) = \int_{-\infty}^\infty f(x)dx.$$

(b) Find $I(f)$ and $I(g)$ for $f(x) = x$ and $g(x) = \sin(x)$.

(c) Show that the Cauchy integral of f may exist even though $\int_{-\infty}^\infty f(x)dx$ diverges.

Difference Equations to **D**ifferential Equations

Section 4.7

More on Area

In Section 4.1 we motivated the definition of the definite integral with the idea of finding the area of a region in the plane. However, to solve the problem we restricted to a very special type of region, namely, a region lying between the graph of a function f and an interval on the x-axis. We will now consider the more general problem of the area of a region lying between the graphs of two functions.

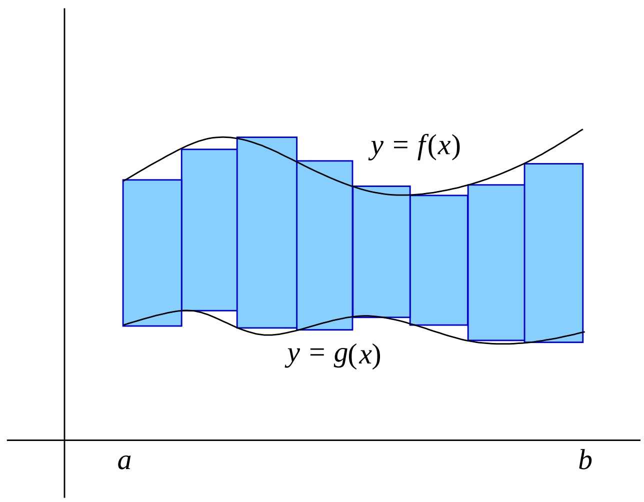


Figure 4.7.1 Approximating the area between $y = f(x)$ and $y = g(x)$

Suppose f and g are functions defined on an interval $[a, b]$ with $g(x) \leq f(x)$ for all x in $[a, b]$. We suppose that f and g are integrable on $[a, b]$, from which it follows that the function k defined by

$$k(x) = f(x) - g(x)$$

is also integrable on $[a, b]$. Let R be the region lying between the graphs of f and g over the interval $[a, b]$ and let A be the area of R. In other words, A is the area of the region of the plane bounded by the curves $y = f(x)$, $y = g(x)$, $x = a$, and $x = b$. We begin with an approximation for A. First, we divide $[a, b]$ into n intervals of equal length

$$\Delta x = \frac{b - a}{n}$$

and let $a = x_0 < x_1 < x_2 < x_3 < \cdots < x_n = b$ be the endpoints of these intervals. Next, for $i = 1, 2, 3, \ldots, n$, let R_i be the region lying between the graphs of f and g over the

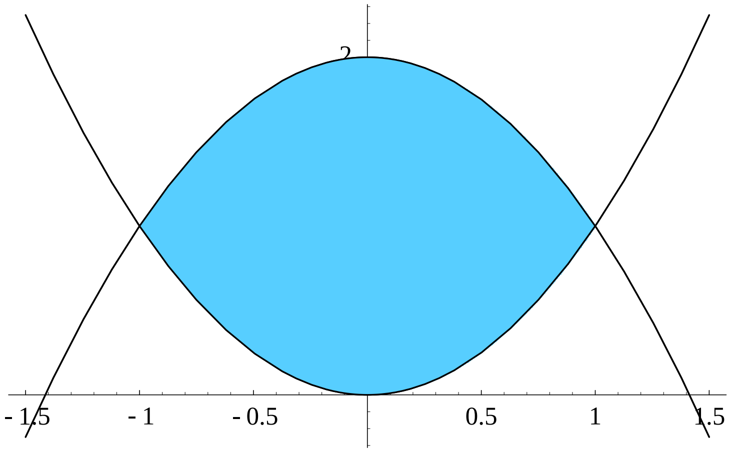


Figure 4.7.2 Region bounded by the graphs of $y = 2 - x^2$ and $y = x^2$

interval $[x_{i-1}, x_i]$. If A_i is the area of R_i, then

$$A = \sum_{i=1}^{n} A_i. \tag{4.7.1}$$

Now $f(x_i) - g(x_i)$ is the distance between the graphs of f and g at x_i, and so

$$(f(x_i) - g(x_i))\Delta x$$

should approximate A_i reasonably well when Δx is small. Thus

$$\sum_{i=1}^{n}(f(x_i) - g(x_i))\Delta x = \sum_{i=1}^{n} k(x_i)\Delta x \tag{4.7.2}$$

will approximate A. Moreover, we should expect that this approximation will improve as Δx decreases, that is, as n increases, and so we should have

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} k(x_i)\Delta x. \tag{4.7.3}$$

But now the right-hand side of (4.7.2) is a Riemann sum, in particular, the right-hand rule sum, and so the right-hand side of (4.7.3) converges to the definite integral of k on $[a, b]$. Hence we have

$$A = \int_a^b k(x)dx = \int_a^b (f(x) - g(x))dx. \tag{4.7.4}$$

Example Let R be the region bounded by the curves $y = 2 - x^2$ and $y = x^2$, as shown in Figure 4.7.2. Note that these curves intersect when

$$2 - x^2 = x^2,$$

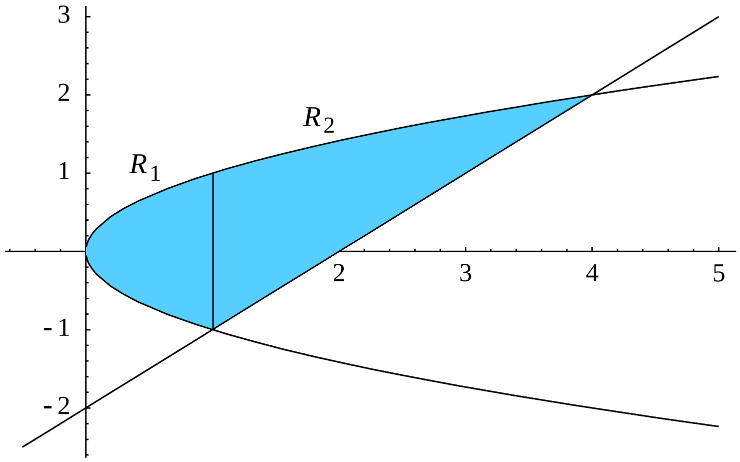


Figure 4.7.3 Region bounded by the graphs of $x = y^2$ and $x = y + 2$

which implies that $2x^2 = 2$, that is, $x = -1$ or $x = 1$. Hence the two curves intersect at $(-1, 1)$ and $(1, 1)$, and so we may describe R as the region between the curves $y = 2 - x^2$ and $y = x^2$ which lies above the interval $[-1, 1]$. Thus if A is the area of R, we have

$$\begin{aligned}
A &= \int_{-1}^{1} ((2 - x^2) - x^2)dx \\
&= \int_{-1}^{1} (2 - 2x^2)dx \\
&= \left(2x - \frac{2}{3}x^3\right)\Big|_{-1}^{1} \\
&= \left(2 - \frac{2}{3}\right) - \left(-2 + \frac{2}{3}\right) \\
&= \frac{8}{3}.
\end{aligned}$$

Example Let R be the region bounded by the curves $x = y^2$ and $x = y + 2$. These two curves intersect when

$$y^2 = y + 2,$$

which implies that

$$0 = y^2 - y - 2 = (y - 2)(y + 1).$$

Hence the two curves intersect when $y = -1$ and $y = 2$, that is, at the points $(1, -1)$ and $(4, 2)$. However, looking at Figure 4.7.3, we see that not all of R lies over the interval $[1, 4]$. In fact, R may be broken up into two regions, $R_1$and R_2, where R_1 is the region between the curves $y = \sqrt{x}$ and $y = -\sqrt{x}$ over the interval $[0, 1]$ and R_2 is the region between the curves $y = \sqrt{x}$ and $y = x - 2$ over the interval $[1, 4]$. Thus, if A is the area of R, A_1 is the area of R_1, and A_2 is the area of R_2, then

$$\begin{aligned} A &= A_1 + A_2 \\ &= \int_0^1 (\sqrt{x} - (-\sqrt{x})) + \int_1^4 (\sqrt{x} - (x-2))dx \\ &= \int_0^1 2\sqrt{x}\, dx + \int_1^4 (\sqrt{x} - x + 2)dx \\ &= \frac{4}{3}x^{\frac{3}{2}}\bigg|_0^1 + \left(\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{2}x^2 + 2x\right)\bigg|_1^4 \\ &= \frac{4}{3} + \left(\frac{16}{3} - 8 + 8\right) - \left(\frac{2}{3} - \frac{1}{2} + 2\right) \\ &= \frac{9}{2}. \end{aligned}$$

The region R in the previous example may also be described as the region lying between the curves $x = y^2$ and $x = y+2$ over the interval $[-1, 2]$ on the y-axis. In general, analogous to our development above, if f and g are functions defined on an interval $[c, d]$ on the y-axis with $g(y) \le f(y)$ for all y in $[c, d]$, then the area A of the region bounded by $x = f(y)$, $x = g(y)$, $y = c$, and $y = d$ (see Figure 4.7.4), is given by

$$A = \int_c^d (f(y) - g(y))dy. \tag{4.7.5}$$

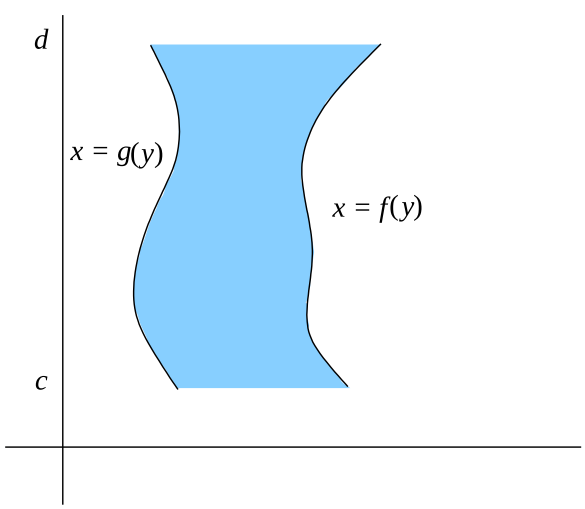


Figure 4.7.4 Region between the curves $x = f(y)$ and $x = g(y)$

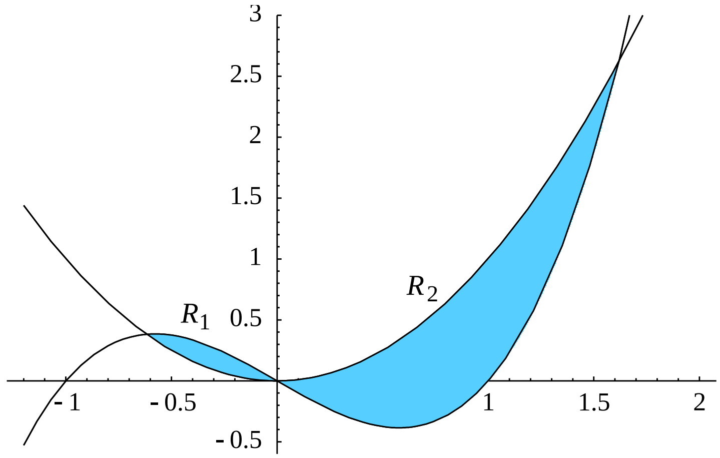


Figure 4.7.5 Region bounded by the graphs of $y = x^3 - x$ and $y = x^2$

In particular, for our previous example we have

$$\begin{aligned} A &= \int_{-1}^{2} (y + 2 - y^2)dy \\ &= \left(\frac{1}{2}y^2 + 2y - \frac{1}{3}y^3 \right) \Big|_{-1}^{2} \\ &= \left(2 + 4 - \frac{8}{3} \right) - \left(\frac{1}{2} - 2 + \frac{1}{3} \right) \\ &= \frac{9}{2}. \end{aligned}$$

In this case, the second method for solving the problem is a little simpler than the first; in general, it is often useful to look at a problem both ways and evaluate using the simpler of the two approaches.

Example Let R be the region bounded by the curves $y = x^3 - x$ and $y = x^2$. These curves intersect when

$$x^3 - x = x^2,$$

that is, when

$$0 = x^3 - x^2 - x = x(x^2 - x - 1).$$

Hence the curves intersect when $x = 0$,

$$x = \frac{1 - \sqrt{5}}{2},$$

or

$$x = \frac{1 + \sqrt{5}}{2},$$

where the latter two values were found using the quadratic formula. From the graphs in Figure 4.7.5, we see that R may be divided into two regions, R_1 and R_2, where R_1 extends from $x = \frac{1-\sqrt{5}}{2}$ to $x = 0$ and R_2 extends from $x = 0$ to $x = \frac{1+\sqrt{5}}{2}$. Note that in R_1 we have $x^3 - x \geq x^2$, whereas $x^2 \geq x^3 - x$ in R_2. Thus, if A is the area of R, A_1 is the area of R_1, and A_2 is the area of R_2, then

$$\begin{aligned} A &= A_1 + A_2 \\ &= \int_{\frac{1-\sqrt{5}}{2}}^{0} (x^3 - x - x^2)dx + \int_{0}^{\frac{1+\sqrt{5}}{2}} (x^2 - x^3 + x)dx \\ &= \left(\frac{1}{4}x^4 - \frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Bigg|_{\frac{1-\sqrt{5}}{2}}^{0} + \left(\frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{2}x^2\right)\Bigg|_{0}^{\frac{1+\sqrt{5}}{2}} \\ &= \frac{13}{12}. \end{aligned}$$

Problems

1. Find the area of the region bounded by the curves $y = x$ and $y = x^2$.
2. Find the area of the region bounded by the curves $y = \sqrt{x}$ and $y = \frac{1}{2}x$.
3. Find the area of the region bounded by the curves $y = x^2$ and $y = x + 2$.
4. Find the area of the region bounded by the curves $y = \cos(x)$ and $y = x^2$.
5. Find the area of the region bounded by the curves $y = \sin(x)$ and $y = x^2$.
6. Find the area of one of the regions lying between the curves $y = \cos(x)$ and $y = \sin(x)$ between two consecutive points of intersection.
7. Find the area of the region in the first quadrant bounded by $y = \cos(x)$, $y = \sin(x)$, and $x = 0$.
8. Find the area of one of the regions lying between the curves $y = \cos^2(x)$ and $y = \sin^2(x)$ between two consecutive points of intersection.
9. Let R be the region bounded by the curves $y = x^2$ and $y = 2 - x$.
   (a) Set up an integral to find the area of R using functions of x.
   (b) Set up an integral to find the area of R using functions of y.
   (c) Evaluate the simpler of the integrals in (a) and (b).
10. Find the area of the region bounded by the curves $x = y^2 - 1$ and $x = 1 - y^2$.
11. Find the area of the region bounded by the curves $x = y^2$ and $x = 6 - y$.
12. Find the area of the region bounded by the curves $y = x^3 - 2x$ and $y = x^2$.
13. Find the area of the region bounded by the curves $y = x^4 - 4x^2$ and $y = 3x^3$.
14. To estimate the surface area of a lake, 21 measurements of the width of the lake are made at points spaced 50 yards apart from one end of the lake to the other. Suppose the measurements are, in order, 0, 50, 100, 120, 180, 240, 300, 250, 220, 295, 305, 265, 240, 275, 225, 180, 120, 90, 63, 40, and 0, all measured in yards. Use Simpson's rule to approximate the surface area of the lake.

Difference Equations to **D**ifferential Equations

Section 4.8

Distance, Position, and the Length of Curves

Although we motivated the definition of the definite integral with the notion of area, there are many applications of integration to problems unrelated to the computation of area. Depending on the context, the definite integral of a function f from a to b could represent the total mass of a wire, the total electric charge on such a wire, or the probability that a light bulb will fail sometime in the time interval from a to b. In this section we will consider three applications of definite integrals: finding the distance traveled by an object over an interval of time if we are given its velocity as a function of time, finding the position of an object at any time if we are given its initial position and its velocity as a function of time, and finding the length of a curve.

Distance

Suppose the function v is continuous on the interval $[a, b]$ and, for any $a \leq t \leq b$, $v(t)$ represents the velocity at time t of an object traveling along a line. Divide $[a, b]$ into n time intervals of equal length

$$\Delta t = \frac{b-a}{n}$$

with endpoints $a = t_0 < t_1 < t_2 < \cdots < t_n = b$. Then, for $j = 1, 2, 3, \ldots, n$, $|v(t_{j-1})|$ is the speed of the object at the beginning of the jth time interval. Hence, for small enough Δt, $|v(t_{j-1})|\Delta t$ will give a good approximation of the distance the object will travel during the jth time interval. Thus if D represents the total distance the object travels from time $t = a$ to time $t = b$, then

$$D \approx \sum_{j=1}^{n} |v(t_{j-1})|\Delta t. \tag{4.8.1}$$

Moreover, we expect that as Δt decreases, or, equivalently, as n increases, this approximation should approach the exact value of D. That is, we should have

$$D = \lim_{n\to\infty} \sum_{j=1}^{n} |v(t_{j-1})|\Delta t. \tag{4.8.2}$$

Now the right-hand side of (4.8.1) is a Riemann sum (in particular, a left-hand rule sum) which approximates the definite integral

$$\int_a^b |v(t)|dt.$$

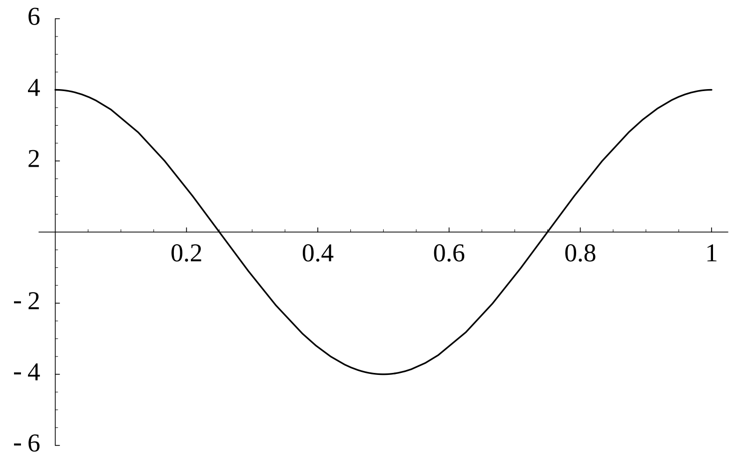


Figure 4.8.1 Graph of the velocity function $v(t) = 4\cos(2\pi t)$

Hence this integral is the value of the limit in (4.8.2), and so we have

$$D = \lim_{n\to\infty} \sum_{j=1}^{n} |v(t_{j-1})|\Delta t = \int_a^b |v(t)|dt. \tag{4.8.3}$$

Example Suppose an object is oscillating at the end of a spring so that its velocity at time t is given by $v(t) = 4\cos(2\pi t)$. Then the distance D traveled by the object from time $t = 0$ to time $t = 1$ is given by

$$D = \int_0^1 |4\cos(2\pi t)|dt = 4\int_0^1 |\cos(2\pi t)|dt.$$

Now

$$\cos(2\pi t) \geq 0 \text{ when } 0 \leq t \leq \frac{1}{4} \text{ or } \frac{3}{4} \leq t \leq 1,$$

and

$$\cos(2\pi t) \leq 0 \text{ when } \frac{1}{4} \leq t \leq \frac{3}{4}.$$

Hence

$$|\cos(2\pi t)| = \cos(2\pi t) \text{ when } 0 \leq t \leq \frac{1}{4} \text{ or } \frac{3}{4} \leq t \leq 1,$$

and

$$|\cos(2\pi t)| = -\cos(2\pi t) \text{ when } \frac{1}{4} \leq t \leq \frac{3}{4}.$$

Thus

$$\begin{aligned}
\int_0^1 |\cos(2\pi t)|dt &= \int_0^{\frac{1}{4}} |\cos(2\pi t)|dt + \int_{\frac{1}{4}}^{\frac{3}{4}} |\cos(2\pi t)|dt + \int_{\frac{3}{4}}^1 |\cos(2\pi t)|dt \\
&= \int_0^{\frac{1}{4}} \cos(2\pi t)dt - \int_{\frac{1}{4}}^{\frac{3}{4}} \cos(2\pi t)dt + \int_{\frac{3}{4}}^1 \cos(2\pi t)dt \\
&= \frac{1}{2\pi}\sin(2\pi t)\Big|_0^{\frac{1}{4}} - \frac{1}{2\pi}\sin(2\pi t)\Big|_{\frac{1}{4}}^{\frac{3}{4}} + \frac{1}{2\pi}\sin(2\pi t)\Big|_{\frac{3}{4}}^1 \\
&= \left(\frac{1}{2\pi} - 0\right) - \left(-\frac{1}{2\pi} - \frac{1}{2\pi}\right) + \left(0 + \frac{1}{2\pi}\right) \\
&= \frac{2}{\pi}.
\end{aligned}$$

Hence

$$D = 4\int_0^1 |\cos(2\pi t)|dt = \frac{8}{\pi}.$$

Position

Again suppose v is continuous on $[a, b]$ and $v(t)$ represents, for $a \leq t \leq b$, the velocity at time t of an object moving on a line. Let $x(t)$ be the position of the object at time t and suppose we know the value of $x(a)$, the position of the object at the beginning of the time interval. It follows that

$$\dot{x}(t) = \frac{d}{dt}x(t) = v(t), \tag{4.8.4}$$

and so, by the Fundamental Theorem of Integral Calculus, for any t between a and b,

$$\int_a^t v(s)ds = x(t) - x(a). \tag{4.8.5}$$

Thus we have

$$x(t) = \int_a^t v(s)ds + x(a) \tag{4.8.6}$$

for $a \leq t \leq b$. In other words, if we are given the velocity of an object for every time t in the interval $[a, b]$ and the position of the object at time $t = a$, then we may use (4.8.6) to compute the position of the object at any time t in $[a, b]$.

Example As in the previous example, consider an object oscillating at the end of a spring so that its velocity is given by $v(t) = 4\cos(2\pi t)$. If $x(t)$ is the position of the object at time t and, initially, $x(0) = 3$, then

$$x(t) = \int_0^t 4\cos(2\pi s)ds + 3 = \frac{2}{\pi}\sin(2\pi s)\Big|_0^t + 3 = \frac{2}{\pi}\sin(2\pi t) + 3.$$

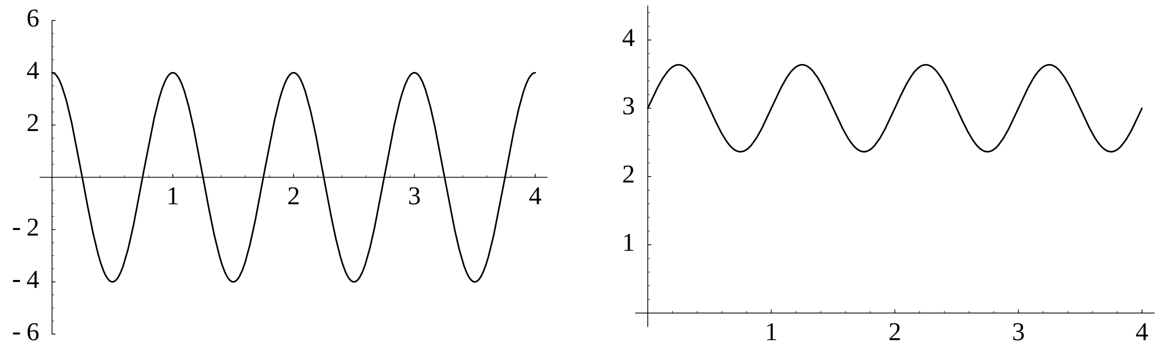


Figure 4.8.2 Velocity $v(t) = 4\cos(2\pi t)$ and position $x(t) = \dfrac{2}{\pi}\sin(2\pi t) + 3$

You should compare the graphs of the velocity function v and the position function x in Figure 4.8.2. Note that the object will oscillate between $3 - \frac{2}{\pi}$ and $3 + \frac{2}{\pi}$. In particular, the distance between these two extremes is $\frac{4}{\pi}$, and so the object will travel a distance of $\frac{8}{\pi}$ during a complete oscillation, in agreement with our computation in the previous example.

Example Suppose the velocity of an object at time t is given by $v(t) = 4\sin(t^2)$. If $x(t)$ is the position of the object at time t and its position at time 0 is $x(0) = -1$, then

$$x(t) = \int_0^t \sin(s^2)ds - 1.$$

However, unlike the previous example, there does not exist a simple antiderivative for v; hence, the best we can do is approximate $x(t)$ for a specified value of t using numerical integration. For example, we can compute numerically that

$$x(2) = \int_0^2 4\sin(s^2)ds - 1 = 2.219,$$

where we have rounded the result to the third decimal place. If we do this for enough points, we can plot the graph of x, as shown in Figure 4.8.3. Again, you should compare this graph with the graph of v, also shown in Figure 4.8.3.

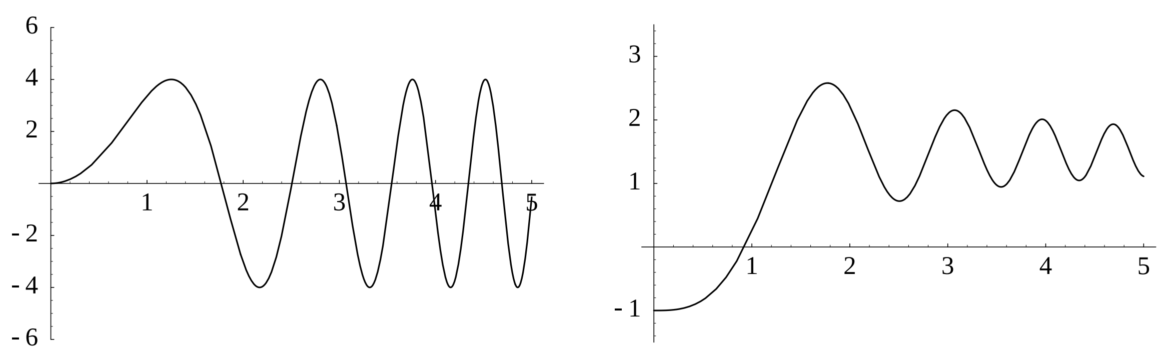


Figure 4.8.3 Velocity $v(t) = 4\sin(t^2)$ and position $x(t) = \int_0^t 4\sin(s^2)ds - 1$

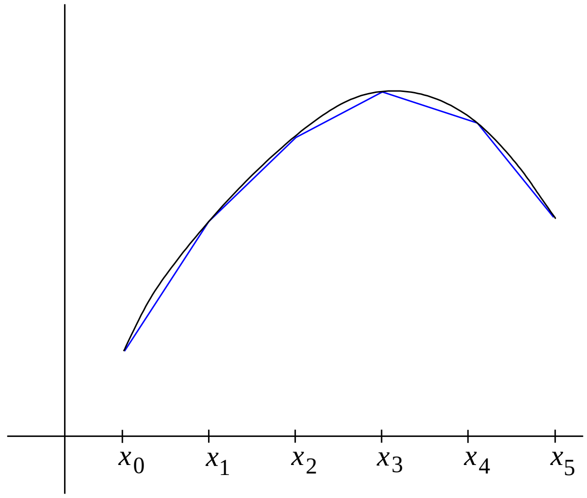


Figure 4.8.4 Approximating a curve with line segments

Length of a curve

Here we will consider the problem of finding the length of a curve which is the graph of some differentiable function. So suppose the function f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Let C be the graph of f and let L be the length of C. As we have done previously, we will first describe a method for finding good approximations to L. To begin, divide $[a, b]$ into n intervals of equal length

$$\Delta x = \frac{b-a}{n}$$

with endpoints $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. For $j = 1, 2, 3, \ldots, n$, we can approximate the length of the piece of C lying over the jth interval by the distance between the endpoints of this piece, as shown in Figure 4.8.4. That is, since the endpoints of the jth piece are $(x_{j-1}, f(x_{j-1}))$ and $(x_j, f(x_j))$, we can approximate the length of the piece of C lying over the interval $[x_{j-1}, x_j]$ by

$$\sqrt{(x_j - x_{j-1})^2 + (f(x_j) - f(x_{j-1}))^2}.$$

Since $\Delta x = x_j - x_{j-1}$,

$$\begin{aligned}\sqrt{(x_j - x_{j-1})^2 + (f(x_j) - f(x_{j-1}))^2} &= \sqrt{(\Delta x)^2 + (f(x_j) - f(x_{j-1}))^2}\\ &= \sqrt{(\Delta x)^2\left(1 + \frac{(f(x_j) - f(x_{j-1}))^2}{(\Delta x)^2}\right)} \qquad (4.8.7)\\ &= \Delta x\sqrt{1 + \left(\frac{f(x_j) - f(x_{j-1})}{\Delta x}\right)^2}.\end{aligned}$$

Hence, when n is large (equivalently, when Δx is small), a good approximation for L is given by

$$L \approx \sum_{j=1}^{n} \sqrt{1 + \left(\frac{f(x_j) - f(x_{j-1})}{\Delta x}\right)^2}\ \Delta x. \tag{4.8.8}$$

Moreover, we expect that

$$L = \lim_{n\to\infty} \sum_{j=1}^{n} \sqrt{1 + \left(\frac{f(x_j) - f(x_{j-1})}{\Delta x}\right)^2}\ \Delta x, \tag{4.8.9}$$

provided this limit exists. By the Mean Value Theorem, for each $j = 1, 2, 3, \ldots$, there exists a point c_j in the interval (x_{j-1}, x_j) such that

$$f'(c_j) = \frac{f(x_j) - f(x_{j-1})}{\Delta x}. \tag{4.8.10}$$

Hence

$$L = \lim_{n\to\infty} \sum_{j=1}^{n} \sqrt{1 + (f'(c_j))^2}\ \Delta x. \tag{4.8.11}$$

Now the sum in (4.8.11) is a Riemann sum for the integral

$$\int_a^b \sqrt{1 + (f'(x))^2}\ dx,$$

and so the limit, if it exists, converges to the value of this integral. Thus the length of C is given by

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\ dx. \tag{4.8.12}$$

Example Let L be the length of the graph of $f(x) = x^{\frac{3}{2}}$ on the interval $[0, 1]$, as shown in Figure 4.8.5. Then

$$f'(x) = \frac{3}{2}x^{\frac{1}{2}},$$

so

$$\begin{aligned} L &= \int_0^1 \sqrt{1 + \left(\frac{3}{2}x^{\frac{1}{2}}\right)^2}\ dx \\ &= \int_0^1 \sqrt{1 + \frac{9}{4}x}\ dx \\ &= \frac{8}{27}\left(1 + \frac{9}{4}x\right)^{\frac{3}{2}}\Bigg|_0^1 \\ &= \frac{13\sqrt{13} - 8}{27} = 1.4397, \end{aligned}$$

where we have rounded the result to four decimal places.

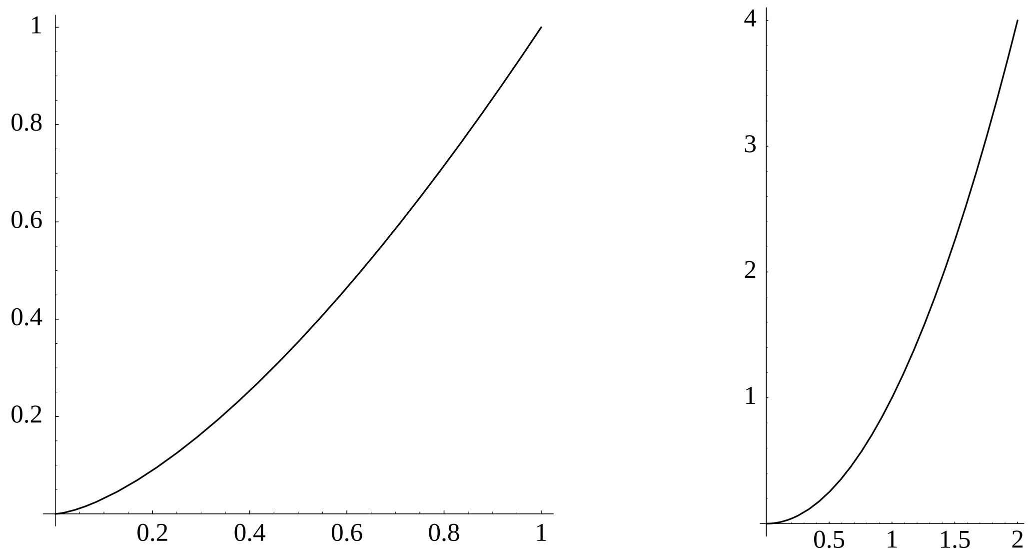


Figure 4.8.5 Graphs of $y = x^{\frac{3}{2}}$ and $y = x^2$

Example Let L be the length of the parabola $y = x^2$ from $(0, 0)$ to $(2, 4)$, as shown in Figure 4.8.5. Then

$$\frac{dy}{dx} = 2x,$$

so

$$L = \int_0^2 \sqrt{1 + (2x)^2}\, dx = \int_0^2 \sqrt{1 + 4x^2}\, dx.$$

At this point we do not have the techniques to evaluate this integral exactly using the Fundamental Theorem (although we will see such techniques in Chapter 6); however, we may use a computer algebra system to find that

$$L = \sqrt{17} + \frac{1}{4}\sinh^{-1}(4) = \sqrt{17} + \frac{1}{4}(\log(4 + \sqrt{17})),$$

where $\log(x)$ is the natural logarithm of x and $\sinh^{-1}(x)$ is the inverse hyperbolic sine of x. Since we will not study either of these functions until Chapter 6, we will use a numerical approximation to give us $L = 4.6468$ to four decimal places, the same answer we would obtain by using numerical integration to evaluate the integral.

Example To find the length L of one arch of the curve $y = \sin(x)$, as shown in Figure 4.8.6, we need to evaluate

$$L = \int_0^{\pi} \sqrt{1 + \cos^2(x)}\, dx,$$

an integral which is even more difficult than the one in the previous example. However, using numerical integration, we find that $L = 3.8202$ to four decimal places.

The last two examples illustrate some of the difficulties in finding the length of a curve. In general, the integrals involved in these problems require more sophisticated techniques than we have available at this time, and frequently require the use of numerical techniques.

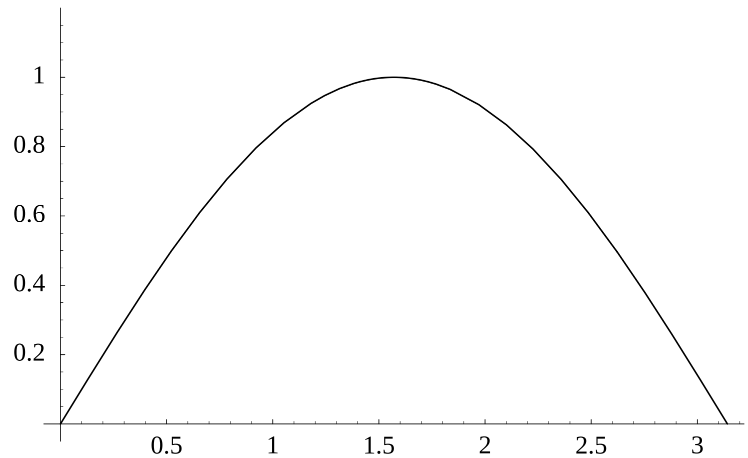


Figure 4.8.6 Graph of $y = \sin(x)$ over the interval $[0, \pi]$

Problems

1. For each of the following, assume that $v(t)$ is the velocity at time t of an object moving on a line and find the distance traveled by the object over the given time period.

   (a) $v(t) = 32t$ over $0 \leq t \leq 3$

   (b) $v(t) = -32t + 16$ over $0 \leq t \leq 3$

   (c) $v(t) = t^2 - t - 6$ over $0 \leq t \leq 2$

   (d) $v(t) = t^2 - t - 6$ over $0 \leq t \leq 4$

   (e) $v(t) = 2\sin(2t)$ over $0 \leq t \leq \pi$

   (f) $v(t) = 3\cos(2\pi t)$ over $0 \leq t \leq 2$

2. Suppose the velocity of a falling object is given by $v(t) = -32t$ feet per second. If the object is at a height of 100 feet at time $t = 0$, find the height of the object at an arbitrary time t.

3. Suppose $x(t)$ and $v(t)$ are the position and velocity, respectively, at time t of an object moving on a line. If $x(0) = 5$ and $v(t) = 3t^2 - 6$, find $x(t)$.

4. If an object of mass m is connected to a spring, pulled a distance x_0 away from its equilibrium position and released, then, ignoring the effects of friction, the velocity of the object at time t will be given by

$$v(t) = -x_0\sqrt{\frac{k}{m}}\sin\left(\sqrt{\frac{k}{m}}\,t\right),$$

   where k is a constant that depends on the strength of the spring. Find $x(t)$, the position of the object at time t.

5. Show that if $\dot{x}(t) = f(t)$ and f is continuous on $[a, b]$, then

$$x(t) = \int_a^t f(s)ds + x(a).$$

6. For each of the following, use the result from Problem 5 to find $x(t)$.

   (a) $\dot{x}(t) = 3t^3 + 6t - 17$ with $x(2) = 4$
   (b) $\dot{x}(t) = 3\cos(6t) - t$ with $x(0) = -1$
   (c) $\dot{x}(t) = \sin^2(2t)$ with $x(0) = 2$
   (d) $\dot{x}(t) = 3t^2 \sin(2t)$ with $x(0) = 0$
   (e) $\dot{x}(t) = \sqrt{1 + 2t}$ with $x(4) = 3$

7. Let $x(t)$, $v(t)$, and $a(t)$ be the height, velocity, and acceleration, respectively, at time t of an object of mass m in free fall near the surface of the earth. Let x_0 and v_0 be the height and velocity, respectively, of the object at time $t = 0$. If we ignore the effects of air resistance, the force acting on the body is $-mg$, where g is a constant ($g = 9.8$ meters per second, or 32 feet per second per second). Thus, by Newton's second law of motion,
$$-mg = ma(t),$$
from which we obtain
$$a(t) = -g.$$
Using Problem 5, show that
$$x(t) = -\frac{1}{2}gt^2 + v_0 t + x_0.$$

8. Suppose an object is projected vertically upward from a height of 100 feet with an initial velocity of 20 feet per second. Use Problem 7 to answer the following questions.

   (a) Find $x(t)$, the height of the object at time t.
   (b) At what time does the object reach its maximum height?
   (c) What is the maximum height reached by the object?
   (d) At what time will the object strike the ground?

9. For each of the following, find the length of the graph of the given function over the given interval.

   (a) $f(x) = 2x^{\frac{3}{2}}$ over $[0, 2]$
   (b) $f(x) = \sin(2x)$ over $\left[0, \frac{\pi}{2}\right]$
   (c) $g(x) = x^3$ over $[-1, 1]$
   (d) $g(t) = \tan(t)$ over $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$
   (e) $f(t) = \sin^2(t)$ over $[0, \pi]$
   (f) $g(\theta) = \sin(\theta^2)$ over $[0, \sqrt{\pi}\,]$

10. A sheet of corrugated aluminum is to be made from a flat sheet of aluminum. Suppose a cross section of the corrugated sheet, when measured in inches, is in the shape of the curve
$$y = 2\sin\left(\frac{\pi}{4}t\right).$$
Find the length of a flat sheet that would be needed to make a corrugated sheet that is 10 feet long.

Difference Equations to **D**ifferential Equations

Section 5.1

Polynomial Approximations

In Chapter 3 we discussed the problem of finding the affine function which best approximates a given function about some point. In particular, we found that the best affine approximation to a function f at a point c is given by

$$T(x) = f'(c)(x - c) + f(c), \tag{5.1.1}$$

provided that f is differentiable at c. In this section and the next, we will extend the ideas of Sections 3.1 and 3.2 to the problem of finding polynomial approximations of any given degree to a function about some specified point. We shall see that many nonlinear functions can be approximated to any desired level of accuracy over a specified interval if we use polynomials of sufficiently high degree. As an example, compare the graphs of $f(x) = \sin(x)$ and

$$P(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{362,880}x^9$$

in Figure 5.1.1. They are almost indistinguishable over the interval $[-\pi, \pi]$. In practical terms, this means there is little difference in working with $P(x)$ instead of $f(x)$ for x in $[-\pi, \pi]$. Moreover, since polynomials are the simplest of functions, involving only the arithmetic operations of addition, subtraction, and multiplication, the substitution of P for f can be a very helpful step in simplifying a problem.

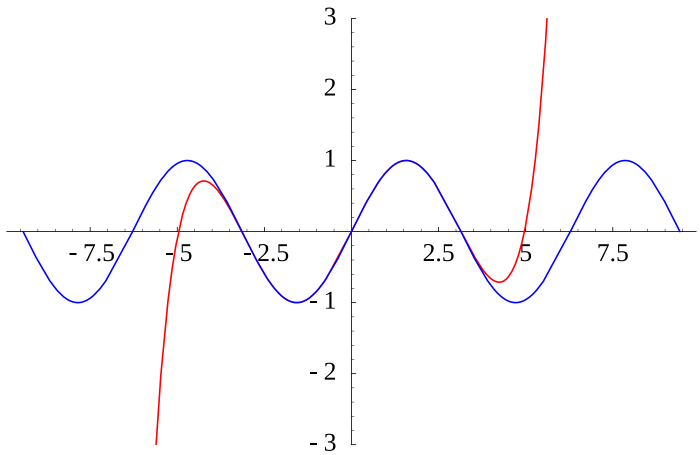


Figure 5.1.1 Graphs of $f(x) = \sin(x)$ and an approximating polynomial

To begin, we need to recall, and then generalize, some definitions and facts from Sections 3.1 and 3.2. First, recall that a function f is said to be $o(h)$ if

$$\lim_{h\to 0} \frac{f(h)}{h} = 0; \tag{5.1.2}$$

a function f is said to be $O(h)$ if there exist constants M and ϵ such that

$$\left|\frac{f(h)}{h}\right| \leq M \tag{5.1.3}$$

whenever $-\epsilon < h < \epsilon$. In particular, we saw that f is $O(h)$ if

$$\lim_{h\to 0} \frac{f(h)}{h}$$

exists. The following definition generalizes to other powers of h this method of characterizing the rate at which a function converges to 0.

Definition For any $n > 0$, a function f is said to be $o(h^n)$ if

$$\lim_{h\to 0} \frac{f(h)}{h^n} = 0. \tag{5.1.4}$$

For any $n > 0$, a function f is said to be $O(h^n)$ if there exist constants M and ϵ such that

$$\left|\frac{f(h)}{h^n}\right| \leq M \tag{5.1.5}$$

whenever $-\epsilon < h < \epsilon$.

Similar to our result in Section 3.1, f is $O(h^n)$ if

$$\lim_{h\to 0} \frac{f(h)}{h^n}$$

exists.

As before, we use this notation as a means of comparing the rates at which functions approach 0. As h approaches 0, a function which is $O(h^n)$ approaches 0 as least as fast as h^n does. Note that for $n > m > 0$,

$$\lim_{h\to 0} \frac{h^n}{h^m} = \lim_{h\to 0} h^{n-m} = 0 \tag{5.1.6}$$

since $n - m > 0$, and so h^n goes to 0 faster than h^m as h approaches 0. Thus if $n > m > 0$, as h goes to 0, a function which is $O(h^n)$ approaches 0 faster than does a function which

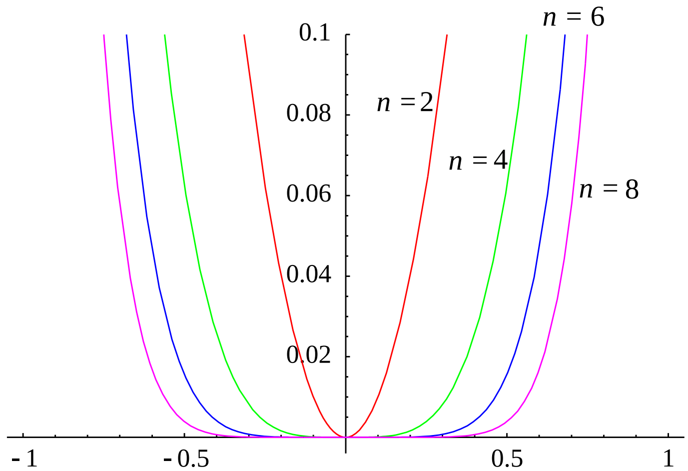


Figure 5.1.2 Graphs of $f(h) = h^n$ for $n = 2, 4, 6$, and 8

is $O(h^m)$ but not $O(h^n)$. Figure 5.1.2 illustrates this fact with the graphs of $f(h) = h^n$ for $n = 2, 4, 6$ and 8.

Example Since

$$\lim_{h \to 0} \frac{\sin(h)}{h} = 1,$$

it follows that $\sin(h)$ is $O(h)$.

Example Since

$$\lim_{h \to 0} \frac{\sin^2(h)}{h} = \lim_{h \to 0} \frac{\sin(h)}{h} \lim_{h \to 0} \sin(h) = (1)(0) = 0,$$

it follows that $\sin^2(h)$ is $o(h)$.

Example Since

$$\lim_{h \to 0} \frac{\sin^2(h)}{h^2} = \lim_{h \to 0} \frac{\sin(h)}{h} \lim_{h \to 0} \frac{\sin(h)}{h} = (1)(1) = 1,$$

it follows that $\sin^2(h)$ is $O(h^2)$.

Example Since

$$\lim_{h \to 0} \frac{\sin^3(h)}{h^3} = \lim_{h \to 0} \frac{\sin(h)}{h} \lim_{h \to 0} \frac{\sin(h)}{h} \lim_{h \to 0} \frac{\sin(h)}{h} = (1)(1)(1) = 1,$$

it follows that $\sin^3(h)$ is $O(h^3)$.

Hence, for example, we would say that as h goes to 0, $\sin^2(h)$ approaches 0 faster than h, but at about the same rate as h^2.

Now suppose that f is $O(h^n)$ for some $n > 0$. This means that as h goes to 0, f approaches 0 at least as fast as h^n does. It should follow that f goes to 0 faster than h^m, and so is $o(h^m)$, for any $0 < m < n$. To see this, let M and $\epsilon > 0$ be numbers such that

$$\left|\frac{f(h)}{h^n}\right| \leq M \tag{5.1.7}$$

for all h in the interval $(-\epsilon, \epsilon)$. Then

$$\left|\frac{f(h)}{h^m}\right| = |h^{n-m}|\left|\frac{f(h)}{h^n}\right| \leq |h|^{n-m}M \tag{5.1.8}$$

for all h in $(-\epsilon, \epsilon)$. Since

$$\lim_{h \to 0} |h|^{n-m}M = 0, \tag{5.1.9}$$

it follows that

$$\lim_{h \to 0} \left|\frac{f(h)}{h^m}\right| = 0, \tag{5.1.10}$$

and so

$$\lim_{h \to 0} \frac{f(h)}{h^m} = 0. \tag{5.1.11}$$

Thus f is $o(h^m)$.

Proposition If $n > m > 0$ and f is $O(h^n)$, then f is $o(h^m)$.

Example We saw above that $\sin^3(h)$ is $O(h^3)$, from which it now follows, for example, that $\sin^3(h)$ is $o(h^2)$.

Next, recall that if f is a function defined in an open interval about a point c and T is an affine function such that $T(c) = f(c)$ and

$$R(h) = f(c+h) - T(c+h) \tag{5.1.12}$$

is $o(h)$, then T is the best affine approximation to f at c. Moreover, as mentioned above, we saw in Chapter 3 that a function f has a best affine approximation at a point c if and only if f is differentiable at c and, in that case, the best affine approximation is given by

$$T(x) = f(c) + f'(c)(x-c). \tag{5.1.13}$$

Putting (5.1.12) and (5.1.13) together and letting $x = c + h$, or, equivalently, $h = x - c$, we have that

$$f(x) - f(c) - f'(c)(x-c)$$

is $o(x-c)$. We may express this by writing

$$f(x) - f(c) - f'(c)(x-c) = o(x-c), \tag{5.1.14}$$

or, solving for $f(x)$, simply

$$f(x) = f(c) + f'(c)(x-c) + o(x-c). \tag{5.1.15}$$

In words, (5.1.15) says that $f(x)$ is equal to $f(c) + f'(c)(x-c)$ plus some function which is $o(x-c)$, that is, some function which approaches 0 faster than $x-c$ as x approaches c.

Example Let $f(x) = \sqrt{x}$. Then

$$f'(1) = \frac{1}{2},$$

so the best affine approximation to f at 1 is

$$T(x) = 1 + \frac{1}{2}(x-1).$$

That is,

$$\sqrt{x} = 1 + \frac{1}{2}(x-1) + o(x-1).$$

In words, this statement says that $\sqrt{x}$ is equal to

$$1 + \frac{1}{2}(x-1)$$

plus a term of order higher than $x-1$, that is, plus a term which goes to 0 faster than $x-1$ as x approaches 1.

Example Let $f(x) = \sin(x)$. Then $f'(0) = \cos(0) = 1$, so the best affine approximation to f at 0 is

$$T(x) = x.$$

Thus

$$\sin(x) = x + o(x),$$

a fact which is often used in applications to justify replacing the function $\sin(x)$ by the function x for calculations involving only values of x close to 0.

Now suppose that f is twice continuously differentiable on an interval $(c-\delta, c+\delta)$ for some $\delta > 0$; that is, suppose both f' and f'' exist and are continuous on $(c-\delta, c+\delta)$. If T is the best affine approximation to f at c, then, as we have seen,

$$R(h) = f(c+h) - T(c+h) \tag{5.1.16}$$

is $o(h)$. We will show that R is in fact $O(h^2)$. Suppose $0 < \epsilon < \delta$ and $-\epsilon < h < \epsilon$. First, note that

$$f(c+h) - T(c+h) = f(c+h) - f(c) - f'(c)h. \tag{5.1.17}$$

By the Mean Value Theorem, there is a point u between c and $c+h$ such that

$$f(c+h) - f(c) = f'(u)h. \tag{5.1.18}$$

Hence

$$f(c+h) - T(c+h) = f'(u)h - f'(c)h = h(f'(u) - f'(c)). \tag{5.1.19}$$

Applying the Mean Value Theorem again, there exists a point v between c and u such that

$$f'(u) - f'(c) = f''(v)(u-c). \tag{5.1.20}$$

Thus

$$\frac{R(h)}{h^2} = \frac{f(c+h) - T(c+h)}{h^2} = \frac{h(u-c)f''(v)}{h^2} = \frac{(u-c)f''(v)}{h}. \tag{5.1.21}$$

If we let M be the maximum value of $|f''(x)|$ on $[c-\epsilon, c+\epsilon]$ and note that $|u-c| < |h|$, then we see that

$$\left|\frac{R(h)}{h^2}\right| = \frac{|u-c||f''(v)|}{|h|} < \frac{|h|M}{|h|} = M \tag{5.1.22}$$

for all h with $-\epsilon < h < \epsilon$. Hence $R(h)$ is $O(h^2)$.

Proposition If f is twice continuously differentiable on an open interval containing the point c and T is the best affine approximation to f at c, then

$$R(h) = f(c+h) - T(c+h) \tag{5.1.23}$$

is $O(h^2)$.

Letting $x = c+h$, we can rephrase the proposition to say that

$$r(x) = f(x) - T(x) \tag{5.1.24}$$

is $O((x-c)^2)$. Similar to our notation above, we may write

$$f(x) = f(c) + f'(c)(x-c) + O((x-c)^2). \tag{5.1.25}$$

For our previous examples, this means that

$$\sqrt{x} = 1 + \frac{1}{2}(x-1) + O((x-1)^2)$$

and

$$\sin(x) = x + O(x^2).$$

This is the type of formulation that we wish to generalize to higher order polynomial approximations. We will introduce these polynomials, called *Taylor polynomials*, here, but save the verification that they provide the sought-for approximations until the next section.

Taylor polynomials

The best affine approximation T to a function f at a point c may be described as the only first degree polynomial satisfying both $T(c) = f(c)$ and $T'(c) = f'(c)$. This provides a clue as to where to look for higher order polynomial approximations. Namely, given a function f which is n times differentiable at a point c, we will look for a polynomial P_n of degree at most n with the property that $P_n(c) = f(c)$ and the first n derivatives of P_n at c agree with the first n derivatives of f at c. Hence we want to find constants $b_0, b_1, b_2, \ldots, b_n$ so that the polynomial

$$P_n(x) = b_0 + b_1(x-c) + b_2(x-c)^2 + \cdots + b_n(x-c)^n \tag{5.1.26}$$

satisfies

$$P_n^{(j)}(c) = f^{(j)}(c) \tag{5.1.27}$$

for $j = 0, 1, 2, \ldots, n$, where $P_n^{(0)} = P_n$ and, for $j > 0$, $P_n^{(j)}$ is the jth derivative of P_n. Now

$$\begin{aligned}
P_n(c) &= b_0 \\
P_n'(c) &= b_1 \\
P_n''(c) &= 2b_2 \\
P_n'''(c) &= (3)(2)b_3 \\
P_n^{(4)}(c) &= (4)(3)(2)b_4 \\
&\vdots \\
P_n^{(n)}(c) &= n!b_n.
\end{aligned}$$

Thus, to satisfy (5.1.27), we must have

$$\begin{aligned}
f(c) &= b_0 \\
f'(c) &= b_1 \\
f''(c) &= 2b_2 \\
f'''(c) &= 3!b_3 \\
f^{(4)}(c) &= 4!b_4 \\
&\vdots \\
f^{(n)}(c) &= n!b_n.
\end{aligned}$$

Solving for $b_0, b_1, b_2, \ldots, b_n$, we have

$$\begin{aligned}
b_0 &= f(c) \\
b_1 &= f'(c) \\
b_2 &= \frac{f''(c)}{2}
\end{aligned}$$

$$b_3 = \frac{f'''(c)}{3!}$$
$$b_4 = \frac{f^{(4)}(c)}{4!}$$
$$\vdots$$
$$b_n = \frac{f^{(n)}(c)}{n!}.$$

That is,

$$b_j = \frac{f^{(j)}(c)}{j!} \tag{5.1.28}$$

for $j = 0, 1, 2, \ldots, n$. The resulting polynomial is named after Brook Taylor (1685-1731), an English mathematician who was the first to publish work on the related infinite series that we will consider later in this chapter.

Definition Suppose f is n times differentiable at a point c. Then the polynomial

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

is called the *Taylor polynomial of order* n for f at c.

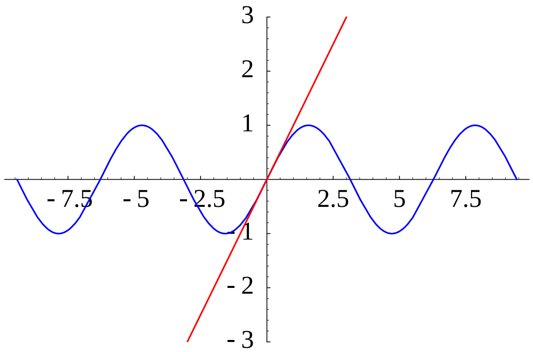
Example Consider $f(x) = \sin(x)$ and $c = 0$. Then

$$\begin{aligned} f'(x) &= \cos(x) \\ f''(x) &= -\sin(x) \\ f'''(x) &= -\cos(x) \\ f''''(x) &= \sin(x). \end{aligned}$$

Notice that, if we were to continue finding higher derivatives, this cycle would repeat itself. Evaluating the function and its derivatives at 0, we obtain

$$\begin{aligned} f(0) &= 0 \\ f'(0) &= 1 \\ f''(0) &= 0 \\ f'''(0) &= -1 \\ f''''(0) &= 0, \end{aligned}$$

a cycle which would repeat itself if we were to continue evaluating higher-order derivatives.



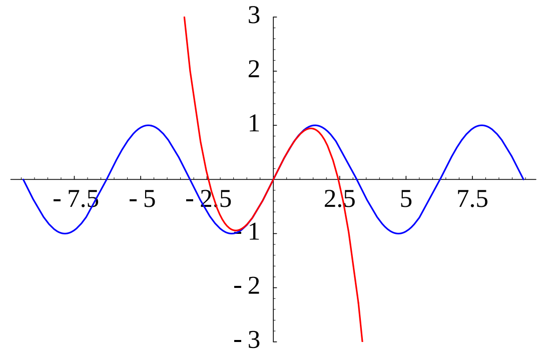


Figure 5.1.3 Graphs of $f(x) = \sin(x)$ with Taylor polynomials P_1 (left) and P_3 (right)

Thus we obtain the following Taylor polynomials for $\sin(x)$ at $x = 0$:

$$\begin{aligned}
P_1(x) &= x \\
P_2(x) &= x \\
P_3(x) &= x - \frac{x^3}{3!} \\
P_4(x) &= x - \frac{x^3}{3!} \\
P_5(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} \\
P_6(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} \\
P_7(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \\
P_8(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \\
P_9(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}.
\end{aligned}$$

The graphs of P_1, P_3, P_5, and P_7, along with the graph of f, are shown in Figures 5.1.3 and 5.1.4. We have already seen the graph of P_9 in Figure 5.1.1. Notice how the Taylor polynomials give increasingly better approximations to $\sin(x)$ as the order increases. Finally, since the values of the derivatives repeat the pattern 0, 1, 0, endlessly, in this case we can write down a simple general expression for the Taylor polynomial of any order. Namely, for any integer $n \geq 0$,

$$P_{2n+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

and $P_{2n+2}(x) = P_{2n+1}(x)$, the latter following from the fact that all the even-order derivatives are 0.

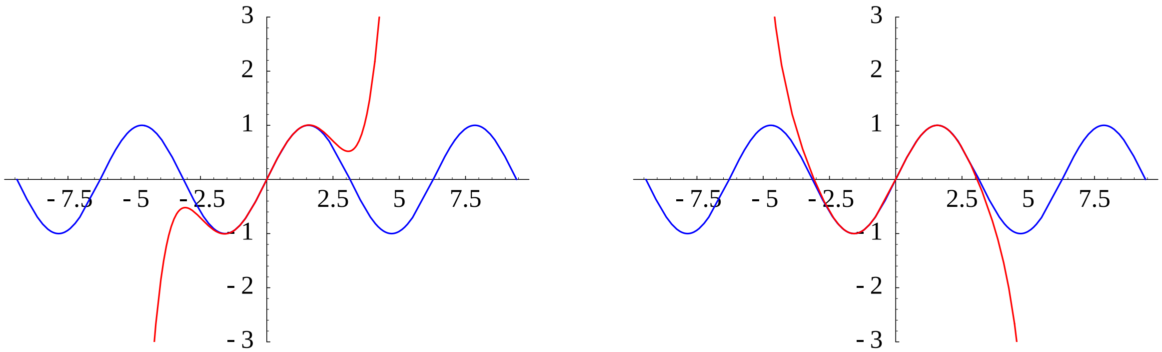


Figure 5.1.4 Graphs of $f(x) = \sin(x)$ with Taylor polynomials P_5 (left) and P_7 (right)

Example Now we will find the Taylor polynomial of order 4 for $g(x) = \sqrt{x}$ at $x = 1$. First we find that

$$\begin{aligned} g'(x) &= \frac{1}{2}x^{-\frac{1}{2}} \\ g''(x) &= -\frac{1}{4}x^{-\frac{3}{2}} \\ g'''(x) &= \frac{3}{8}x^{-\frac{5}{2}} \\ g''''(x) &= -\frac{15}{16}x^{-\frac{7}{2}}. \end{aligned}$$

Hence

$$\begin{aligned} g(1) &= 1 \\ g'(1) &= \frac{1}{2} \\ g''(1) &= -\frac{1}{4} \\ g'''(1) &= \frac{3}{8} \\ g''''(1) &= -\frac{15}{16}. \end{aligned}$$

Thus we have

$$\begin{aligned} P_4(x) &= 1 + \frac{1}{2}(x-1) - \frac{\frac{1}{4}}{2}(x-1)^2 + \frac{\frac{3}{8}}{3!}(x-1)^3 - \frac{\frac{15}{16}}{4!}(x-1)^4 \\ &= 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \frac{5}{128}(x-1)^4. \end{aligned}$$

The graphs of P_4 and g are shown in Figure 5.1.5. As we hoped, $P_4(x)$ provides a good approximation to $\sqrt{x}$ for values of x close to 1. For example, to 8 decimal places,

$$P_4(1.1) = 1.04880859,$$

while

$$\sqrt{1.1} = 1.04880884.$$

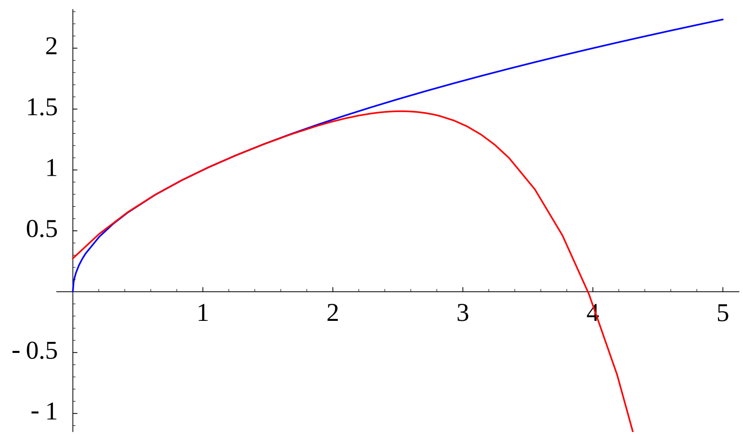


Figure 5.1.5 Graphs of $g(x) = \sqrt{x}$ with Taylor polynomial of order 4

As we should expect, the approximation worsens for x farther away from 1. For example, to 3 decimal places,

$$P_4(2) = 1.398,$$

while

$$\sqrt{2} = 1.414.$$

Although finding a Taylor polynomial of order n for a given function f involves only evaluating the derivatives of f at a specified point, nevertheless, the required computations may become unwieldy, especially if f is itself complicated or n is large. In such cases, a computer algebra system may prove useful. For example, you may find a computer algebra system helpful in working Problems 10 through 12.

In the next section we will see that the Taylor polynomials provide polynomial approximations that generalize best affine approximations. That is, we shall show that, under suitable conditions, if P_n is the Taylor polynomial of order n for f at c, then the remainder function

$$R(h) = f(c+h) - P_n(c+h) \tag{5.1.29}$$

is $O(h^{n+1})$, in agreement with our previous result that the remainder function for the best affine approximation, P_1, is $O(h^2)$.

Problems

1. Show that $f(x) = \tan^2(x)$ is $o(h)$.
2. Show that $g(x) = \tan^2(x)$ is $O(h^2)$.
3. Show that $f(z) = z^2 \sin(z)$ is $o(h^2)$ and $O(h^3)$.
4. Show that $h(t) = 1 - \cos(t)$ is $O(h^2)$.
5. Show that $f(x) = \sin^2(3x)$ is $O(h^2)$.

6. Show that $f(x) = x^{\frac{4}{3}}$ is $o(h)$, but not $O(h^2)$.

7. For each of the following functions, find the Taylor polynomial of order 4 at the given point c.

   (a) $f(x) = \sin(2x)$ at $c = 0$ (b) $g(x) = \cos(x)$ at $c = 0$

   (c) $f(z) = \sqrt{z}$ at $c = 4$ (d) $f(\theta) = \tan(\theta)$ at $\theta = 0$

   (e) $h(x) = \sin(x)$ at $c = \pi$ (f) $g(t) = \cos(2t)$ at $c = \pi$

   (g) $f(x) = \dfrac{1}{x}$ at $c = 1$ (h) $f(x) = 3x^2 + 2x - 9$ at $c = 0$

   (i) $g(x) = \dfrac{1}{1 + x^2}$ at $c = 0$ (j) $h(x) = x^4 + 5x^3 + 4x^2 - 9x - 20$ at $c = 0$

   (k) $g(t) = \dfrac{1}{t^2}$ at $c = 1$ (l) $h(z) = 8z^5 - 3z^3 + 6z$ at $c = 0$

   (m) $x(t) = \sec(t)$ at $c = 0$ (n) $f(x) = 3x^4 - 4x^3 + x^2 - 3x - 2$ at $c = 1$

8. Let P_9 be the 9th order Taylor polynomial for $f(x) = \cos(x)$ at 0. Graph f and P_9 on the same axes. On what interval does $P_9(x)$ appear to give a good approximation to $\cos(x)$?

9. Let P_{13} be the 13th order Taylor polynomial for $f(x) = \sin(x)$ at 0. Graph f and P_{13} on the same axes. On what interval does $P_{13}(x)$ appear to give a good approximation to $\sin(x)$?

10. Let P_n be the nth order Taylor polynomial for $f(x) = \dfrac{1}{x^2 + 1}$ at 0.

    (a) Graph f and P_6 on the same axes. On what interval does $P_6(x)$ appear to give a good approximation to $f(x)$?

    (b) Repeat part (a) for P_{10} and P_{20}.

    (c) Do any of the polynomials in parts (a) and (b) appear to give a good approximation to f on the interval $[1, 2]$?

11. Let P_6 be the 6th order Taylor polynomial for $x(t) = \tan(t)$ at 0. Graph x and P_6 on the same axes and comment.

12. Let P_{15} be the 15th order Taylor polynomial for $g(x) = \sqrt{x}$ at 1. Graph g and P_{15} on the same axes. On what interval does $P_{15}(x)$ appear to give a good approximation to $\sqrt{x}$?

D*ifference Equations*
to
D*ifferential Equations*

Section 5.2

Taylor's Theorem

The goal of this section is to prove that if P_n is the nth order Taylor polynomial for a function f at a point c, then, under suitable conditions, the remainder function

$$R_n(h) = f(c+h) - T(c+h) \tag{5.2.1}$$

is $O(h^{n+1})$. This result is a consequence of *Taylor's theorem*, which we now state and prove.

Taylor's Theorem Suppose f is continuous on the closed interval $[a, b]$ and has $n+1$ continuous derivatives on the open interval (a, b). If x and c are points in (a, b), then

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + r_n(x), \tag{5.2.2}$$

where

$$r_n(x) = \int_c^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt. \tag{5.2.3}$$

That is, if P_n is the nth order Taylor polynomial for f at some point c in (a, b) and x is any point in (a, b), then

$$f(x) = P_n(x) + r_n(x), \tag{5.2.4}$$

where r_n is given by (5.2.3).

We will show that Taylor's theorem follows from the Fundamental Theorem of Integral Calculus combined with repeated applications of integration by parts. Let f be a function satisfying the conditions of the theorem. Since f is an antiderivative of f', by the Fundamental Theorem of Integral Calculus we have

$$f(x) - f(c) = \int_c^x f'(t)dt. \tag{5.2.5}$$

Hence

$$f(x) = f(c) + \int_c^x f'(t)dt, \tag{5.2.6}$$

which is the statement of Taylor's theorem when $n = 0$. For $n = 1$, we perform an integration by parts on the integral in (5.2.6) using

$$\begin{array}{ll} u = f'(t) & dv = dt \\ du = f''(t) & v = -(x-t). \end{array}$$

Note that this is not the most obvious choice for v (certainly, $v = t$ would be a simpler choice), but it is a valid choice and one that leads to the result we desire. Namely, this gives us

$$\begin{aligned} f(x) &= f(c) - f'(t)(x-t)\Big|_c^x + \int_c^x (x-t)f''(t)dt \\ &= f(c) + f'(c)(x-c) + \int_c^x (x-t)f''(t)dt, \end{aligned}$$

which is the statement of Taylor's theorem for the case $n = 1$. For $n = 2$, we perform another integration by parts using

$$\begin{aligned} u &= f''(t) & dv &= (x-t)dt \\ du &= f'''(t) & v &= -\frac{(x-t)^2}{2}, \end{aligned}$$

from which we obtain

$$\begin{aligned} f(x) &= f(c) + f'(c)(x-c) - \frac{f''(t)(x-t)^2}{2}\Big|_c^x + \int_c^x \frac{(x-t)^2}{2} f'''(t)dt \\ &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \int_c^x \frac{(x-t)^2}{2} f'''(t)dt. \end{aligned}$$

Similarly, we obtain Taylor's theorem for $n = 3$ by another integration by parts. This time we have

$$\begin{aligned} u &= f'''(t) & dv &= \frac{(x-t)^2}{2}\,dt \\ du &= f^{(4)}(t) & v &= -\frac{(x-t)^3}{3!}, \end{aligned}$$

which yields

$$\begin{aligned} f(x) &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 - \frac{f'''(t)(x-t)^3}{3!}\Big|_c^x + \int_c^x \frac{(x-t)^3}{3!} f^{(4)}(t)dt \\ &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \int_c^x \frac{(x-t)^3}{3!} f^{(4)}(t)dt. \end{aligned}$$

From this we can see that, for any nonnegative integer n, performing integration by parts n times will yield

$$\begin{aligned} f(x) &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots \\ &\quad + \frac{f^{(n)}(c)}{n!}(x-c)^n + \int_c^x \frac{(x-t)^n}{n!} f^{(n)}(t)dt, \end{aligned} \tag{5.2.7}$$

which is the general statement of Taylor's theorem.

In applying Taylor's theorem, it is seldom the case that the remainder term $r_n(x)$ can be evaluated exactly. In most cases, we try to find an upper bound for $|r_n(x)|$ so that we know what is the worst possible error that we could commit in approximating $f(x)$ by $P_n(x)$. For these purposes, there is an alternative formulation of the remainder term which is often more useful than the one given in Taylor's theorem.

Lagrange's form of the remainder term Using the same notation as in the statement of Taylor's theorem, there exists a number k between c and x such that

$$r_n(x) = \frac{f^{(n+1)}(k)(x-c)^{n+1}}{(n+1)!}. \tag{5.2.8}$$

To show this, we will assume $x > c$, the argument in the case $x < c$ being similar. So let u be the point where $f^{(n+1)}$ attains its maximum value on $[c, x]$ and let v be the point where $f^{(n+1)}$ attains its minimum value on $[c, x]$. Note that we know such points exist because we have assumed $f^{(n+1)}$ to be a continuous function on (a, b), and hence on $[c, x]$. Then we have

$$\frac{(x-t)^n}{n!} f^{(n+1)}(v) \leq \frac{(x-t)^n}{n!} f^{(n+1)}(t) \leq \frac{(x-t)^n}{n!} f^{(n+1)}(u) \tag{5.2.9}$$

for all t in $[c, x]$. Integrating each of the terms in (5.2.9) from c to x, we have

$$f^{(n+1)}(v) \int_c^x \frac{(x-t)^n}{n!}\, dt \leq \int_c^x \frac{(x-t)^n}{n!} f^{(n+1)}(t) dt \leq f^{(n+1)}(u) \int_c^x \frac{(x-t)^n}{n!}\, dt. \tag{5.2.10}$$

Now

$$\int_c^x \frac{(x-t)^n}{n!}\, dt = \left. \frac{(x-t)^{n+1}}{(n+1)!} \right|_c^x = \frac{(x-c)^{n+1}}{(n+1)!} \tag{5.2.11}$$

and

$$\int_c^x \frac{(x-t)^n}{n!} f^{(n+1)}(t) dt = r_n(x), \tag{5.2.12}$$

so (5.2.10) implies that

$$\frac{f^{(n+1)}(v)(x-c)^{n+1}}{(n+1)!} \leq r_n(x) \leq \frac{f^{(n+1)}(u)(x-c)^{n+1}}{(n+1)!}. \tag{5.2.13}$$

Finally, since

$$g(t) = \frac{f^{(n+1)}(t)(x-c)^{n+1}}{(n+1)!} \tag{5.2.14}$$

is a continuous function of t on the interval $[c, x]$, it follows from (5.2.13) and the Intermediate Value Theorem that there exists a number k in $[c, x]$ such that $g(k) = r_n(x)$, that is,

$$r_n(x) = \frac{f^{(n+1)}(k)(x-c)^{n+1}}{(n+1)!}, \tag{5.2.15}$$

which is (5.2.8).

Of course, we cannot calculate $r_n(x)$ exactly without knowing the value of k. However, if we can find a number M such that

$$|f^{(n+1)}(t)| \leq M \tag{5.2.16}$$

for all t between c and x, then (5.2.8) implies that

$$|r_n(x)| \leq \frac{M}{(n+1)!}|x-c|^{n+1}. \tag{5.2.17}$$

Hence, although usually we cannot hope to know the exact amount of error in our approximation, in this case we can at least find an upper bound for the size of the error.

We can now show that $r_n(x)$ is $O((x-c)^{n+1})$, or, equivalently, that

$$R_n(h) = f(c+h) - P_n(c+h) \tag{5.2.18}$$

is $O(h^{n+1})$. First choose $\epsilon > 0$ so that the interval $I = [c-\epsilon, c+\epsilon]$ is contained in the interval (a,b), and let M be the maximum value of $|f^{(n+1)}|$ on I. Then, using (5.2.17), we have

$$|r_n(x)| \leq \frac{M}{(n+1)!}|x-c|^{n+1} \tag{5.2.19}$$

for all x in I. Thus if $|h| \leq \epsilon$,

$$|R_n(h)| \leq \frac{M}{(n+1)!}|h|^{n+1}, \tag{5.2.20}$$

from which it follows that

$$\left|\frac{R_n(h)}{h^{n+1}}\right| \leq \frac{M}{(n+1)!}. \tag{5.2.21}$$

That is, $R_n(h)$ is $O(h^{n+1})$.

Proposition If f satisfies the conditions of Taylor's theorem and P_n is the nth order Taylor polynomial for f at c, then

$$R_n(h) = f(c+h) - P_n(h) \tag{5.2.22}$$

is $O(h^{n+1})$.

Of course, from our previous work we know that this statement implies that $R_n(h)$ is also $o(h^n)$.

With this proposition we may write

$$f(c+h) = f(c) + f'(c)h + \frac{f''(c)}{2!}h^2 + \cdots + \frac{f^{(n)}(c)}{n!}h^n + O(h^{n+1}), \tag{5.2.23}$$

or, in terms of $x = c + h$,

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + O((x-c)^{n+1}), \tag{5.2.24}$$

as long as f is $n+1$ times continuously differentiable on an open interval containing c.

Example Recall that the fifth order Taylor polynomial for $\sin(x)$ at 0 is

$$P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}.$$

Since in this case $P_5 = P_6$, we now know that

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + O(x^7).$$

More explicitly, since

$$\frac{d^7}{dx^7}\sin(x) = -\cos(x),$$

we have

$$\left|\frac{d^7}{dx^7}\sin(x)\right| \leq 1$$

for any value of x. Hence if

$$r_5(x) = \sin(x) - P_5(x),$$

then by (5.2.17) it follows that

$$|r_5(x)| \leq \frac{|x|^7}{7!}$$

for any value of x. For example,

$$|\sin(1) - P_5(1)| \leq \frac{1}{7!} = \frac{1}{5040} = 0.000198,$$

to 6 decimal places. That is, the error in approximating $\sin(1)$ by $P_5(1)$ is no more than 0.000198. In this case the error bound is very close to the actual error, for, to six decimal place accuracy,

$$\sin(1) = 0.841471$$

and

$$P_5(1) = 1 - \frac{1}{6} + \frac{1}{120} = 0.841667,$$

which gives an error of

$$\sin(1) - P_5(1)| = 0.000196.$$

Note that, in general, for any nonnegative integer n,

$$\left|\frac{d^{2n+3}}{dx^{2n+3}}\sin(x)\right| = |\cos(x)|.$$

Thus

$$\left|\frac{d^{2n+3}}{dx^{2n+3}}\sin(x)\right| \leq 1$$

for any value of x. Hence, using the fact that in this case $P_{2n+1} = P_{2n+2}$,

$$|\sin(x) - P_{2n+1}(x)| \leq \frac{|x|^{2n+3}}{(2n+3)!} \tag{5.2.25}$$

for any value of x. With this inequality we can determine the order necessary for a Taylor polynomial to give some desired level of accuracy for a particular approximation. For example, if we wish to estimate $\sin(1.4)$ with an error of less than 0.001 using a Taylor polynomial about 0, then (5.2.25) says we need only find a nonnegative integer n such that

$$\frac{1.4^{2n+3}}{(2n+3)!} < 0.001,$$

in which case $P_{2n+1}(1.4)$ will provide the desired approximation. For $n = 0, 1, 2,$ and 3, we have the following table:

n	$\dfrac{1.4^{2n+3}}{(2n+3)!}$
0	0.4573333
1	0.0448187
2	0.0020915
3	0.0000569

Hence the smallest value of n that will work is $n = 3$; thus to attain the desired level of accuracy we would use

$$P_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}.$$

Checking this to 7 decimal places, we find that

$$P_7(1.4) = 0.9853938$$

and

$$\sin(1.4) = 0.9854497$$

an error of only 0.0000559.

Example Combining our work from Section 5.1 about the Taylor polynomial of order 4 for $f(x) = \sqrt{x}$ at 1 with our new results, we now have

$$\sqrt{x} = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \frac{5}{128}(x-1)^4 + O((x-1)^5).$$

If, for example, we wanted to bound the error involved in using $P_4(1.6)$ as an estimate for $\sqrt{1.6}$, we would first note that

$$f^{(5)}(x) = \frac{105}{32x^{\frac{9}{2}}},$$

which is a decreasing function and hence is maximized on the interval $[1, 1.6]$ at $x = 1$. Thus

$$|f^{(5)}(x)| \leq \frac{105}{32}$$

for all x in $[1, 1.6]$. From (5.2.16) it follows that

$$|\sqrt{1.6} - P_4(1.6)| \leq \frac{\frac{105}{32}}{5!}|1.6 - 1|^5 = 0.0021263,$$

to 7 decimal places. Checking this on a calculator to 7 decimal places, we have

$$\sqrt{1.6} = 1.2649111$$

and

$$P_4(1.6) = 1.2634375,$$

showing that the error in this case is 0.0014736.

If f is indefinitely differentiable on an interval about a point c and P_n is the Taylor polynomial of order n for f at c, then it is frequently the case that $P_1, P_2, P_3, \ldots$ is a sequence of increasingly accurate approximating polynomials for f on some interval I containing the point c. Of course, unless f is itself a polynomial, there is no polynomial P_n in this sequence such that $f(x) = P_n(x)$ for all x in I. Nevertheless, there are many functions for which

$$f(x) = \lim_{n\to\infty} P_n(x) \tag{5.2.26}$$

for all x in some interval I. Such functions are said to be *analytic*. Since a polynomial is just the sum of a finite number of monomials, $\lim\limits_{n\to\infty} P_n(x)$ may be regarded as an infinite sum of monomials, an infinite polynomial. That is, if f is an analytic function, then

$$\begin{aligned} f(x) &= \lim_{n\to\infty} P_n(x) \\ &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n. \end{aligned} \tag{5.2.27}$$

for all x in some interval I containing c. An infinite series of this type is called a *power series*. Since power series have many of the nice properties of polynomials, such as being easy to integrate, a representation of a function f by a power series in this manner can be extremely useful. Although we considered infinite series in Section 1.3, we will need a far more thorough discussion of them before we will be able fully to understand a series like (5.2.27). We will do this in Sections 5.3 through 5.6.

Problems

1. For each of the following functions f, find the Taylor polynomial of order 5 at the point c, use it to approximate $f(a)$, and find an upper bound for the absolute value of the error in the approximation.

   (a) $f(x) = \sin(x)$, $c = 0$, $a = 0.8$
   (b) $f(x) = \cos(x)$, $c = 0$, $a = 1.2$
   (c) $f(x) = \sin(2x)$, $c = 0$, $a = -0.5$
   (d) $f(x) = \sqrt{x}$, $c = 1$, $a = 1.5$
   (e) $f(x) = \sqrt{x}$, $c = 9$, $a = 10$
   (f) $f(x) = x^{\frac{3}{2}}$, $c = 1$, $a = 1.4$
   (g) $f(x) = \dfrac{1}{x}$, $c = 1$, $a = 0.8$
   (h) $f(x) = \sin(x)$, $c = 0$, $a = -1.2$

2. Use a Taylor polynomial to approximate $\sin(0.6)$ with an error of less than 0.0001.

3. Use a Taylor polynomial to approximate $\sin(-1.3)$ with an error of less than 0.00001.

4. Use a Taylor polynomial to approximate $\cos(1.2)$ with an error of less than 0.0001.

5. Find the Taylor polynomial of smallest order that will approximate $\sin(x)$ with an error of less than 0.0005 for all x in $[-2, 2]$.

6. Suppose L is a function defined on $(0, \infty)$ with $L(1) = 0$ and
$$L'(x) = \frac{1}{x}.$$
   (a) Find P_{10}, the Taylor polynomial of order 10 for L at 1.
   (b) Use P_{10} to approximate $L(1.5)$. Find an upper bound for the absolute value of the error of this approximation.
   (c) Find the Taylor polynomial of smallest degree that will approximate $L(x)$ with an error less than 0.0005 for all x in $[1, 1.5]$.

7. Suppose E is a function defined on $(-\infty, \infty)$ with $E(0) = 1$ and $E'(x) = E(x)$ for all x.
   (a) Find P_{10}, the Taylor polynomial of order 10 for E at 0.
   (b) Use P_{10} to approximate $E(1)$.
   (c) Given that $|E(x)| < 3^x$ for all $x > 0$, find an upper bound for the absolute value of the error in the approximation in part (b).
   (d) Find the Taylor polynomial of smallest degree that will approximate $E(x)$ with an error less than 0.0001 for all x in $[0, 2]$.

8. Let P_9 be the 9th order Taylor polynomial for $f(x) = \sin(x)$ at 0. Use P_9 to approximate
$$\int_0^3 \frac{\sin(x)}{x}\, dx.$$

9. (a) Find the 6th order Taylor polynomial for $f(x) = \sin(x^2)$. How is it related to the 3rd order Taylor polynomial for $g(x) = \sin(x)$?

   (b) Find the 7th order Taylor polynomial for

$$h(x) = \int_0^x \sin(t^2)dt.$$

   How is it related to your answer in (a)?

Difference Equations to **D**ifferential Equations

Section 5.3

Infinite Series Revisited

Recall from Section 1.3 that for a given sequence $\{a_n\}$, the sequence $\{s_n\}$ with nth term

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n \tag{5.3.1}$$

is called an *infinite series*. An individual term s_n is called a *partial sum* and we say the series is *convergent*, or has a *sum*, if $\lim_{n\to\infty} s_n$ exists. If the series is not convergent, we say it is *divergent*. We write

$$\lim_{n\to\infty} s_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n. \tag{5.3.2}$$

Example In Section 1.3 we saw that if $a_n = r^n$, $n = 0, 1, 2, \ldots$, then the associated infinite series, called a geometric series, is convergent if and only if $-1 < r < 1$, in which case

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}. \tag{5.3.3}$$

For example,

$$\sum_{n=0}^{\infty} \frac{1}{3^n} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}.$$

Geometric series comprise one of the few classes of series for which we can evaluate sums exactly. For most series we can only approximate the sum by computing the partial sums s_n for sufficiently large values of n. However, before this procedure becomes meaningful, we must first know that the series converges. Hence, in this section, as well as in Sections 5.4, 5.5, and 5.6, one of our primary goals will be the development of methods for determining whether a given series converges or diverges.

We begin by considering several basic properties of infinite series. First, suppose we know that both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series with

$$\sum_{n=1}^{\infty} a_n = L$$

and

$$\sum_{n=1}^{\infty} b_n = M.$$

If s_n is the nth partial sum of $\sum_{n=1}^{\infty} a_n$, t_n is the nth partial sum of $\sum_{n=1}^{\infty} b_n$, and u_n is the nth partial sum of $\sum_{n=1}^{\infty}(a_n + b_n)$, then $u_n = s_n + t_n$. Thus

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}(s_n + t_n) = \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n = L + M. \tag{5.3.4}$$

That is,

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n. \tag{5.3.5}$$

Proposition If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge, then $\sum_{n=1}^{\infty}(a_n + b_n)$ converges and

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n. \tag{5.3.6}$$

Similarly, $\sum_{n=1}^{\infty}(a_n - b_n)$ converges and

$$\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n. \tag{5.3.7}$$

Example From our results above, it follows that

$$\sum_{n=0}^{\infty}\left(\frac{1}{3^n} + \frac{1}{5^n}\right) = \sum_{n=0}^{\infty}\frac{1}{3^n} + \sum_{n=0}^{\infty}\frac{1}{5^n} = \frac{1}{1-\frac{1}{3}} + \frac{1}{1-\frac{1}{5}} = \frac{3}{2} + \frac{5}{4} = \frac{11}{4}.$$

Now suppose $\sum_{n=1}^{\infty} a_n$ is a convergent series,

$$\sum_{n=1}^{\infty} a_n = L,$$

and k is any constant. If s_n is the nth partial sum of $\sum_{n=1}^{\infty} a_n$ and t_n is the nth partial sum of the series $\sum_{n=1}^{\infty} ka_n$, then $t_n = ks_n$. Hence

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} ks_n = k\lim_{n\to\infty} s_n = kL. \tag{5.3.8}$$

That is,

$$\sum_{n=1}^{\infty} ka_n = k\sum_{n=1}^{\infty} a_n. \tag{5.3.9}$$

Proposition If $\sum_{n=1}^{\infty} a_n$ converges and k is any constant, then $\sum_{n=1}^{\infty} a_n$ converges and

$$\sum_{n=1}^{\infty} ka_n = k\sum_{n=1}^{\infty} a_n. \tag{5.3.10}$$

Example We have

$$\sum_{n=1}^{\infty} \frac{10}{2^n} = \frac{10}{2} \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 5 \sum_{n=0}^{\infty} \frac{1}{2^n} = 5 \left(\frac{1}{1 - \frac{1}{2}} \right) = 10.$$

Notice that if $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} ka_n$ must also diverge for any constant $k \neq 0$. This follows because, if, on the contrary, $\sum_{n=1}^{\infty} ka_n$ converged, then, by the previous proposition, so would $\sum_{n=1}^{\infty} a_n$ since

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{k}(ka_n). \tag{5.3.11}$$

Proposition If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} ka_n$ diverges for any $k \neq 0$.

Example In Section 1.3 we saw that the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges. It follows that both

$$\sum_{n=1}^{\infty} \frac{1}{3n} = \sum_{n=1}^{\infty} \frac{1}{3} \left(\frac{1}{n} \right)$$

and

$$\sum_{n=1}^{\infty} \frac{9}{20n} = \sum_{n=1}^{\infty} \frac{9}{20} \left(\frac{1}{n} \right)$$

are divergent series.

It is also important to note that since

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{m-1} a_n + \sum_{n=m}^{\infty} a_n \tag{5.3.12}$$

for any positive integer m, the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{n=m}^{\infty} a_n$ converges. In other words, convergence or divergence of a series is never determined by the behavior of any finite number of terms.

Example It follows from the previous example that

$$\sum_{n=200}^{\infty} \frac{9}{20n}$$

diverges.

Example The series

$$\sum_{n=4}^{\infty} \frac{3}{5^n}$$

converges. Moreover,

$$\sum_{n=4}^{\infty} \frac{3}{5^n} = \sum_{n=4}^{\infty} \frac{3}{5^4}\left(\frac{1}{5^{n-4}}\right) = \frac{3}{625}\sum_{n=0}^{\infty} \frac{1}{5^n} = \frac{3}{625}\left(\frac{1}{1-\frac{1}{5}}\right) = \frac{3}{500}.$$

Now suppose the series $\sum_{n=1}^{\infty} a_n$ converges with

$$\sum_{n=1}^{\infty} a_n = L.$$

Let $s_n = a_1 + a_2 + a_3 + \cdots + a_n$ be the nth partial sum of $\sum_{n=1}^{\infty} a_n$. Now

$$a_n = s_n - s_{n-1},$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i = \sum_{i=1}^{\infty} a_i = L,$$

and

$$\lim_{n\to\infty} s_{n-1} = \lim_{n\to\infty} \sum_{i=1}^{n-1} a_i = \sum_{i=1}^{\infty} a_i = L,$$

Hence

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = L - L = 0. \tag{5.3.13}$$

That is, the nth term of a convergent series must have a limit of 0.

Proposition If $\sum_{n=1}^{\infty} a_n$ converges, then

$$\lim_{n\to\infty} a_n = 0. \tag{5.3.14}$$

Note that this result only demonstrates a consequence of a series converging, and so does not provide a criterion to determine convergence. However, it may be useful in showing that certain series are divergent. Namely, if either the sequence $\{a_n\}$ does not have a limit or

$$\lim_{n\to\infty} a_n \neq 0,$$

then the series $\sum_{n=1}^{\infty} a_n$ must diverge. This result is often called the *nth term test for divergence.*

Example The series

$$\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$$

diverges since

$$\lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = \cos(0) = 1.$$

Example The series $\sum_{n=1}^{\infty}(-1)^n$ diverges since $\{(-1)^n\}$ does not have a limit.

Example Note that

$$\lim_{n\to} \frac{1}{n} = 0,$$

yet the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges.

p-series

In the next section we will consider a method for determining the convergence or divergence of a series by comparing a given series with a series which is already known to converge or diverge. In order to make significant use of such a result it is necessary to have a supply of series whose convergence or divergence is already known. So far geometric series are the only series we have studied in any detail. Now we will consider the class of series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \tag{5.3.15}$$

where p is a fixed constant. Such series are called *p-series*. The following proposition contains our main result.

Proposition The p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \tag{5.3.16}$$

converges for $p > 1$ and diverges for $p \leq 1$.

To demonstrate this result, we shall consider four cases. First, suppose $p \leq 0$. Then

$$\lim_{n\to\infty} \frac{1}{n^p} = \begin{cases} \infty, & \text{if } p < 0, \\ 1, & \text{if } p = 0. \end{cases} \tag{5.3.17}$$

Thus the series diverges by the nth term test for divergence.

Next, consider $0 < p < 1$. Note that for any $n > 0$, the partial sum

$$s_n = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p}, \tag{5.3.18}$$

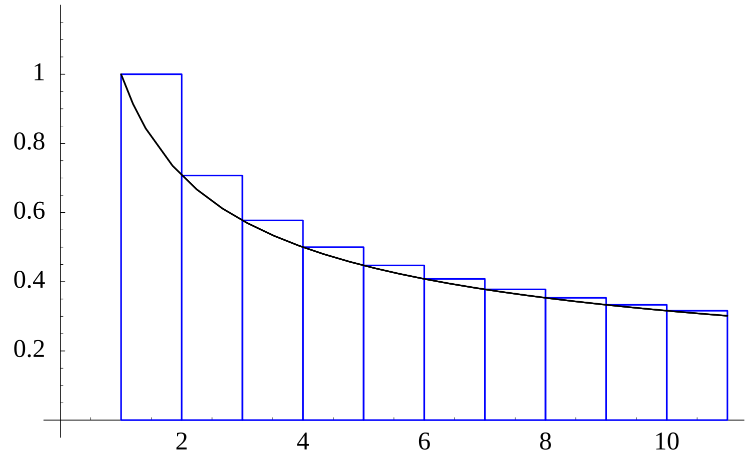


Figure 5.3.1 Rectangles for left-hand rule approximation for $\int_1^{11} \frac{1}{\sqrt{x}}\, dx$

is a left-hand rule approximation, using intervals of length 1, for the integral

$$\int_1^{n+1} \frac{1}{x^p}\, dx. \tag{5.3.19}$$

See Figure 5.3.1 for the case $p = \frac{1}{2}$ and $n = 10$. Since

$$f(x) = \frac{1}{x^p}$$

is a decreasing function on the interval $[1, n+1]$, s_n is an upper sum for the integral (5.3.19), and hence

$$s_n \geq \int_1^{n+1} \frac{1}{x^p}\, dx. \tag{5.3.20}$$

Now

$$\int_1^{n+1} \frac{1}{x^p}\, dx = \left.\frac{x^{1-p}}{1-p}\right|_1^{n+1} = \frac{(n+1)^{1-p} - 1}{1-p}. \tag{5.3.21}$$

Thus

$$s_n \geq \frac{(n+1)^{1-p} - 1}{1-p}. \tag{5.3.22}$$

But, since $1 - p > 0$,

$$\lim_{n\to\infty} \frac{(n+1)^{1-p} - 1}{1-p} = \infty. \tag{5.3.23}$$

Hence $\{s_n\}$ is an unbounded, increasing sequence, and so, from a result in Section 1.2,

$$\lim_{n\to\infty} s_n = \infty. \tag{5.3.24}$$

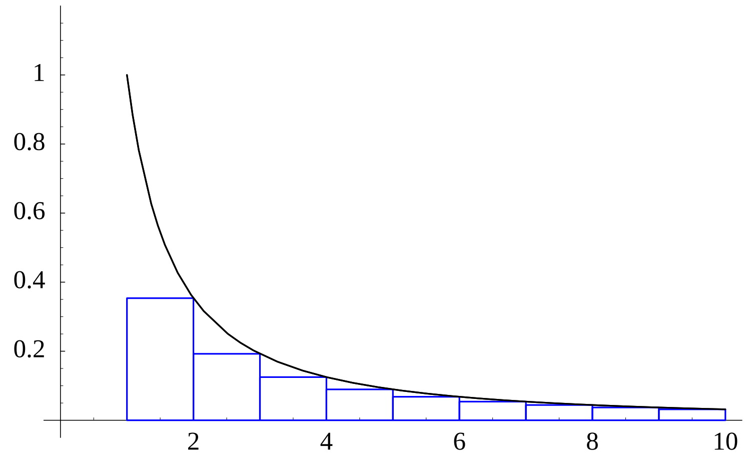


Figure 5.3.2 Rectangles for right-hand rule approximation for $\displaystyle\int_1^{10} \frac{1}{x^{1.5}}\, dx$

In other words,

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

is a divergent series.

For $p = 1$, the p-series is the harmonic series and so diverges.

Finally, consider $p > 1$. If for any integer $n > 1$ we let

$$t_n = \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots + \frac{1}{n^p}, \tag{5.3.25}$$

then t_n is a right-hand rule approximation, using intervals of length 1, for the integral

$$\int_1^n \frac{1}{x^p}\, dx. \tag{5.3.26}$$

See Figure 5.3.2 for the case $p = 1.5$ and $n = 10$. Since

$$f(x) = \frac{1}{x^p}$$

is a decreasing function on the interval $[1, n]$, t_n is a lower sum for the integral (5.3.26), and hence

$$t_n \leq \int_1^n \frac{1}{x^p}\, dx. \tag{5.3.27}$$

Now

$$\begin{aligned}\int_1^n \frac{1}{x^p}\,dx &\le \int_1^\infty \frac{1}{x^p}\,dx \\ &= \lim_{n\to\infty}\int_1^n \frac{1}{x^p}\,dx \\ &= \lim_{n\to\infty} \left.\frac{x^{1-p}}{1-p}\right|_1^n \\ &= \lim_{n\to\infty}\frac{n^{1-p}-1}{1-p} \\ &= \frac{1}{p-1},\end{aligned} \tag{5.3.28}$$

where the final equality follows from the fact that, since $p > 1$,

$$\lim_{n\to\infty} n^{1-p} = \lim_{n\to\infty}\frac{1}{n^{p-1}} = 0. \tag{5.3.29}$$

Thus

$$t_n \le \frac{1}{p-1}. \tag{5.3.30}$$

Now if s_n is the nth partial sum of

$$\sum_{n=1}^{\infty}\frac{1}{n^p},$$

then $s_1 = 1$ and $s_n = 1 + t_n$, $n = 2, 3, 4, \ldots$. Hence

$$s_n \le 1 + \frac{1}{p-1} = \frac{p}{p-1} \tag{5.3.31}$$

for $n = 1, 2, 3, \ldots$. Thus $\{s_n\}$ is a bounded, increasing sequence, and so, from a result in Section 1.2, must have a limit. That is, $\lim\limits_{n\to\infty} s_n$ exists and

$$\sum_{n=1}^{\infty}\frac{1}{n^p}$$

is a convergent series.

Example The series

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$$

diverges because it is a p-series with $p = \frac{1}{2}$. Moreover, it now follows from our earlier results that the series

$$\sum_{n=1}^{\infty}\frac{3}{2\sqrt{n}}$$

and

$$\sum_{n=10}^{\infty} \frac{1}{\sqrt{n}}$$

both diverge as well.

Example The series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges because it is a p-series with $p = 2$. Similar to our last example, it now follows from our earlier results that the series

$$\sum_{n=1}^{\infty} \frac{35}{6n^2}$$

and

$$\sum_{n=20}^{\infty} \frac{7}{5n^2}$$

both converge as well. Moreover, from (5.3.31), we know that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \leq \frac{2}{2-1} = 2.$$

In Problem 5 in Section 4.6, you were asked to show that the integral

$$\int_1^{\infty} \frac{1}{x^p}\, dx$$

diverges for $p < 1$ and converges for $p > 1$. In Section 6.2 we will see that

$$\int_1^{\infty} \frac{1}{x}\, dx$$

diverges as well (see also Problem 5 of this section). Combining these facts with our results about p-series, it follows that

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if

$$\int_1^{\infty} \frac{1}{x^p}\, dx$$

converges. This should not be surprising considering the intimate relationship we have seen between the partial sums of the series and the left-hand and right-hand rule approximations for the integral. The essential ingredient in making these connections was that the function

$$f(x) = \frac{1}{x^p}$$

is continuous, positive, and decreasing on the interval $[1, \infty)$ when $p > 0$. In fact, it can be shown, using arguments similar to those given above, that if g is a continuous, decreasing function on $[1, \infty)$ with $g(x) > 0$ for all $x \geq 1$, then

$$\sum_{n=1}^{\infty} g(n)$$

converges if and only if

$$\int_{1}^{\infty} g(x)dx$$

converges. You are asked to verify this result, known as the *integral test*, in Problem 4.

Problems

1. For each of the following infinite series, decide whether the series converges or diverges and explain your answer. If the series is a convergent geometric series, find its sum.

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{3}{5^n}$

   (b) $\displaystyle\sum_{n=0}^{\infty} \left(4 - \frac{1}{2^n}\right)$

   (c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{5}{2^n} + \frac{2}{3^n}\right)$

   (d) $\displaystyle\sum_{n=1}^{\infty} (-3)^n$

   (e) $\displaystyle\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)$

   (f) $\displaystyle\sum_{n=1}^{\infty} n \sin\left(\frac{1}{n}\right)$

   (g) $\displaystyle\sum_{n=10}^{\infty} \frac{(-1)^n}{1000}$

   (h) $\displaystyle\sum_{n=2}^{\infty} \left(-\frac{3}{7}\right)^n$

2. For each of the following infinite series, decide whether the series converges or diverges and explain your answer.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{4}{n^3}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^{15}}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{5}{n^4}\right)$

   (d) $\displaystyle\sum_{n=21}^{\infty} \frac{3}{\sqrt{n}}$

   (e) $\displaystyle\sum_{n=1}^{\infty} n^{-\frac{1}{3}}$

   (f) $\displaystyle\sum_{n=1}^{\infty} \sqrt{\frac{1}{n^3}}$

   (g) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{234\sqrt{n}}$

   (h) $\displaystyle\sum_{n=5}^{\infty} \frac{210 - \sqrt{n}}{n^2}$

3. Give an example of divergent series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ for which the series
$$\sum_{n=1}^{\infty}(a_n + b_n)$$
converges.

4. Prove the integral test. That is, show that if g is a continuous decreasing function on $[1, \infty)$ with $g(x) > 0$ for all $x \geq 1$, then
$$\sum_{n=1}^{\infty} g(n)$$
converges if and only if
$$\int_1^{\infty} g(x)dx$$
converges.

5. Use the integral test to determine the convergence or divergence of each of the following.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+3}$ (b) $\displaystyle\sum_{n=2}^{\infty} \frac{5}{\sqrt{n-1}}$

   (c) $\displaystyle\int_1^{\infty} \frac{1}{x}\, dx$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{3n}{\sqrt{n^2-1}}$

6. Find three different examples of divergent series $\sum_{n=1}^{\infty} a_n$ with the property that $\lim_{n\to\infty} a_n = 0$.

7. The following argument has been used to show that
$$\sum_{n=0}^{\infty}(-1)^n = \frac{1}{2}:$$
Let
$$L = \sum_{n=0}^{\infty}(-1)^n.$$
Then
$$L = \sum_{n=0}^{\infty}(-1)^n = 1 + \sum_{n=1}^{\infty}(-1)^n = 1 - \sum_{n=0}^{\infty}(-1)^n = 1 - L.$$
Thus $L = 1 - L$, and so $L = \frac{1}{2}$. Where is the fallacy in this argument?

Difference Equations to Differential Equations

Section 5.4

Infinite Series: The Comparison Test

In this section we continue our discussion of the convergence properties of infinite series. Now that we have two classes of series, namely, the geometric series and the p-series, for which classification as either convergent or divergent is relatively easy, it is reasonable to develop tests for convergence based on comparing a given series with a series of known behavior. We will see this idea first in the *comparison test*, which we will later generalize with the *limit comparison test*.

To begin, suppose $\sum_{n=1}^{\infty} a_n$ is a convergent series with $a_n \geq 0$ for all n and $\sum_{n=1}^{\infty} b_n$ is a series with $0 \leq b_n \leq a_n$ for all n. Let

$$L = \sum_{n=1}^{\infty} a_n.$$

If s_n is the nth partial sum of $\sum_{n=1}^{\infty} a_n$ and t_n is the nth partial sum of $\sum_{n=1}^{\infty} b_n$, then

$$t_n \leq s_n \tag{5.4.1}$$

for $n = 1, 2, 3, \ldots$. Since $a_n \geq 0$ for all n, the sequence $\{s_n\}$ is increasing; hence

$$s_n \leq L \tag{5.4.2}$$

for all n. Since $b_n \geq 0$, $n = 1, 2, 3, \ldots$, $\{t_n\}$ is also an increasing sequence which, by (5.4.1) and (5.4.2), is bounded above by L. Hence $\lim\limits_{n\to\infty} t_n$ exists, showing that $\sum_{n=1}^{\infty} b_n$ converges. Moreover, from (5.4.1),

$$\sum_{n=1}^{\infty} b_n \leq \sum_{n=1}^{\infty} a_n. \tag{5.4.3}$$

Now suppose that $\sum_{n=1}^{\infty} a_n$ is a divergent series with $a_n \geq 0$ for all n and $\sum_{n=1}^{\infty} b_n$ is a series with $a_n \leq b_n$ for all n. If s_n is the nth partial sum of $\sum_{n=1}^{\infty} a_n$ and t_n is the nth partial sum of $\sum_{n=1}^{\infty} b_n$, then

$$t_n \geq s_n \tag{5.4.4}$$

for $n = 1, 2, 3, \ldots$. Since $a_n \geq 0$ for all n, $\{s_n\}$ is an increasing sequence; thus, since $\sum_{n=1}^{\infty} a_n$ diverges, it follows that

$$\lim_{n\to\infty} s_n = \infty.$$

Hence, from (5.4.4), and the fact that $\{t_n\}$ is also an increasing sequence, we have

$$\lim_{n\to\infty} t_n = \infty,$$

that is, $\sum_{n=1}^{\infty} b_n$ diverges.

The preceding results are summarized in the comparison test.

Comparison Test Suppose $a_n \geq 0$ for $n = 1, 2, 3, \ldots$ and $\sum_{n=1}^{\infty} a_n$ converges. If $0 \leq b_n \leq a_n$ for $n = 1, 2, 3, \ldots$, then $\sum_{n=1}^{\infty} b_n$ converges and

$$\sum_{n=1}^{\infty} b_n \leq \sum_{n=1}^{\infty} a_n. \tag{5.4.5}$$

Suppose $a_n \geq 0$ for $n = 1, 2, 3, \ldots$ and $\sum_{n=1}^{\infty} a_n$ diverges. If $a_n \leq b_n$ for $n = 1, 2, 3, \ldots$, then $\sum_{n=1}^{\infty} b_n$ diverges.

In other words, if the terms of the series $\sum_{n=1}^{\infty} b_n$ are nonnegative and smaller than the terms of a series which converges, then $\sum_{n=1}^{\infty} b_n$ must converge; if the terms of the series $\sum_{n=1}^{\infty} b_n$ are larger than those of a divergent series with nonnegative terms, then $\sum_{n=1}^{\infty} b_n$ must diverge.

Example The series

$$\sum_{m=1}^{\infty} \frac{1}{n^2+1}$$

converges since

$$0 < \frac{1}{n^2+1} < \frac{1}{n^2}$$

for $n = 1, 2, 3, \ldots$, and

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is a convergent series (namely, a p-series with $p = 2$).

Example The series

$$\sum_{n=1}^{\infty} \frac{1}{2n-1}$$

diverges since

$$\frac{1}{2n-1} > \frac{1}{2n} > 0$$

for $n = 1, 2, 3, \ldots$, and

$$\sum_{n=1}^{\infty} \frac{1}{2n}$$

is a divergent series (since it is a multiple of the harmonic series).

Example The series

$$\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^2}$$

converges since

$$0 \leq \frac{\sin^2(n)}{n^2} \leq \frac{1}{n^2}$$

for $n = 1, 2, 3, \ldots$, and

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is, as in a previous example, a convergent series.

Example The series

$$\sum_{n=1}^{\infty} \frac{n+1}{n^2}$$

diverges since

$$\frac{n+1}{n^2} = \left(\frac{n+1}{n}\right)\frac{1}{n} > \frac{1}{n} > 0$$

for $n = 1, 2, 3, \ldots$, and

$$\sum_{n=1}^{\infty} \frac{1}{n},$$

the harmonic series, is a divergent series.

Example The series

$$\sum_{n=1}^{\infty} \frac{1}{n3^n}$$

converges since

$$0 < \frac{1}{n3^n} \leq \frac{1}{3^n}$$

for $n = 1, 2, 3, \ldots$, and

$$\sum_{n=1}^{\infty} \frac{1}{3^n}$$

is a convergent series (namely, a geometric series with ratio $\frac{1}{3}$).

We know that if a series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim\limits_{n\to\infty} a_n = 0$; however, we have seen numerous examples, such as the harmonic series, which show the latter condition is not sufficient to guarantee convergence. To ensure that a series with nth term $a_n \geq 0$ satisfying $\lim\limits_{n\to\infty} a_n = 0$ converges, we need additional information about the rate at which the terms are approaching 0; namely, we need to know that a_n approaches 0 fast enough to guarantee that the sequence of partial sums, although increasing, is nevertheless bounded.

The problem lies in determining how to measure rates of convergence to 0 and how to decide what rates are sufficient for convergence. We have already seen that the comparison test, by comparing a_n with the terms of a series of known behavior, provides one way to measure whether or not a_n is approaching 0 fast enough for the sum to converge. However, finding a series to use for the comparison can be somewhat tricky, even when the behavior of the series is relatively obvious. For example, the series

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$$

should converge, since for large values of n there is very little difference between

$$\frac{1}{n^2 - 1}$$

and

$$\frac{1}{n^2}.$$

But a direct comparison will not work, since

$$\frac{1}{n^2 - 1} > \frac{1}{n^2}$$

for $n = 2, 3, 4, \ldots$. Hence it would be useful to have a method for describing the rate at which a sequence $\{a_n\}$ converges to 0 and a test which exploits this description. We will supply the former with the next definition, a version of the "O" and "o" notation adapted for sequences, and the latter with the limit comparison test.

Definition If $\{a_n\}$ and $\{b_n\}$ are sequences with

$$\lim_{n \to \infty} \frac{b_n}{a_n} = 0, \tag{5.4.6}$$

then we say b_n is $o(a_n)$. If $\{a_n\}$ and $\{b_n\}$ are sequences and there exists an integer N and a constant M such that

$$\left| \frac{b_n}{a_n} \right| \leq M \tag{5.4.7}$$

for all $n > N$, then we say b_n is $O(a_n)$.

Similar to our earlier results, if

$$\lim_{n \to \infty} \frac{b_n}{a_n}$$

exists, then b_n is $O(a_n)$ (see Problem 5). Analogous to our earlier use of this notation, if

$$\lim_{n \to \infty} a_n = 0,$$

$$\lim_{n\to\infty} b_n = 0,$$

and b_n is $o(a_n)$, then b_n is approaching 0 faster than a_n as $n \to \infty$; if

$$\lim_{n\to\infty} a_n = 0,$$

$$\lim_{n\to\infty} b_n = 0,$$

and b_n is $O(a_n)$, then b_n is approaching 0 at least as fast as a_n as $n \to \infty$. Of course, if b_n is $o(a_n)$, then b_n is also $O(a_n)$.

Example $\frac{1}{3n^2+4}$ is $O\left(\frac{1}{n^2}\right)$ since

$$\lim_{n\to\infty} \frac{\dfrac{1}{3n^2+4}}{\dfrac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{3n^2+4} = \lim_{n\to\infty} \frac{1}{3+\dfrac{4}{n^2}} = \frac{1}{3}.$$

Example $\frac{1}{n^4+6}$ is $o\left(\frac{1}{n^3}\right)$ since

$$\lim_{n\to\infty} \frac{\dfrac{1}{n^4+6}}{\dfrac{1}{n^3}} = \lim_{n\to\infty} \frac{n^3}{n^4+6} = \lim_{n\to\infty} \frac{\dfrac{1}{n}}{1+\dfrac{6}{n^4}} = 0.$$

Now consider two series, $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, where $a_n > 0$ and $b_n \geq 0$ for all n, b_n is $O(a_n)$, and $\sum_{n=1}^{\infty} a_n$ converges. Then there is an integer N and a constant M such that

$$\frac{b_n}{a_n} \leq M \tag{5.4.8}$$

for all $n > N$. Hence

$$b_n \leq M a_n \tag{5.4.9}$$

for all $n > N$, so the series

$$\sum_{n=N+1}^{\infty} b_n$$

converges by comparison with the convergent series

$$\sum_{n=N+1}^{\infty} M a_n.$$

Thus $\sum_{n=1}^{\infty} b_n$ is also a convergent series.

Next consider two series, $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, where $a_n \geq 0$ and $b_n > 0$ for all n, a_n is $O(b_n)$, and $\sum_{n=1}^{\infty} a_n$ diverges. Then there exists an integer N and a constant $M > 0$ such that

$$\frac{a_n}{b_n} \leq M \tag{5.4.10}$$

for all $n > N$. Hence

$$b_n \geq \frac{a_n}{M} \tag{5.4.11}$$

for all $n > N$, so the series

$$\sum_{n=N+1}^{\infty} b_n$$

diverges by comparison with the divergent series

$$\sum_{n=N+1}^{\infty} \frac{a_n}{M}.$$

Thus $\sum_{n=1}^{\infty} b_n$ is also a divergent series.

The preceding results are summarized in the limit comparison test.

Limit Comparison Test Suppose $a_n > 0$ for $n = 1, 2, 3, \ldots$ and $\sum_{n=1}^{\infty} a_n$ converges. If $b_n \geq 0$ for $n = 1, 2, 3, \ldots$ and b_n is $O(a_n)$, then $\sum_{n=1}^{\infty} b_n$ also converges. Suppose $a_n \geq 0$ for $n = 1, 2, 3, \ldots$ and $\sum_{n=1}^{\infty} a_n$ diverges. If $b_n > 0$ for $n = 1, 2, 3, \ldots$ and a_n is $O(b_n)$, then $\sum_{n=1}^{\infty} b_n$ also diverges.

In other words, if the terms of the series $\sum_{n=1}^{\infty} b_n$ are nonnegative and approach 0 at least as fast as the terms of a convergent series with positive terms, then $\sum_{n=1}^{\infty} b_n$ converges; if the terms of a divergent series are nonnegative and approach 0 at least as fast as the terms of the series $\sum_{n=1}^{\infty} b_n$, which are positive, then $\sum_{n=1}^{\infty} b_n$ diverges.

Example Consider the series

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}.$$

As mentioned above, we would expect this series to behave very much like the convergent series

$$\sum_{n=2}^{\infty} \frac{1}{n^2}.$$

Now

$$\lim_{n \to \infty} \frac{\dfrac{1}{n^2 - 1}}{\dfrac{1}{n^2}} = \lim_{n \to \infty} \frac{n^2}{n^2 - 1} = \lim_{n \to \infty} \frac{1}{1 - \dfrac{1}{n^2}} = 1,$$

so $\frac{1}{n^2-1}$ is $O\left(\frac{1}{n^2}\right)$. Hence

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$$

converges by the limit comparison test.

Example Consider the series

$$\sum_{n=1}^{\infty} \frac{2n^2}{n^3+2}.$$

Since the nth term of this series is a rational function of n with a numerator of degree 2 and a denominator of degree 3, we might expect this series to behave much like the divergent series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

Now

$$\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{2n^2}{n^3+2}} = \lim_{n\to\infty} \frac{n^3+2}{2n^3} = \lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{n^3}\right) = \frac{1}{2},$$

so $\frac{1}{n}$ is $O\left(\frac{2n^2}{n^3+2}\right)$. Hence

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}.$$

diverges by the limit comparison test.

Problems

1. For each of the following infinite series, decide whether the series converges or diverges and explain your answer.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{2^n}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3+2}$

   (c) $\displaystyle\sum_{n=3}^{\infty} \frac{2}{\sqrt{n-2}}$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$

   (e) $\displaystyle\sum_{n=1}^{\infty} \pi^{-n}$ (f) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^4}$

   (g) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{3^n} - n^{-\frac{3}{2}}\right)$ (h) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+2}$

2. For each of the following infinite series, decide whether the series converges or diverges and explain your answer.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sin^2(n)}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+2n}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{3n-1}{2n^4+2}$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n2^n}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)3^{-n}}{n}$ (f) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{2n}$

(g) $\displaystyle\sum_{n=23}^{\infty} \frac{3n^5+1}{13n^7-2}$ (h) $\displaystyle\sum_{n=2}^{\infty} \frac{n+1}{n^2-2}$

3. (a) Give an example of a convergent series $\sum_{n=1}^{\infty} a_n$ and a divergent series $\sum_{n=1}^{\infty} b_n$ with the property that $b_n \le a_n$ for all n.
   (b) Give an example of a divergent series $\sum_{n=1}^{\infty} a_n$ and a convergent series $\sum_{n=1}^{\infty} b_n$ with the property that $a_n \le b_n$ for all n.
   (c) Comment on why the comparison test does not apply to the series in (a) and (b).
4. Explain why

$$\int_1^{\infty} \frac{1}{4x^5-2}\, dx$$

converges.

5. Show that if

$$\lim_{n\to\infty} \frac{b_n}{a_n}$$

exists, then b_n is $O(a_n)$.

Difference Equations to **D**ifferential Equations

Section 5.5

Infinite Series: The Ratio Test

In the last section we saw that we could demonstrate the convergence of a series $\sum_{n=1}^{\infty} a_n$, where $a_n \geq 0$ for all n, by showing that a_n approaches 0 as $n \to \infty$ as fast as the terms of another series with nonnegative terms which is already known to converge. Both of the techniques developed in Section 5.4, the comparison test and the limit comparison test, proved to be very useful; however, they both suffer from the drawback of requiring that we first find a series of known behavior which allows for the proper comparison with the series under consideration. In this section we shall consider another test for convergence, the *ratio test*, which determines whether or not the terms of a series are approaching 0 at a rate sufficient for the series to converge without reference to any other series. Although this test does not require knowledge of any other series, it has the limitation of being inconclusive in certain circumstances. Unfortunately, there is no single test for convergence which is useful under all conditions.

The ratio test determines if the terms of a given series are approaching 0 at a rate sufficient for convergence by considering the ratio between successive terms of the series. Specifically, suppose $a_n > 0$ for $n = 1, 2, 3, \ldots$ and

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \rho. \tag{5.5.1}$$

If $\rho < 1$, then there is an integer N and a number r with $\rho < r < 1$ such that

$$\frac{a_{n+1}}{a_n} < r \tag{5.5.2}$$

for all $n > N$. Then

$$a_{n+1} < r a_n \tag{5.5.3}$$

for all $n > N$, so

$$\begin{aligned} a_{N+2} &< r a_{N+1}, \\ a_{N+3} &< r a_{N+2} < r^2 a_{N+1}, \\ a_{N+4} &< r a_{N+3} < r^3 a_{N+1}, \end{aligned}$$

and, in general, for any integer $m > 2$,

$$a_{N+m} < r^{m-1} a_{N+1}. \tag{5.5.4}$$

That is,

$$\left| \frac{a_{N+m}}{r^{m-1}} \right| < a_{N+1} \tag{5.5.5}$$

for $m = 2, 3, 4, \ldots$. Letting $n = N + m$, in which case $m = n - N$, we have

$$\left|\frac{a_n}{r^{n-N-1}}\right| < a_{N+1} \tag{5.5.6}$$

for all $n > N + 1$. Thus a_n is $O(r^{n-N-1})$. Moreover,

$$\sum_{n=1}^{\infty} r^{n-N-1} = r^{-N} \sum_{n=1}^{\infty} r^{n-1} \tag{5.5.7}$$

converges since $\sum_{n=1}^{\infty} r^{n-1}$ is a geometric series and $0 < r < 1$. Thus $\sum_{n=1}^{\infty} a_n$ converges by the limit comparison test.

Now suppose $\rho > 1$, in which we include the possibility that $\rho = \infty$. Then there is an integer N such that

$$\frac{a_{n+1}}{a_n} > 1 \tag{5.5.8}$$

for all $n > N$. Hence $a_{n+1} > a_n$ for all $n > N$, and so

$$a_{N+1} < a_{N+2} < a_{N+3} < a_{N+4} < \cdots. \tag{5.5.9}$$

In particular, $a_n > a_{N+1}$ for $n = N + 2, N + 3, N + 4, \ldots$. It follows that either $\lim_{n\to\infty} a_n$ does not exist or

$$\lim_{n\to\infty} a_n > a_{N+1} > 0. \tag{5.5.10}$$

Thus $\sum_{n=1}^{\infty} a_n$ diverges by the nth term test for divergence.

We now summarize the above results.

Ratio Test Suppose $a_n > 0$ for $n = 1, 2, 3, \ldots$ and

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}. \tag{5.5.11}$$

Then $\sum_{n=1}^{\infty} a_n$ converges if $\rho < 1$ and diverges if $\rho > 1$.

The examples below will show that the ratio test is inconclusive if $\rho = 1$. Namely, the third example considers a divergent series for which $\rho = 1$ and the fourth example considers a convergent series for which $\rho = 1$. Hence some other test will be necessary to determine the behavior of any series for which the ratio test yields $\rho = 1$.

Example For the series

$$\sum_{n=1}^{\infty} \frac{n}{3^n},$$

if we let

$$a_n = \frac{n}{3^n},$$

$n = 1, 2, 3, \ldots$, then

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{n+1}{3^{n+1}}}{\dfrac{n}{3^n}} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)\left(\frac{3^n}{3^{n+1}}\right) = \frac{1}{3} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = \frac{1}{3}.$$

Thus $\rho < 1$ and the series converges.

Example For the series

$$\sum_{n=1}^{\infty} \frac{5^n}{n+2},$$

if we let

$$a_n = \frac{5^n}{n+2},$$

$n = 1, 2, 3, \ldots$, then

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{5^{n+1}}{n+3}}{\dfrac{5^n}{n+2}} = \lim_{n\to\infty} \left(\frac{n+2}{n+3}\right)\left(\frac{5^{n+1}}{5^n}\right) = 5 \lim_{n\to\infty} \frac{1 + \dfrac{2}{n}}{1 + \dfrac{3}{n}} = 5.$$

Thus $\rho > 1$ and the series diverges.

Example For the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n},$$

the ratio test yields

$$\rho = \lim_{n\to\infty} \frac{\dfrac{1}{n+1}}{\dfrac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1 + \dfrac{1}{n}} = 1.$$

This shows that it is possible for a series to diverge when $\rho = 1$.

Example For the convergent p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^2},$$

the ratio test yields

$$\rho = \lim_{n\to\infty} \frac{\dfrac{1}{(n+1)^2}}{\dfrac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = \lim_{n\to\infty} \left(\frac{1}{1 + \dfrac{1}{n}}\right)^2 = 1.$$

This shows that is possible for a series to converge when $\rho = 1$.

Example For the series

$$\sum_{n=1}^{\infty} \frac{3^n}{n!},$$

if we let

$$a_n = \frac{3^n}{n!},$$

$n = 1, 2, 3, \ldots$, then

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{3^{n+1}}{(n+1)!}}{\dfrac{3^n}{n!}} = \lim_{n\to\infty} \left(\frac{n!}{(n+1)!}\right)\left(\frac{3^{n+1}}{3^n}\right) = \lim_{n\to\infty} \frac{3}{n+1} = 0.$$

Thus $\rho < 1$ and the series converges.

Problems

1. For each of the following infinite series, decide whether the series converges or diverges and explain your answer.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{n!}$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{7^{n+2}}{(n+1)!}$

   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n!}$ (f) $\displaystyle\sum_{n=1}^{\infty} \frac{4}{n2^n}$

   (g) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$ (h) $\displaystyle\sum_{n=1}^{\infty} \frac{\pi^n}{5n+2}$

2. For each of the following infinite series, decide whether the series converges or diverges and explain your answer.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{4n-2}}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{5^n}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n+5}}{(2n)!}$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$

   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{n!n!}{(2n)!}$ (f) $\displaystyle\sum_{n=1}^{\infty} \frac{3n+2}{7^{2n}}$

   (g) $\displaystyle\sum_{n=1}^{\infty} \frac{3n+1}{2n-1}$ (h) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{5^n} - \frac{3^n}{5^n}\right)$

3. Define
$$f(x)=\sum_{n=1}^{\infty}\frac{x^{2n}}{n}.$$
   (a) Find the domain of f. That is, find all values of x for which the series
   $$\sum_{n=1}^{\infty}\frac{x^{2n}}{n}$$
   converges.

   (b) Plot an approximation to the graph of f on the domain found in (a) using
   $$f(x)\approx\sum_{n=1}^{100}\frac{x^{2n}}{n}.$$

4. Define
$$g(t)=\sum_{n=0}^{\infty}\frac{t^{2n}}{n!}.$$
   (a) Find the domain of g. That is, find all values of t for which the series
   $$\sum_{n=0}^{\infty}\frac{t^{2n}}{n!}$$
   converges.

   (b) Plot an approximation to the graph of g on the interval $[-2,2]$ using
   $$g(t)\approx\sum_{n=0}^{50}\frac{t^{2n}}{n!}.$$

5. Suppose the terms of the series $\sum_{n=1}^{\infty}a_n$ satisfy the difference equation
$$a_{n+1}=\frac{(n+1)a_n}{2n}$$
with $a_1=10$. Does this series converge or diverge? Explain.

6. Suppose, for $n=1,2,3,\ldots$, $a_n\geq 0$ and
$$\alpha=\lim_{n\to\infty}\sqrt[n]{a_n}.$$
Show that $\sum_{n=1}^{\infty}a_n$ converges if $\alpha<1$ and diverges if $\alpha>1$. This result is known as the *root test*.

Difference Equations to **D**ifferential Equations

Section 5.6

Infinite Series: Absolute Convergence

At this point we have limited our study of series primarily to those series having nonnegative terms, the only exceptions being some geometric series and series which are multiples of series with nonnegative terms. In this section we shall consider the more general question of series with negative as well as positive terms.

An important consideration when looking at the behavior of an arbitrary series

$$\sum_{n=1}^{\infty} a_n \tag{5.6.1}$$

is the behavior of the related series

$$\sum_{n=1}^{\infty} |a_n|. \tag{5.6.2}$$

Of course, if all the terms of (5.6.1) are nonnegative, then (5.6.1) and (5.6.2) are the same series. In any case, (5.6.2) has all nonnegative terms, so we may use our results of the last three sections to help determine whether or not it converges. Suppose that, by one method or another, we have shown that (5.6.2) converges. Then, since

$$0 \le a_n + |a_n| \le 2|a_n| \tag{5.6.3}$$

for any n, we know, by the comparison test, that the series

$$\sum_{n=1}^{\infty} (a_n + |a_n|) \tag{5.6.4}$$

converges. Hence

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n| \tag{5.6.5}$$

converges since it is the difference of two convergent series. That is, the convergence of (5.6.2) implies the convergence of (5.6.1).

Proposition If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Definition The series $\sum_{n=1}^{\infty} a_n$ is said to *converge absolutely* if the series $\sum_{n=1}^{\infty} |a_n|$ converges.

With this terminology, the previous proposition says that any series which converges absolutely also converges. We shall see later that the converse of this statement does not hold; namely, there are series which converge, but do not converge absolutely.

Example The series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25} + \cdots$$

converges absolutely since the series

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges. In particular, it follows that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

converges.

Example The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots,$$

known as the *alternating harmonic series*, is not absolutely convergent since

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

is the harmonic series, which diverges. Hence the previous proposition does not provide any information on the behavior of the alternating harmonic series itself. We shall see below that in fact the alternating harmonic series converges even though it is not absolutely convergent.

In general, determining whether a series which is not absolutely convergent is convergent or divergent is a difficult problem. However, there is one particular type of series for which we have, under certain conditions, a simple test. These series are the *alternating series*, the series which, like those in the previous examples, alternate in sign from one term to the next.

Definition A series in which the terms are alternately positive and negative is called an *alternating series.*

Now suppose $\sum_{n=1}^{\infty} a_n$ is an alternating series which satisfies the following two conditions:

(1) $|a_{n+1}| \leq |a_n|$ for $n = 1, 2, 3, \ldots,$

(2) $\lim\limits_{n\to\infty} |a_n| = 0.$

For the sake of the discussion we will assume that $a_1 > 0$, although that will not affect our conclusion. If s_n is the nth partial sum of this series, then

$$s_1 = a_1, \tag{5.6.6}$$

and, since $a_2 < 0$,

$$s_2 = a_1 + a_2 = s_1 + a_2 < s_1. \tag{5.6.7}$$

Next, since $a_3 > 0$,

$$s_3 = a_1 + a_2 + a_3 = s_2 + a_3 > s_2. \tag{5.6.8}$$

Moreover, condition (1) implies $a_2 + a_3 \leq 0$, from which it follows that

$$s_3 = a_1 + a_2 + a_3 = s_1 + a_2 + a_3 \leq s_1. \tag{5.6.9}$$

Thus we have $s_2 \leq s_3 \leq s_1$. Next,

$$s_4 = s_3 + a_4 < s_3 \tag{5.6.10}$$

since $a_4 < 0$ and

$$s_4 = s_2 + a_3 + s_4 \geq s_2 \tag{5.6.11}$$

since $a_3 + a_4 \geq 0$. Thus $s_2 \leq s_4 \leq s_3 \leq s_1$. For the next step,

$$s_5 = s_4 + a_5 > s_4 \tag{5.6.12}$$

since $a_5 > 0$ and

$$s_5 = s_3 + a_4 + a_5 \leq s_3 \tag{5.6.13}$$

since $a_4 + a_5 \leq 0$. Thus $s_2 \leq s_4 \leq s_5 \leq s_3 \leq s_1$. Continuing in this way, we see that

$$s_2 \leq s_4 \leq s_6 \leq s_5 \leq s_3 \leq s_1 \tag{5.6.14}$$

and

$$s_2 \leq s_4 \leq s_6 \leq s_7 \leq s_5 \leq s_3 \leq s_1. \tag{5.6.15}$$

In general, for any positive integer n,

$$s_2 \leq s_4 \leq \cdots \leq s_{2n} \leq \cdots \leq s_{2n-1} \leq \cdots \leq s_5 \leq s_3 \leq s_1. \tag{5.6.16}$$

That is, for $n = 1, 2, 3, \ldots$, $\{s_{2n}\}$ is a bounded increasing sequence and $\{s_{2n-1}\}$ is a bounded decreasing sequence. Thus both sequences have limits, say

$$\lim_{n\to\infty} s_{2n} = L \tag{5.6.17}$$

and

$$\lim_{n\to\infty} s_{2n-1} = M. \tag{5.6.18}$$

But then

$$L - M = \lim_{n\to\infty} s_{2n} - \lim_{n\to\infty} s_{2n-1} = \lim_{n\to\infty}(s_{2n} - s_{2n-1}) = \lim_{n\to\infty} a_{2n} = 0,$$

where the final equality follows from condition (2). Hence $L = M$, so

$$\lim_{n\to\infty} s_n = L. \tag{5.6.19}$$

In other words, $\sum_{n=1}^{\infty} a_n$ converges. This conclusion, known as *Leibniz's theorem*, gives a simple criterion for determining the convergence of some alternating series.

Leibniz's theorem Suppose $\sum_{n=1}^{\infty} a_n$ is an alternating series for which $|a_{n+1}| \leq |a_n|$ for $n = 1, 2, 3, \ldots$. If

$$\lim_{n\to\infty} |a_n| = 0, \tag{5.6.20}$$

then $\sum_{n=1}^{\infty} a_n$ converges.

Example The alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

satisfies the conditions of Leibniz's theorem: If we let

$$a_n = \frac{(-1)^{n+1}}{n},$$

$n = 1, 2, 3, \ldots$, then

$$|a_{n+1}| = \frac{1}{n+1} < \frac{1}{n} = |a_n|$$

and

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Thus, as we claimed earlier, the alternating harmonic series converges.

Definition A series which converges but does not converge absolutely is said to *converge conditionally.*

The previous example shows that the alternating harmonic series is an example of a series which converges conditionally.

From the discussion prior to Leibniz's theorem, we see that if $\sum_{n=1}^{\infty} a_n$ satisfies the conditions of Leibniz's theorem, $a_1 > 0$, s_n is its nth partial sum, and

$$s = \sum_{n=1}^{\infty} a_n, \tag{5.6.21}$$

then we must have

$$s_2 \le s_4 \le \cdots \le s_6 \le \cdots \le s \le \cdots \le s_5 \le s_3 \le s_1. \tag{5.6.22}$$

Note that for any positive integer n we have $s_{n+1} \le s \le s_n$ if n is odd and $s_n \le s \le s_{n+1}$ if n is even. Thus, in either case,

$$|s - s_n| \le |s_{n+1} - s_n| = |a_{n+1}|, \tag{5.6.23}$$

a result which also holds if $a_1 < 0$

Proposition Suppose $\sum_{n=1}^{\infty} a_n$ is a convergent alternating series for which $|a_{n+1}| \le |a_n|$ for $n = 1, 2, 3, \ldots$. If

$$s = \sum_{n=1}^{\infty} a_n \tag{5.6.24}$$

and

$$s_n = \sum_{j=1}^{n} a_j, \tag{5.6.25}$$

then, for any $n = 1, 2, 3, \ldots$,

$$|s - s_n| \le |a_{n+1}|. \tag{5.6.26}$$

Hence for those alternating series which satisfy the conditions of the proposition, the error committed in approximating the sum of the series by a particular partial sum is no greater in absolute value than the absolute value of the next term in the series.

Example For the alternating harmonic series, if

$$s = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

and

$$s_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n+1}}{n} = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j},$$

then

$$|s - s_n| \le \frac{1}{n+1}$$

for $n = 1, 2, 3, \ldots$. For example

$$s_{100} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{99} - \frac{1}{100} = 0.688172,$$

so

$$|s - s_{100}| \le \frac{1}{101} = 0.009901,$$

where both results have been rounded to 6 decimal places. In other words, the sum of the alternating harmonic series differs from 0.688172 by less than 0.009901. In fact, since the next term in the series is positive, we know that s must lie between 0.688172 and

$$0.688172 + 0.009901 = 0.698073.$$

We will see in Section 6.2 that the sum of the alternating harmonic series is exactly the natural logarithm of 2, which, to 6 decimal places, is 0.693147

Problems

1. For each of the following infinite series, answer the questions: Does the series converge absolutely? Does the series converge conditionally? Does the series converge?

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2-1}{4n^2+2}$ (d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n}{n!}$

   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+3}}$ (f) $\displaystyle\sum_{n=3}^{\infty} (-1)^n \pi^n$

   (g) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n!}{2^n}$ (h) $\displaystyle\sum_{n=1}^{\infty} \left(-\frac{2}{3}\right)^{n-1}$

2. For each of the following infinite series, answer the questions: Does the series converge absolutely? Does the series converge conditionally? Does the series converge?

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n^2+1)}{3n^5-2}$ (b) $\displaystyle\sum_{n=0}^{\infty} -\frac{3}{5^n}$

   (c) $\displaystyle\sum_{n=0}^{\infty} \frac{3^{2n}}{(2n)!}$ (d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!}$

   (e) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(2n)!}$ (f) $\displaystyle\sum_{n=14}^{\infty} \frac{(-2)^n}{\sqrt{n+1}}$

   (g) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(n+1)}{2n-1}$ (h) $\displaystyle\sum_{n=2}^{\infty} \frac{1-n}{2n^2}$

3. (a) Approximate

$$s = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$$

   using

$$s_{15} = \sum_{n=0}^{15} \frac{(-1)^n}{n!}.$$

(b) Find an upper bound for the error in approximating s by s_{15}.

(c) Find the smallest n such that the absolute value of the error in approximating s by

$$s_n = \sum_{j=0}^{n} \frac{(-1)^j}{j!}$$

is less than 0.000001. What is this approximation?

4. (a) Approximate

$$s = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n2^n}$$

by

$$s_{50} = \sum_{n=1}^{50} \frac{(-1)^{n+1}}{n2^n}.$$

(b) Find an upper bound for the absolute value of the error in approximating s by s_{50}.

(c) Find the smallest n such that the absolute value of the error in approximating s by

$$s_n = \sum_{j=1}^{n} \frac{(-1)^{j+1}}{j2^j}$$

is less than 0.0001. What is this approximation?

5. In our development of Leibniz's theorem, we assumed that $a_1 > 0$. Discuss the changes which must be made in the discussion if $a_1 < 0$.

Difference Equations to **D**ifferential Equations

Section 5.7

Power Series

We are now in a position to pick up the story we left off in Section 5.2: the extension of Taylor polynomials to Taylor series. We shall see that a Taylor series is a type of infinite series whose nth partial sum is a Taylor polynomial. Such series are examples of power series, objects that we will study in this section before considering Taylor series in Section 5.8.

Definition An infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots \tag{5.7.1}$$

is called a *power series* in x about c.

Example The infinite series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

is a power series in x about 0. Note that if we let

$$b_n = \left|\frac{x^n}{n!}\right| = \frac{|x|^n}{n!}$$

for $n = 0, 1, 2, \ldots$, then

$$\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \lim_{n\to\infty} \frac{\dfrac{|x|^{n+1}}{(n+1)!}}{\dfrac{|x|^n}{n!}} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0$$

for any value of x. That is, by the ratio test, the series is absolutely convergent, and hence convergent, for any value of x. Thus if we define a function, called the *exponential function*, by

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{5.7.2}$$

then this function is defined for all values of x. We shall have much more to say about this function, which may be thought of as the simplest "infinite" polynomial which is defined for all real numbers, in Chapter 6.

Notice that the convergence of (5.7.2) for all x implies, by the nth term test for divergence, that

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \tag{5.7.3}$$

for any value of x. We have seen particular cases of this limit in the past, but this is the first time we have had a simple proof that it is always 0.

Example Recall that the Taylor polynomial of order $2n+1$ for $\sin(x)$ at 0 is

$$P_{2n+1}(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!}.$$

Hence $P_{2n+1}(x)$ is a partial sum of the power series

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}. \tag{5.7.4}$$

If, for $k = 0, 1, 2, \ldots$, we let

$$b_k = \left| \frac{(-1)^k x^{2k+1}}{(2k+1)!} \right| = \frac{|x|^{2k+1}}{(2k+1)!},$$

then

$$\lim_{k\to\infty} \frac{b_{k+1}}{b_k} = \lim_{k\to\infty} \frac{\frac{|x|^{2k+3}}{(2k+3)!}}{\frac{|x|^{2k+1}}{(2k+1)!}} = \lim_{k\to\infty} \frac{|x|^2}{(2k+3)(2k+2)} = 0$$

for all values of x. Thus, by the ratio test, (5.7.4) is absolutely convergent, and hence convergent, for all values of x. Moreover, from our work in Section 5.2, we know that

$$|\sin(x) - P_{2n+1}(x)| \le \frac{|x|^{2n+3}}{(2n+3)!}$$

for all values of x. Since (using (5.7.3))

$$\lim_{n\to\infty} \frac{|x|^{2n+3}}{(2n+3)!} = 0,$$

it follows that

$$\lim_{n\to\infty} |\sin(x) - P_{2n+1}(x)| = 0$$

for all values of x. Hence

$$\sin(x) = \lim_{n\to\infty} P_{2n+1}(x)$$

for all values of x. That is, for any value of x,

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}. \tag{5.7.5}$$

Example The series $\sum_{n=0}^{\infty} x^n$ is a power series in x about 0. From our work on geometric series, we know that this series will converge absolutely when $-1 < x < 1$ and will diverge otherwise. In fact, we have seen that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

for $-1 < x < 1$.

These examples show that some functions may be expressed as power series. Such functions are examples of *analytic functions*, which we now define.

Definition If f is a function for which there exists constants $a_0, a_1, a_2, \ldots$ such that

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n \tag{5.7.6}$$

for all values of x in some open interval about c, then we say f is *analytic at* c. If for some $h > 0$ the equality (5.7.6) holds for all x in the interval $I = (c-h, c+h)$, then we say f is *analytic on* I and we call

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

a *power series representation* of f on I.

Example From the previous examples we see that

$$f(x) = \exp(x)$$

and

$$g(x) = \sin(x)$$

are analytic on $(-\infty, \infty)$ and

$$h(x) = \frac{1}{1-x}$$

is analytic on $(-1, 1)$.

Before we can work effectively with power series we need to consider their convergence behavior. First note that the power series

$$\sum_{n=0}^{\infty} a_n (x-c)^n \tag{5.7.7}$$

converges at $x = c$ since in that case all terms after the first are 0. Next, suppose the series converges at a point $x = c + r$, where $r > 0$.That is, suppose $\sum_{n=0}^{\infty} a_n r^n$ converges. Then, by the nth term test for divergence,

$$\lim_{n\to\infty} a_n r^n = 0. \tag{5.7.8}$$

In particular, there exists an integer N such that

$$|a_n r^n| < 1 \tag{5.7.9}$$

for all $n > N$. Hence for any x and $n > N$,

$$\frac{|a_n (x-c)^n|}{\left|\dfrac{x-c}{r}\right|^n} = |a_n r^n| < 1. \tag{5.7.10}$$

In particular, $|a_n (x-c)^n|$ is

$$O\left(\left|\frac{x-c}{r}\right|^n\right).$$

Now if $c - r < x < c + r$, then

$$\left|\frac{x-c}{r}\right| < 1$$

and

$$\sum_{n=0}^{\infty} \left|\frac{x-c}{r}\right|^n$$

is a convergent geometric series. Thus, by the limit comparison test,

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

converges absolutely. In other words, we have shown that if (5.7.7) converges at $c+r$ with $r > 0$, then it converges absolutely for all x in $(c - r, c + r)$. The same argument works to show that if (5.7.7) converges at $c - r$ with $r > 0$, then it converges absolutely for all x in $(c - r, c + r)$. Letting R be the largest real number such that (5.7.7) converges absolutely for all x for which $|x - c| < R$, where we allow $R = \infty$ if (5.7.7) converges for all x, it follows that

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

converges absolutely on $(c - R, c + R)$ ($(-\infty, \infty)$ if $R = \infty$) and diverges for all x with $|x - c| > R$.

Proposition For a power series

$$\sum_{n=0}^{\infty} a_n (x-c)^n,$$

there exists an R, with $R = 0$, $R > 0$, or $R = \infty$, such that the series converges absolutely for all x satisfying $|x - c| < R$ and diverges for all x satisfying $|x - c| > R$.

Definition With the notation of the previous proposition, the interval $(c - R, c + R)$ ($(-\infty, \infty)$ if $R = \infty$) is called the *interval of convergence* and R is called the *radius of convergence* of the power series.

Note that the proposition does not say anything about the behavior of the series at $x = c - R$ or $x = c + R$. In fact, any type of behavior is possible at the endpoints of the interval of convergence; for a given series, these points must be checked individually for convergence. Moreover, although the proposition does not provide a method for finding the interval of convergence of a series, the next examples illustrate that the ratio test is very useful in this regard.

Example Consider the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n 2^n}. \tag{5.7.11}$$

If we let

$$b_n = \left| \frac{(-1)^{n+1} x^n}{n 2^n} \right| = \frac{|x|^n}{n 2^n},$$

then

$$\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \lim_{n\to\infty} \frac{\dfrac{|x|^{n+1}}{(n+1)2^{n+1}}}{\dfrac{|x|^n}{n 2^n}} = \lim_{n\to\infty} \left(\frac{n}{n+1} \right) \frac{|x|}{2} = \frac{|x|}{2}.$$

Hence, by the ratio test, (5.7.11) is absolutely convergent when

$$\frac{|x|}{2} < 1,$$

that is, when $-2 < x < 2$. Thus the radius of convergence is $R = 2$ and the interval of convergence is $(-2, 2)$. Now at $x = -2$, (5.7.11) becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-2)^n}{n 2^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-1)^n 2^n}{n 2^n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n} = \sum_{n=1}^{\infty} -\frac{1}{n},$$

which is a multiple of the harmonic series and hence divergent. At $x = 2$, (5.7.11) becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^n}{n 2^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n},$$

which is the alternating harmonic series and hence convergent, although not absolutely convergent. Putting this together, we see that the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n2^n}$$

converges absolutely for all x in $(-2, 2)$, converges conditionally at $x = 2$, and diverges for all other x.

Example In the first example of this section we used the ratio test to show that the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{n!}$$

converges absolutely for all values of x. Hence in this case the interval of convergence is $(-\infty, \infty)$ and the radius of convergence is $R = \infty$.

Example Consider the power series $\sum_{n=1}^{\infty} n!x^n$. If we let

$$b_n = |n!x^n| = n!|x|^n,$$

then

$$\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \lim_{n\to\infty} \frac{(n+1)!|x|^{n+1}}{n!|x|^n} = \lim_{n\to\infty} (n+1)|x| = \begin{cases} 0, & \text{if } x = 0, \\ \infty, & \text{if } x \neq 0. \end{cases}$$

Hence, by the ratio test, this power series converges only when $x = 0$. Accordingly, the radius of convergence is $R = 0$.

Example Consider the power series

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}. \tag{5.7.12}$$

If we let

$$b_n = \left| \frac{(x-1)^n}{n} \right| = \frac{|x-1|^n}{n},$$

then

$$\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \lim_{n\to\infty} \frac{\dfrac{|x-1|^{n+1}}{n+1}}{\dfrac{|x-1|^n}{n}} = \lim_{n\to\infty} \left(\frac{n}{n+1} \right) |x-1| = |x-1|.$$

Using the ratio test, we see that (5.7.12) converges absolutely when $|x - 1| < 1$. Thus the radius of convergence is $R = 1$ and the interval of convergence is $(0, 2)$. At $x = 0$, (5.7.12) becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n},$$

which is a multiple of the alternating harmonic series and so converges conditionally. At $x = 2$, (5.7.12) becomes

$$\sum_{n=1}^{\infty} \frac{1}{n},$$

which is the harmonic series and so diverges. Hence the power series

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$$

converges absolutely for x in the interval $(0, 2)$, converges conditionally at $x = 0$, and diverges for all other x.

A power series resembles a polynomial; in fact, often it is convenient to think of a power series as a polynomial of infinite degree. Among the many nice properties of polynomials is the ease with which they may be differentiated and integrated. Our next result states that power series may be differentiated and integrated term by term in the same manner as polynomials. Although we have the tools to provide justifications for these statements, they are technical and perhaps best left to a more advanced text.

Differentiation and integration of power series Suppose the radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

is $R > 0$ and let

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

for x in $(c - R, c + R)$. Then

$$f'(x) = \sum_{n=0}^{\infty} \frac{d}{dx} a_n (x-c)^n = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1} \tag{5.7.13}$$

for all x in $(c - R, c + R)$ and

$$\int_c^b f(x)dx = \sum_{n=0}^{\infty} \left(\int_c^b a_n (x-c)^n dx \right) = \sum_{n=0}^{\infty} \frac{a_n (b-c)^{n+1}}{n+1} \tag{5.7.14}$$

for all b in $(c - R, c + R)$.

Example Recall that the interval of convergence of

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

is $(-\infty, \infty)$. Hence

$$\begin{aligned}\frac{d}{dx}\exp(x) &= \frac{d}{dx}\sum_{n=0}^{\infty}\frac{x^n}{n!}\\ &= \sum_{n=0}^{\infty}\frac{d}{dx}\left(\frac{x^n}{n!}\right)\\ &= \sum_{n=1}^{\infty}\frac{nx^{n-1}}{n!}\\ &= \sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!}\\ &= \sum_{n=0}^{\infty}\frac{x^n}{n!}\\ &= \exp(x)\end{aligned}$$

for all x in $(-\infty, \infty)$. That is, the function $\exp(x)$ is its own derivative. We will have much more to say about this interesting property of the exponential function in Chapter 6.

Example From our work above we know that

$$\sin(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

for all x in $(-\infty, \infty)$. Now

$$\int_0^x \sin(t)dt = -\cos(t)\Big|_0^x = -\cos(x) + 1$$

for any x. However, we also know that

$$\begin{aligned}\int_0^x \sin(t)dt &= \int_0^x\left(\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n+1}}{(2n+1)!}\right)dt\\ &= \sum_{n=0}^{\infty}\int_0^x \frac{(-1)^n t^{2n+1}}{(2n+1)!}\,dt\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n t^{2n+2}}{(2n+2)(2n+1)!}\Bigg|_0^x\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{(2n+2)!}\\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n-1} x^{2n}}{(2n)!}.\end{aligned}$$

Hence

$$\begin{aligned}\cos(x) &= 1 - \int_0^x \sin(t)dt \\ &= 1 - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{2n}}{(2n)!} \\ &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n}x^{2n}}{(2n)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{n}x^{2n}}{(2n)!}\end{aligned}$$

for all x in $(-\infty, \infty)$. Thus we have found a power series representation of $\cos(x)$ on $(-\infty, \infty)$. In particular, $\cos(x)$is analytic on $(-\infty, \infty)$.

To close this section we note that a power series representation of a function about a specific point c is unique. To see this, suppose

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n \tag{5.7.15}$$

on $(c-R, c+R)$, where $R > 0$ is the radius of convergence of the power series. We need to show that the coefficients a_n, $n = 0, 1, 2, \ldots$, are uniquely determined by f. To start,

$$\begin{aligned}f(c) &= \sum_{n=0}^{\infty} a_n (c-c)^n \\ &= a_0 + a_1(c-c) + a_2(c-c)^2 + a_3(c-c)^3 + \cdots \\ &= a_0,\end{aligned} \tag{5.7.16}$$

so

$$a_0 = f(c). \tag{5.7.17}$$

Next,

$$f'(c) = \sum_{n=1}^{\infty} n a_n (c-c)^{n-1} = a_1, \tag{5.7.18}$$

so

$$a_1 = f'(c). \tag{5.7.19}$$

For a_2 we have

$$f''(c) = \sum_{n=2}^{\infty} n(n-1) a_n (c-c)^{n-2} = 2a_2, \tag{5.7.20}$$

so

$$a_2 = \frac{f''(c)}{2}. \tag{5.7.21}$$

In general, for $k = 0, 1, 2, \ldots$,

$$f^{(k)}(c) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(c-c)^{n-k} = k!a_k, \tag{5.7.22}$$

from which it follows that

$$a_k = \frac{f^{(k)}(c)}{k!}. \tag{5.7.23}$$

As a consequence, the power series representation of f about c is uniquely determined by the values of the derivatives of f at c.

Proposition Suppose

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n \tag{5.7.24}$$

on $(c-R, c+R)$, where $R > 0$ is the radius of convergence of the power series. Then

$$a_n = \frac{f^{(n)}(c)}{n!} \tag{5.7.25}$$

for $n = 0, 1, 2, \ldots$.

Note that the coefficients a_n as given by (5.7.25) are the same as the coefficients used in the definition of the Taylor polynomial of f at c. This observation leads immediately to the question of extending Taylor polynomials to Taylor series, the topic of Section 5.8.

Our final example of this section illustrates how (5.7.25) may be used to find the derivatives of f at c if we already know a power series representation for f about c.

Example In a previous example we saw that if

$$f(x) = \frac{1}{1-x},$$

then

$$f(x) = \sum_{n=0}^{\infty} x^n$$

for all x in $(-1, 1)$. In this series, the coefficient of x^n is 1 for all n, so, by the previous proposition,

$$1 = \frac{f^{(n)}(0)}{n!}$$

for $n = 0, 1, 2, \ldots$. That is, $f^{(n)}(0) = n!$ for all n.

Problems

1. For each of the following power series, find the interval of convergence and determine the behavior of the series at the endpoints of the interval. State clearly where the series converges absolutely, where it converges conditionally, and where it diverges. Also, for each series write out the first 5 nonzero terms.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}$

   (c) $\displaystyle\sum_{n=1}^{\infty} nx^n$

   (d) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$

   (e) $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^{n+1}}{n+1}$

   (f) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^n}$

   (g) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^2}$

   (h) $\displaystyle\sum_{n=1}^{\infty} n^3 x^{2n}$

2. For each of the following power series, find the interval of convergence and determine the behavior of the series at the endpoints of the interval. State clearly where the series converges absolutely, where it converges conditionally, and where it diverges. Also, for each series write out the first 5 nonzero terms.

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{(x-3)^n}{n!}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n5^n}$

   (c) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{2^n}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-6)^n}{n3^n}$

   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{x^{2n+1}}{n}$

   (f) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{2n}}{(2n)!}$

   (g) $\displaystyle\sum_{n=1}^{\infty} 3^n x^n$

   (h) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}$

3. (a) Using the fact that

   $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

   for $-1 < x < 1$, find a power series representation about 0 for

   $$f(x) = \frac{1}{1+x}$$

   on $(-1, 1)$.

   (b) Use your result from (a) to find $f^{(35)}(0)$.

(c) Use your result from (a) to find a power series representation about 0 for

$$\int_0^x \frac{1}{1+t}\,dt$$

on $(-1,1)$. Determine where the series converges absolutely, where it converges conditionally, and where it diverges.

(d) Use your result from (c) to find an infinite series representation for

$$\int_0^1 \frac{1}{1+t}\,dt.$$

Use this series to estimate the integral with an error of no more than 0.001 in absolute value.

4. Use the power series representations of $\sin(x)$ and $\cos(x)$ about 0 to prove the following identities.

   (a) $\sin(-x) = -\sin(x)$

   (b) $\cos(-x) = \cos(x)$

   (c) $\dfrac{d}{dx}\sin(x) = \cos(x)$

   (d) $\dfrac{d}{dx}\cos(x) = -\sin(x)$

5. Using the power series representation of $\cos(x)$ about 0, find an infinite series representation of $\cos(1)$. Use the infinite series to estimate $\cos(1)$ with an error of no more than 0.000001.

6. Use the fact that
$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$$
to find a power series representation about 0 for
$$g(x) = \frac{1}{(1-x)^2}.$$
Find the interval of convergence for this power series and determine the behavior of the series at the endpoints.

7. Use your result from Problem 6 to evaluate
$$\sum_{n=1}^{\infty} \frac{n}{2^{n-1}}.$$

8. (a) A fair coin is tossed repeatedly. In Section 1.3 we saw that the probability that a head appears for the first time on the nth toss is
$$P_n = \frac{1}{2^n}$$

for $n = 1, 2, 3, \ldots$. The average number of tosses before the first head appears is then given by

$$A = \sum_{n=1}^{\infty} nP_n.$$

Use Problem 7 to find A.

(b) A manufacturer of circuit boards tests every board as it comes off the assembly line. If the probability that a board passes the test is p and the probability that it fails is $q = 1 - p$, then the probability that n boards are tested before the first defective board is encountered is $P_n = p^{n-1}q$. The average number of boards tested before finding a defective one is then

$$A = \sum_{n=1}^{\infty} nP_n.$$

Find A.

Difference Equations to **D**ifferential Equations

Section 5.8

Taylor Series

In this section we will put together much of the work of Sections 5.1-5.7 in the context of a discussion of Taylor series. We begin with two definitions.

Definition If f is a function such that $f^{(n)}$ is continuous on an open interval (a,b) for $n = 0, 1, 2, \ldots$, then we say f is C^∞ on (a,b).

Definition If f is C^∞ on an interval (a,b) and c is a point in (a,b), then the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots \quad (5.8.1)$$

is called the *Taylor series* for f about c.

A Taylor series is a power series constructed from a given function in the same manner as a Taylor polynomial. As with any power series about c, the Taylor series for a function f about c converges at $x = c$, but does not necessarily converge at any other points. If it does converge for other values of x, it will converge absolutely on an interval $(c-R, c+R)$, where R is the radius of convergence. However, even if the series converges at $x \neq c$, it need not converge to $f(x)$. That is, a function may be C^∞ without being analytic. (See Problem 12 of Section 6.1 for an example.) If the Taylor series does converge to $f(x)$ for all x in the interval of convergence, then it is the unique power series representation for f on this interval.

If P_n is the nth order Taylor polynomial for f at c, then P_n is a partial sum of the Taylor series for f about c. Hence to show that the Taylor series converges to f at x, we need to show that

$$f(x) = \lim_{n\to\infty} P_n(x). \quad (5.8.2)$$

Equivalently, we need to show that

$$\lim_{n\to\infty} r_n(x) = 0, \quad (5.8.3)$$

where

$$r_n(x) = f(x) - P_n(x). \quad (5.8.4)$$

In this regard, the error bounds for $r_n(x)$ developed in Section 5.2 can be very useful.

Example For any $n = 0, 1, 2, \ldots$, if P_{2n+1} is the Taylor polynomial of order $2n+1$ for $f(x) = \sin(x)$ at 0, then

$$P_{2n+1}(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!}.$$

In Section 5.2 we saw that if

$$r_{2n+1}(x) = \sin(x) - P_{2n+1}(x),$$

then

$$|r_{2n+1}(x)| \leq \frac{|x|^{2n+3}}{(2n+3)!}$$

for any value of x. In Section 5.7 we saw that, for any x,

$$\lim_{n\to\infty} \frac{|x|^{2n+3}}{(2n+3)!} = 0,$$

so

$$\lim_{n\to\infty} |r_{2n+1}(x)| = 0.$$

Hence

$$\sin(x) = \lim_{n\to\infty} P_{2n+1}(x)$$

for all x. That is,

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \tag{5.8.5}$$

for all x. Thus the Taylor series for $\sin(x)$ about 0 provides a power series representation for $\sin(x)$ on the interval $(-\infty, \infty)$. Note that this example is essentially a restatement of our second example in Section 5.7.

In many cases showing

$$\lim_{n\to\infty} r_n(x) = 0 \tag{5.8.6}$$

is difficult. However, since power series representations are unique, if we are able to find a power series representation for a given function by manipulating some other known representation, then we know that this series is the Taylor series for that function. This is in fact the way many Taylor series representations are found in practice.

Example Since

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

for $-1 < x < 1$, it follows that

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots$$

for $-1 < -x < 1$, that is, $-1 < x < 1$. Hence we have found a Taylor series representation for

$$f(x) = \frac{1}{1+x}$$

on $(-1, 1)$.

Example Similar to the previous example, we have

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \cdots$$

for $-1 < x^2 < 1$, that is, $-1 < x < 1$. Thus we have found a Taylor series representation for

$$f(x) = \frac{1}{1+x^2}$$

on $(-1, 1)$.

Example In Section 5.7 we saw how the relationship

$$\cos(x) = 1 - \int_0^x \sin(t)dt$$

combined with the Taylor series representation

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

yields

$$\cos(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \tag{5.8.7}$$

for all values of x. Thus (5.8.7) is the Taylor series representation for $\cos(x)$ about 0 on $(-\infty, \infty)$.

Example Since

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

for all values of x, it follows that

$$\frac{\sin(x)}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

for all $x \neq 0$. In fact, if we define

$$f(x) = \begin{cases} \dfrac{\sin(x)}{x}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0, \end{cases}$$

then the Taylor series representation for f about 0 on $(-\infty, \infty)$ is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots \tag{5.8.8}$$

Example Since

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

for $-1 < x < 1$,

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \frac{d}{dx} x^n = \sum_{n=1}^{\infty} n x^{n-1}$$

for $-1 < x < 1$. But

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2},$$

so we have the Taylor series representation

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

for all x in $(-1, 1)$.

The final two examples of this section will illustrate the use of Taylor series in solving problems that we could not handle before.

Example Define

$$f(x) = \begin{cases} \dfrac{\sin(x)}{x}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$$

Then, as we saw above,

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

is the Taylor series representation for f about 0 on $(-\infty, \infty)$. Now f is continuous on $(-\infty, \infty)$ and so has an antiderivative on $(-\infty, \infty)$, but, as we have mentioned before, this antiderivative is not expressible in terms of the elementary functions of calculus. However, by the Fundamental Theorem of Calculus, the function

$$\mathrm{Si}(x) = \int_0^x f(t)dt, \tag{5.8.9}$$

called the *sine integral function*, is an antiderivative of f. Moreover, even though we cannot express this integral in terms of the elementary functions, we can find its Taylor series representation. That is,

$$\begin{aligned}
\mathrm{Si}(x) &= \int_0^x \left(\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n+1)!} \right) dt \\
&= \sum_{n=0}^{\infty} \int_0^x \frac{(-1)^n t^{2n}}{(2n+1)!}\, dt \\
&= \sum_{n=0}^{\infty} \left. \frac{(-1)^n t^{2n+1}}{(2n+1)(2n+1)!} \right|_0^x \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!} \\
&= x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!} + \cdots \qquad (5.8.10)
\end{aligned}$$

for all values of x. In particular,

$$\mathrm{Si}(1) = \int_0^1 \frac{\sin(x)}{x}\, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!} = 1 - \frac{1}{3 \cdot 3!} + \frac{1}{5 \cdot 5!} - \frac{1}{7 \cdot 7!} + \cdots.$$

Since this is an alternating series which satisfies the conditions of Leibniz's theorem, if

$$s_n = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)(2k+1)!},$$

then

$$|\mathrm{Si}(1) - s_n| \le \frac{1}{(2n+3)(2n+3)!}.$$

For example, if we want to approximate Si(1) with an error of no more than 0.0001, we note that for $n = 1$ we have, to 6 decimal places,

$$\frac{1}{(2n+3)(2n+3)!} = \frac{1}{5 \cdot 5!} = \frac{1}{600} = 0.001667,$$

while for $n = 2$ we have

$$\frac{1}{(2n+3)(2n+3)!} = \frac{1}{7 \cdot 7!} = \frac{1}{35,280} = 0.000028.$$

Thus

$$s_2 = 1 - \frac{1}{3 \cdot 3!} + \frac{1}{5 \cdot 5!} = 0.946111$$

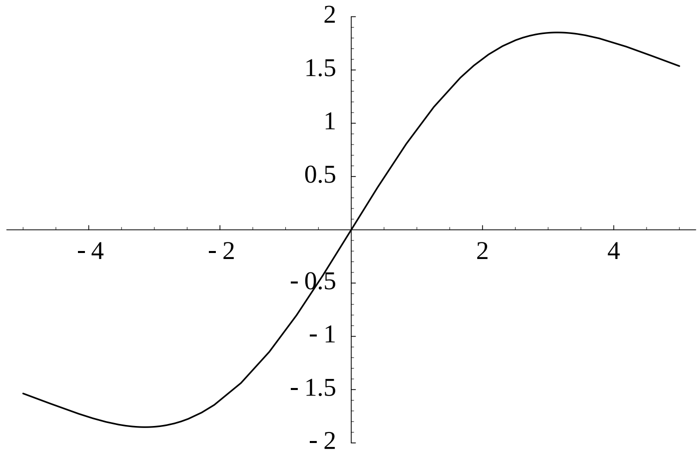


Figure 5.8.1 Taylor polynomial approximation to the graph of $y = \mathrm{Si}(x)$

differs from Si(1) by no more than 0.000028. In fact, since the next term in the series is negative, Si(1) must lie between 0.946111 and

$$0.946111 - 0.000028 = .946083.$$

In particular, we know that

$$\mathrm{Si}(1) = 0.9461$$

to 4 decimal places. Of course, this particular result could also be obtained using numerical integration. However, the point is that (5.8.10) gives us much more; it not only gives us an easy method to evaluate $\mathrm{Si}(x)$ for any value of x to any desired level of accuracy, but it also gives us an algebraic representation of the sine integral function which can be used in applications in much the same way that polynomials are used. In Figure 5.8.1 we have used the Taylor polynomial

$$P_{11}(x) = x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!} + \frac{x^9}{9 \cdot 9!} - \frac{x^{11}}{11 \cdot 11!}$$

to approximate the graph of $\mathrm{Si}(x)$ on the interval $[-5, 5]$. Note that on this interval

$$|\mathrm{Si}(x) - P_{11}(x)| \leq \frac{5^{13}}{13 \cdot 13!} = 0.0151$$

to 4 decimal places, certainly accurate enough for the purposes of our graph.

Example Using

$$\frac{1}{x} = \frac{1}{1 - (1 - x)}$$

and

$$\frac{1}{1 - x} = \sum_{n-0}^{\infty} x^n$$

for $-1 < x < 1$, we have

$$\frac{1}{x} = \sum_{n=0}^{\infty}(1-x)^n = \sum_{n=0}^{\infty}(-1)^n(x-1)^n \tag{5.8.11}$$

for $-1 < 1-x < 1$, that is, $0 < x < 2$. Hence (5.8.11) gives the Taylor series representation for

$$f(x) = \frac{1}{x}$$

about 1. Similar to our work in the previous example, we may now find an antiderivative for f on $(0,2)$ by integration. Namely,

$$\begin{aligned}
\int_1^x \frac{1}{t}\,dt &= \int_1^x \left(\sum_{n=0}^{\infty}(-1)^n(t-1)^n\right)dt \\
&= \sum_{n=0}^{\infty}\int_1^x (-1)^n(t-1)^n dt \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n(t-1)^{n+1}}{n+1}\bigg|_1^x \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n(x-1)^{n+1}}{n+1} \\
&= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots
\end{aligned}$$

provides a Taylor series representation for an antiderivative of f on the interval $(0,2)$. In Chapter 6 we will call this function the *natural logarithm function*, denoted $\log(x)$, although there we will use other means in order to define it on the interval $(0,\infty)$. In particular, note that this series converges at $x = 2$ as well, giving us, with this definition of $\log(x)$,

$$\log(2) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}.$$

Hence $\log(2)$ is the sum of the alternating harmonic series, a number for which we found an approximation in Section 5.6.

Problems

1. Show directly that

$$\cos(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

for all x in $(-\infty,\infty)$.

2. Using any method, find Taylor series representations about 0 for the following functions. State the interval on which the representation is valid. Also, write out the first five nonzero terms of each series.
   (a) $\cos(x^2)$
   (b) $\sin(2x)$
   (c) $\dfrac{1}{1-t^2}$
   (d) $\dfrac{1}{2x-1}$
   (e) $\dfrac{1}{(1+t)^2}$
   (f) $\dfrac{1}{1+4x^2}$
   (g) $f(x) = \begin{cases} \dfrac{1-\cos(x)}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0 \end{cases}$

3. (a) Use the identity
$$\cos^2(x) = \frac{1+\cos(2x)}{2}$$
   to find the Taylor series representation for $\cos^2(x)$ about 0. On what interval is this representation valid?
   (b) What is the Taylor polynomial of order 8 for $\cos^2(x)$ at 0?

4. (a) Use Problem 3 and the identity
$$\sin^2(x) = 1 - \cos^2(x)$$
   to find the Taylor series representation for $\sin^2(x)$ about 0. On what interval is this representation valid?
   (b) What is the Taylor polynomial of order 8 for $\sin^2(x)$ at 0?

5. (a) Use the Taylor series representation about 0 for $\sin(x)$ to find the Taylor series representation for $\sin(x^2)$ about 0. On what interval is this representation valid?
   (b) What is the Taylor polynomial of order 10 for $\sin(x^2)$ at 0?
   (c) Find the Taylor series representation about 0 for
$$S(x) = \int_0^x \sin(t^2)dt.$$
   On what interval is this representation valid?
   (d) What is the Taylor polynomial of order 11 for $S(x)$ at 0?
   (e) Approximate $S(1)$ with an error of less than 0.00001.

6. Let P_n be the Taylor polynomial of order n at 0 for
$$f(x) = \frac{1}{1+x^2}.$$
   Plot f, P_2, P_4, and P_{10} together over the interval $[-1.5, 1.5]$. Why do the Taylor polynomials not give a good approximation to $f(x)$ when $|x| > 1$?

7. Find $\left.\dfrac{d^9}{dx^9}\mathrm{Si}(x)\right|_{x=0}$.

Difference Equations to **D**ifferential Equations

Section 5.9

Some Limit Calculations

In this section we will discuss the use of Taylor polynomials in computing certain types of limits. Although this material could have been treated directly after Section 5.2, we have saved it until now so as not to break into the development of Taylor series. To illustrate the ideas of this section, we begin with two examples, the first of which is already well-known to us.

Example Consider the problem of evaluating

$$\lim_{x \to 0} \frac{\sin(x)}{x}.$$

The reason this limit presents a problem is that, although the function in question is a quotient of two continuous functions, both the numerator and the denominator approach 0 as x approaches 0. Now from our work on Taylor polynomials we know that

$$\sin(x) = x + o(x),$$

so

$$\frac{\sin(x)}{x} = \frac{x + o(x)}{x} = 1 + \frac{o(x)}{x}.$$

But, by definition,

$$\lim_{x \to 0} \frac{o(x)}{x} = 0.$$

Thus

$$\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{x + o(x)}{x} = \lim_{x \to 0} \left(1 + \frac{o(x)}{x}\right) = 1.$$

Example The limit

$$\lim_{x \to 0} \frac{1 - \cos(x)}{x^2}$$

presents the same type of problem. Using the Taylor polynomial of order 2 for $\cos(x)$, we know that

$$\cos(x) = 1 - \frac{x^2}{2} + o(x^2).$$

Hence

$$\begin{aligned}\lim_{x\to 0}\frac{1-\cos(x)}{x^2} &= \lim_{x\to 0}\frac{1-\left(1-\dfrac{x^2}{2}+o(x^2)\right)}{x^2}\\ &= \lim_{x\to 0}\frac{\dfrac{x^2}{2}-o(x^2)}{x^2}\\ &= \lim_{x\to 0}\left(\frac{1}{2}-\frac{o(x^2)}{x^2}\right)\\ &= \frac{1}{2}.\end{aligned}$$

The point in both of these examples was to use the fact that if f is $n+1$ times continuously differentiable on an interval about the point c, then, as we saw in Section 5.2,

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + o((x-c)^n). \tag{5.9.1}$$

Hence if f and g are both $n+1$ times continuously differentiable on an interval about c, $f^{(n)}(c) \neq 0$, $f^{(k)}(c) = 0$ for $k = 0, 1, 2, \ldots, n-1$, and $g^{(k)}(c) = 0$ for $k = 0, 1, 2, \ldots, n-1$, then

$$f(x) = \frac{f^{(n)}(c)}{n!}(x-c)^n + o((x-c)^n) \tag{5.9.2}$$

and

$$g(x) = \frac{g^{(n)}(c)}{n!}(x-c)^n + o((x-c)^n). \tag{5.9.3}$$

Hence

$$\begin{aligned}\lim_{x\to c}\frac{g(x)}{f(x)} &= \lim_{x\to c}\frac{\dfrac{g^{(n)}(c)}{n!}(x-c)^n + o((x-c)^n)}{\dfrac{f^{(n)}(c)}{n!}(x-c)^n + o((x-c)^n)}\\ &= \lim_{x\to c}\frac{\dfrac{g^{(n)}(c)}{n!} + \dfrac{o((x-c)^n)}{(x-c)^n}}{\dfrac{f^{(n)}(c)}{n!} + \dfrac{o((x-c)^n)}{(x-c)^n}}\\ &= \frac{\dfrac{g^{(n)}(c)}{n!}}{\dfrac{f^{(n)}(c)}{n!}}\\ &= \frac{g^{(n)}(c)}{f^{(n)}(c)}.\end{aligned} \tag{5.9.4}$$

That is, under the specified conditions, the value of the limit is equal to the ratio of the nth derivatives of g and f evaluated at c. In addition, if it were the case that, for some $k < n$, $g^{(i)}(c) = 0$ for $i = 1, 2, \ldots, k-1$ and $g^{(k)}(c) \neq 0$, then we would have

$$\begin{aligned}
\frac{g(x)}{f(x)} &= \frac{\dfrac{g^{(k)}(c)}{k!}(x-c)^k + o((x-c)^k)}{\dfrac{f^{(n)}(c)}{n!}(x-c)^n + o((x-c)^n)} \\
&= \frac{\dfrac{g^{(k)}(c)}{k!(x-c)^{n-k}} + \dfrac{o((x-c)^k)}{(x-c)^n}}{\dfrac{f^{(n)}(c)}{n!} + \dfrac{o((x-c)^n)}{(x-c)^n}} \\
&= \frac{\dfrac{1}{(x-c)^{n-k}}\left(\dfrac{g^{(k)}(c)}{k!} + \dfrac{o((x-c)^k)}{(x-c)^k}\right)}{\dfrac{f^{(n)}(c)}{n!} + \dfrac{o((x-c)^n}{(x-c)^n}}. \qquad (5.9.5)
\end{aligned}$$

Since the denominator of this last expression has a limit as x approaches c, but the numerator does not, it follows that in this case $\dfrac{g(x)}{f(x)}$ would not have a limit as x approaches c. That is, in this case $f(x)$ would approach 0 as $x \to c$ at a rate faster than $g(x)$, implying that the limit of the ratio would not exist.

In practice, we do not use the conclusions of the preceding paragraph, but rather apply the procedure outlined. That is, to evaluate

$$\lim_{x\to c} \frac{g(x)}{f(x)},$$

where both

$$\lim_{x\to c} g(x) = 0$$

and

$$\lim_{x\to c} f(x) = 0,$$

we replace both f and g by their respective Taylor polynomial expansions about c, expanded to the first nonzero term, and evaluate the limit as illustrated above.

Example To find

$$\lim_{x\to 0} \frac{\sin(x^2)}{x\tan(x)},$$

we note that

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

implies

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots,$$

so

$$\sin(x^2) = x^2 + o(x^2).$$

Moreover, using the first degree Taylor polynomial for $\tan(x)$ we have

$$\tan(x) = x + o(x).$$

Thus

$$\lim_{x\to 0} \frac{\sin(x^2)}{x\tan(x)} = \lim_{x\to 0} \frac{x^2 + o(x^2)}{x(x+o(x))} = \lim_{x\to 0} \frac{1 + \dfrac{o(x^2)}{x^2}}{1 + \dfrac{o(x)}{x}} = \frac{1+0}{1+0} = 1.$$

Example To evaluate

$$\lim_{x\to 0} \frac{x - \sin(x)}{x(1-\cos(x))},$$

we first note that

$$\sin(x) = x - \frac{x^3}{3!} + o(x^3)$$

and

$$\cos(x) = 1 - \frac{x^2}{2} + o(x^2).$$

Then

$$x - \sin(x) = x - \left(x - \frac{x^3}{3!} + o(x^3)\right) = \frac{x^3}{3!} - o(x^3)$$

and

$$x(1-\cos(x)) = x\left(1 - \left(1 - \frac{x^2}{2} + o(x^2)\right)\right) = \frac{x^3}{2} - xo(x^2).$$

Thus

$$\lim_{x\to 0} \frac{x-\sin(x)}{x(1-\cos(x))} = \lim_{x\to 0} \frac{\dfrac{x^3}{3!} - o(x^3)}{\dfrac{x^3}{2} - xo(x^2)} = \lim_{x\to 0} \frac{\dfrac{1}{6} - \dfrac{o(x^3)}{x^3}}{\dfrac{1}{2} - \dfrac{o(x^2)}{x^2}} = \frac{\dfrac{1}{6} - 0}{\dfrac{1}{2} - 0} = \frac{1}{3}.$$

Example To evaluate

$$\lim_{x\to 0} \frac{\sin(x^2)}{x^4},$$

we first note that, as above,

$$\sin(x^2) = x^2 + o(x^2).$$

Hence

$$\lim_{x\to 0} \frac{\sin(x^2)}{x^4} = \lim_{x\to 0} \frac{x^2 + o(x^2)}{x^4} = \lim_{x\to 0} \left(\frac{1}{x^2} + \frac{o(x^2)}{x^4}\right) = \lim_{x\to 0} \frac{1}{x^2}\left(1 + \frac{o(x^2)}{x^2}\right) = \infty,$$

where the final equality follows after noting that

$$\lim_{x\to 0}\left(1+\frac{o(x^2)}{x^2}\right)=1,$$

while

$$\lim_{x\to 0}\frac{1}{x^2}=\infty.$$

The essence of (5.9.4) is also captured in the following statement, known as *l'Hôpital's rule.*

l'Hôpital's rule If f and g are twice continuously differentiable on an interval about the point c and both $g(c)=0$ and $f(c)=0$, then

$$\lim_{x\to c}\frac{g(x)}{f(x)}=\lim_{x\to c}\frac{g'(x)}{f'(x)}. \tag{5.9.6}$$

This is equivalent to our previous result, assuming the conditions specified at that time, because repeated applications of l'Hôpital's rule yield

$$\lim_{x\to c}\frac{g(x)}{f(x)}=\lim_{x\to c}\frac{g'(x)}{f'(x)}=\lim_{x\to c}\frac{g''(x)}{f''(x)}=\cdots=\lim_{x\to c}\frac{g^{(n)}(x)}{f^{(n)}(x)}=\frac{g^{(n)}(c)}{f^{(n)}(c)}, \tag{5.9.7}$$

which is (5.9.4). As before, if or some $k<n$, $g^{(i)}(c)=0$ for $i=0,1,2,\ldots,k-1$ and $g^{(k)}(c)\neq 0$, then

$$\lim_{x\to c}\frac{g^{(k)}(x)}{f^{(k)}(x)}$$

does not exist and $\dfrac{g(x)}{f(x)}$ does not have a limit as x approaches c.

Example We will illustrate l'Hôpital's rule first with another well-known limit. Namely,

$$\lim_{x\to 0}\frac{1-\cos(x)}{x}=\lim_{x\to 0}\frac{\frac{d}{dx}(1-\cos(x))}{\frac{d}{dx}x}=\lim_{x\to 0}\frac{\sin(x)}{1}=0.$$

Example As an illustration of how it may be necessary to apply l'Hôpital's rule more than once, we have

$$\begin{aligned}
\lim_{x\to 0}\frac{\sin(x)-x}{x^3} &= \lim_{x\to 0}\frac{\dfrac{d}{dx}(\sin(x)-x)}{\dfrac{d}{dx}x^3} \\
&= \lim_{x\to 0}\frac{(\cos(x)-1)}{3x^2} \\
&= \lim_{x\to 0}\frac{\dfrac{d}{dx}(\cos(x)-1)}{\dfrac{d}{dx}3x^2} \\
&= \lim_{x\to 0}\frac{-\sin(x)}{6x} \\
&= -\frac{1}{6}\lim_{x\to 0}\frac{\sin(x)}{x} \\
&= -\frac{1}{6}.
\end{aligned}$$

Note that this particular problem could have been done more quickly using the fact that

$$\sin(x)-x = -\frac{x^3}{6}+o(x^3).$$

Although we will not do so here, it is possible to demonstrate that l'Hôpital's rule is more widely applicable than what we have indicated so far. In particular, we may also apply l'Hôpital's rule to one-sided limits and to limits as x approaches ∞ or x approaches $-\infty$, provided, of course, that both $g(x)$ and $f(x)$ are approaching 0 and are twice continuously differentiable on the appropriate intervals. The following examples illustrate these applications.

Example Using l'Hôpital's rule, we have

$$\begin{aligned}
\lim_{x\to\pi^+}\frac{\sin(x)}{\sqrt{x-\pi}} &= \lim_{x\to\pi^+}\frac{\dfrac{d}{dx}\sin(x)}{\dfrac{d}{dx}\sqrt{x-\pi}} \\
&= \lim_{x\to\pi^+}\frac{\cos(x)}{\dfrac{1}{2\sqrt{x-\pi}}} \\
&= \lim_{x\to\pi^+}2\cos(x)\sqrt{x-\pi} \\
&= 0.
\end{aligned}$$

Example Using l'Hôpital's rule, we have

$$\begin{aligned}
\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) &= \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} \\
&= \lim_{x\to\infty} \frac{\frac{d}{dx}\sin\left(\frac{1}{x}\right)}{\frac{d}{dx}\frac{1}{x}} \\
&= \lim_{x\to\infty} \frac{-\frac{1}{x^2}\cos\left(\frac{1}{x}\right)}{-\frac{1}{x^2}} \\
&= \lim_{x\to\infty} \cos\left(\frac{1}{x}\right) \\
&= 1.
\end{aligned}$$

Notice we could have computed this limit by substituting $h = \frac{1}{x}$, thus obtaining

$$\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) = \lim_{h\to 0^+} \frac{\sin(h)}{h} = 1.$$

Finally, it is also possible to demonstrate that l'Hôpital's rule applies when both the numerator and the denominator are approaching ∞. That is, if f and g are twice continuously differentiable at c and both

$$\lim_{x\to c} f(x) = \infty$$

and

$$\lim_{x\to c} g(x) = \infty,$$

then

$$\lim_{x\to c} \frac{g(x)}{f(x)} = \lim_{x\to c} \frac{g'(x)}{f'(x)}. \tag{5.9.8}$$

As before, this also applies for one-sided limits and for limits as x approaches ∞ or $-\infty$. Moreover, one or both of $g(x)$ and $f(x)$ may be approaching $-\infty$.

Example Using l'Hôpital's rule,

$$\begin{aligned}
\lim_{x\to\infty} \frac{3x+1}{\sqrt{x^2+4}} &= \lim_{x\to\infty} \frac{\dfrac{d}{dx}(3x+1)}{\dfrac{d}{dx}\sqrt{x^2+4}} \\
&= \lim_{x\to\infty} \frac{3}{\dfrac{x}{\sqrt{x^2+4}}} \\
&= \lim_{x\to\infty} \frac{3\sqrt{x^2+4}}{x} \\
&= \lim_{x\to\infty} 3\sqrt{\frac{x^2+4}{x^2}} \\
&= \lim_{x\to\infty} 3\sqrt{1+\frac{4}{x^2}} \\
&= 3.
\end{aligned}$$

Of course, we could have also computed this limit by dividing both the numerator and denominator by x to obtain

$$\lim_{x\to\infty} \frac{3x+1}{\sqrt{x^2+4}} = \lim_{x\to\infty} \frac{3+\dfrac{1}{x}}{\dfrac{\sqrt{x^2+4}}{x}} = \lim_{x\to\infty} \frac{3+\dfrac{1}{x}}{\sqrt{\dfrac{x^2+4}{x^2}}} = \lim_{x\to\infty} \frac{3+\dfrac{1}{x}}{\sqrt{1+\dfrac{4}{x^2}}} = 3.$$

Problems

1. Use Taylor polynomials to find the following limits.

   (a) $\displaystyle\lim_{x\to 0} \frac{\sin(3x)}{x}$

   (b) $\displaystyle\lim_{t\to 0} \frac{t-\sin(t)}{t^2}$

   (c) $\displaystyle\lim_{x\to 0} \frac{\cos(x)-1+\dfrac{x^2}{2}}{x^4}$

   (d) $\displaystyle\lim_{x\to 0} \frac{\sin(x^2)}{\sin^2(x)}$

   (e) $\displaystyle\lim_{u\to 0} \frac{\tan(u)}{\sin(u)}$

   (f) $\displaystyle\lim_{t\to 0} \frac{\sin(t)-t}{t^3}$

   (g) $\displaystyle\lim_{y\to 0} \frac{\tan(3y)}{\tan(5y)}$

   (h) $\displaystyle\lim_{x\to 0} \frac{\tan(x^2)}{\sin(x^2)}$

   (i) $\displaystyle\lim_{x\to 0} \frac{\sqrt{1+x}-1}{3x}$

   (j) $\displaystyle\lim_{t\to 0} \frac{\sqrt{1+t}-1-\dfrac{t}{2}}{3t^2}$

2. Use l'Hôpital's rule to evaluate the following limits.

   (a) $\displaystyle\lim_{x\to 0}\frac{\sin(5x)}{3x}$ (b) $\displaystyle\lim_{t\to 0}\frac{1-\cos(3t)}{t^2}$

   (c) $\displaystyle\lim_{x\to 0}\frac{1-\sec(x)}{x}$ (d) $\displaystyle\lim_{t\to \frac{\pi}{4}}\frac{1-\tan(t)}{\cos(2t)}$

   (e) $\displaystyle\lim_{x\to 0^+}\frac{\sin(2x)}{\sqrt{x}}$ (f) $\displaystyle\lim_{x\to 0}\frac{1-\cos(x)}{\sin(x)}$

   (g) $\displaystyle\lim_{x\to \infty}x\sin\left(\frac{1}{x^2}\right)$ (h) $\displaystyle\lim_{x\to 0}\frac{3x^2}{\sin^2(x)}$

3. Evaluate the following limits using any method you prefer.

   (a) $\displaystyle\lim_{x\to \infty}\frac{3x^2-2x+1}{16x^2+2}$ (b) $\displaystyle\lim_{x\to 0}\frac{\tan(x^2)}{\sin^2(x)}$

   (c) $\displaystyle\lim_{x\to 0}\frac{1-\dfrac{\sin(x)}{x}}{3x^2}$ (d) $\displaystyle\lim_{x\to 0}\frac{(1+x)^{\frac{1}{3}}-1}{x}$

   (e) $\displaystyle\lim_{x\to 0}\frac{(1+x)^{\frac{1}{3}}-1-\dfrac{x}{3}}{x^2}$ (f) $\displaystyle\lim_{t\to 0}\frac{1-\cos(t)}{t\sin(2t)}$

   (g) $\displaystyle\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}$ (h) $\displaystyle\lim_{x\to \pi}\frac{\cos(x)+1}{x-\pi}$

   (i) $\displaystyle\lim_{t\to 0}\frac{\cos(t)-1+\dfrac{t^2}{2}}{t^2\sin(t^2)}$ (j) $\displaystyle\lim_{u\to 0}\frac{\sin(u^2)-u^2}{u^4(1-\cos(u))}$

4. Let $g(x)=\begin{cases} x^2\sin\left(\dfrac{1}{x}\right), & \text{if } x\neq 0,\\ 0, & \text{if } x=0.\end{cases}$

   (a) Show that $g'(0)=0$, and hence that $g(x)=o(x)$.

   (b) Use the preceding result and the fact that $\tan(x)=x+o(x)$ to show that

   $$\lim_{x\to 0}\frac{x^2\sin\left(\dfrac{1}{x}\right)}{\tan(x)}=0.$$

   (c) Letting $f(x)=\tan(x)$, show that

   $$\lim_{x\to 0}\frac{g(x)}{f(x)}\neq\lim_{x\to 0}\frac{g'(x)}{f'(x)}.$$

   Which condition in the statement of l'Hôpital's rule does not hold for this example?

Section 6.1

The Exponential Function

At this point we have seen all the major concepts of calculus: derivatives, integrals, and power series. For the rest of the book we will be concerned with how these ideas apply in various circumstances. In particular, in this chapter we will introduce the remaining elementary functions of calculus: the exponential function, the natural logarithm function, the inverse trigonometric functions, and the hyperbolic trigonometric functions. As they are introduced, we will discuss related issues involving derivatives, integrals, and power series, as well as applications to the physical world.

We will begin by considering the exponential function. We first saw this function in Section 5.7, but we will redefine it here for completeness.

Definition The exponential function, with value at x denoted by $\exp(x)$, is defined by

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots. \tag{6.1.1}$$

We saw in Section 5.7 that this series converges absolutely for all values of x; hence the domain of the exponential function is $(-\infty, \infty)$. We should also note that $\exp(0) = 1$.

Using the properties of power series, it is an easy matter to compute the derivative of the exponential function:

$$\frac{d}{dx}\exp(x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{d}{dx}\left(\frac{x^n}{n!}\right) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \exp(x).$$

Proposition

$$\frac{d}{dx}\exp(x) = \exp(x). \tag{6.1.2}$$

Example Using the chain rule, we have

$$\frac{d}{dx}\exp(4x) = 4\exp(x).$$

Example Similarly,

$$\frac{d}{dx}\exp(x^2) = 2x\exp(x^2).$$

In fact, the exponential function is the only function f for which both $f(0) = 1$ and $f'(x) = f(x)$ for all x. To see this, we first demonstrate a more general property. Suppose

f is any function for which $f(0) = c$ and $f'(x) = kf(x)$ for all x, where c and k are constants. Then it follows that

$$f''(x) = \frac{d}{dx}(kf(x)) = kf'(x) = k^2 f(x),$$
$$f'''(x) = \frac{d}{dx}(k^2 f(x)) = k^2 f'(x) = k^3 f(x),$$

and, in general,

$$f^{(n)}(x) = k^n f(x) \tag{6.1.3}$$

for $n = 0, 1, 2, \ldots$. Hence

$$f^{(n)}(0) = k^n c \tag{6.1.4}$$

for $n = 0, 1, 2, \ldots$. Thus the Taylor series for f about 0 is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{ck^n}{n!} x^n = c \sum_{n=0}^{\infty} \frac{(kx)^n}{n!} = c \exp(kx), \tag{6.1.5}$$

where the final equality follows from the definition of the exponential function. Now, as a consequence of Taylor's theorem, if P_n is the nth order Taylor polynomial for f at 0, then

$$|f(x) - P_n(x)| \leq \frac{M}{(n+1)!} |x|^{n+1}, \tag{6.1.6}$$

where M is the maximum value of $|f^{(n+1)}|$ on the closed interval from 0 to x. But

$$f^{(n+1)}(x) = k^{n+1} f(x),$$

so

$$M = |k|^{n+1} L$$

where L is the maximum value of $|f|$ on the closed interval from 0 to x. Hence

$$|f(x) - P_n(x)| \leq \frac{|k|^{n+1} L}{(n+1)!} |x|^{n+1} = \frac{L|kx|^{n+1}}{(n+1)!}. \tag{6.1.7}$$

As we have seen before,

$$\lim_{n \to \infty} \frac{L|kx|^{n+1}}{(n+1)!} = 0 \tag{6.1.8}$$

for any value of x, so it follows that

$$f(x) = \lim_{n \to \infty} P_n(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \tag{6.1.9}$$

for all x. In other words, f has a Taylor series representation, and so, using (6.1.5), we have

$$f(x) = c \exp(x).$$

Proposition If f is a function for which $f(0) = c$ and $f'(x) = kf(x)$ for all x, where c and k are constants, then

$$f(x) = c\exp(x) \tag{6.1.10}$$

for all x.

In particular, if we let $c = 1$ and $k = 1$ in this proposition, then $f(x) = \exp(x)$. In many ways, it is this property that makes the exponential function one of the most important functions in mathematics.

Now consider a function f defined by $f(x) = \exp(x + b)$ for some constant b. Then

$$f'(x) = \exp(x + b) = f(x),$$

so, by the previous proposition, we must have $f(x) = c\exp(x)$ for all x, where $c = f(0) = \exp(b)$. That is, for all values of x,

$$\exp(x + b) = f(x) = \exp(b)\exp(x).$$

This demonstrates a fundamental algebraic property of the exponential function: For any numbers a and b,

$$\exp(a + b) = \exp(a)\exp(b). \tag{6.1.11}$$

It follows from (6.1.11) that for any number a,

$$\exp(a)\exp(-a) = \exp(a - a) = \exp(0) = 1.$$

That is,

$$exp(-a) = \frac{1}{\exp(a)} \tag{6.1.12}$$

More generally, using both (6.1.11) and (6.1.12), we have

$$\exp(a - b) = \exp(a)\exp(-b) = \frac{\exp(a)}{\exp(b)}. \tag{6.1.13}$$

for any numbers a and b, another important algebraic property of the exponential function.

We shall soon see that the number $\exp(1)$ plays a special role in this discussion.

Definition The value of the exponential function at 1 is denoted by e. That is,

$$e = \exp(1) = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \cdots. \tag{6.1.14}$$

It may be shown, although not easily, that e is an irrational number. Much more easily (see Problem 5), it may be shown that, to 5 decimal places, e is given by 2.71828. The use of the letter e to denote this number originates with Leonhard Euler (1707-1783), one of the most prolific mathematicians of all time.

Notice that for any positive integer n,

$$\exp(n) = \exp(\underbrace{1+1+\cdots+1}_{n \text{ times}}) = \underbrace{\exp(1)\exp(1)\cdots\exp(1)}_{n \text{ times}} = (\exp(1))^n = e^n \tag{6.1.15}$$

and

$$\exp(-n) = \frac{1}{\exp(n)} = \frac{1}{e^n} = e^{-n}. \tag{6.1.16}$$

Combining this with $\exp(0) = 1$, we have

$$\exp(n) = e^n \tag{6.1.17}$$

for all integers n. Moreover, for any integer $n \neq 0$,

$$\begin{aligned}\left(\exp\left(\frac{1}{n}\right)\right)^n &= \underbrace{\exp\left(\frac{1}{n}\right)\exp\left(\frac{1}{n}\right)\cdots\exp\left(\frac{1}{n}\right)}_{n \text{ times}} \\ &= \exp\underbrace{\left(\frac{1}{n}+\frac{1}{n}+\cdots+\frac{1}{n}\right)}_{n \text{ times}} \\ &= \exp(1) = e,\end{aligned}$$

showing that

$$\exp\left(\frac{1}{n}\right) = e^{\frac{1}{n}}. \tag{6.1.18}$$

Hence if m and n are integers with $n \neq 0$, then

$$\exp\left(\frac{m}{n}\right) = \exp\underbrace{\left(\frac{1}{n}+\frac{1}{n}+\cdots+\frac{1}{n}\right)}_{m \text{ times}} = \left(\exp\left(\frac{1}{n}\right)\right)^m = \left(e^{\frac{1}{n}}\right)^m = e^{\frac{m}{n}}. \tag{6.1.19}$$

The next proposition summarizes these facts.

Proposition For any rational number r,

$$\exp(r) = e^r. \tag{6.1.20}$$

That is, evaluating the exponential function at a rational number r is equivalent to raising e to the rth power. A natural question at this point is to ask whether the same result holds for irrational numbers. A little thought shows that this question is not meaningful; although we know what it means to raise a number to a rational power (namely, for integers m and n,

$$a^{\frac{m}{n}} = \sqrt[n]{a^m},$$

that is, $a^{\frac{m}{n}}$ is the nth root of the mth power of a), we have never defined what it means to raise a number to an irrational power. For example, at this point we do not have a meaning to associate with the symbol 2^{π}. We will now take the first step toward remedying this situation by defining e^s for an irrational number s.

Definition If s is an irrational number, then we define

$$e^s = \exp(s). \tag{6.1.21}$$

With this definition we can now say that

$$\exp(x) = e^x \tag{6.1.22}$$

for any real number x. The properties of the exponential function stated in (6.1.11) and (6.1.13) may be restated as

$$e^{x+y} = e^x e^y \tag{6.1.23}$$

and

$$e^{x-y} = \frac{e^x}{e^y} \tag{6.1.24}$$

for any real numbers x and y. Hence exponents behave in this new situation exactly the way we should expect them to behave.

From our previous result that

$$\frac{d}{dx}\exp(x) = \exp(x),$$

it now follows that

$$\frac{d}{dx}e^x = e^x. \tag{6.1.25}$$

From this differentiation rule we obtain the indefinite integral

$$\int e^x dx = e^x + c. \tag{6.1.26}$$

Example Using the chain rule, we have

$$\frac{d}{dx}e^{2x} = 2e^{2x}.$$

Example Using the product and chain rules,

$$\begin{aligned}\frac{d}{dx}\left(3xe^{4x^2}\right) &= 3x\frac{d}{dx}\left(e^{4x^2}\right) + e^{4x^2}\frac{d}{dx}(3x)\\ &= (3x)\left(8xe^{4x^2}\right) + \left(e^{4x^2}\right)(3)\\ &= (3+24x^2)e^{4x^2}.\end{aligned}$$

Example Since

$$\frac{d}{dx}e^{-4x} = -4e^{-4x},$$

it follows that

$$\int e^{-4x}dx = -\frac{1}{4}e^{-4x} + c.$$

Notice the similarity between the evaluation of the integral in the last example and the evaluation of the integral

$$\int \cos(-4x)dx = -\frac{1}{4}\sin(-4x) + c.$$

In fact, just as, for any $a \neq 0$,

$$\int \cos(ax)dx = \frac{1}{a}\sin(ax) + c,$$

we have

$$\int e^{ax}dx = \frac{1}{a}e^{ax} + c. \tag{6.1.27}$$

Example To evaluate $\int 3xe^{2x^2}dx$, we use the substitution

$$\begin{aligned} u &= 2x^2 \\ du &= 4xdx. \end{aligned}$$

Then $\frac{1}{4}du = xdx$, so

$$\int 3xe^{2x^2}dx = \frac{3}{4}\int e^u du = \frac{3}{4}e^u + c = \frac{3}{4}e^{2x^2} + c.$$

Example To evaluate $\int 2xe^x dx$, we use integration by parts with

$$\begin{array}{ll} u = 2x & dv = e^x dx \\ du = 2dx & v = e^x. \end{array}$$

Then

$$\int 2xe^x dx = 2xe^x - \int 2e^x dx = 2xe^x - 2e^x + c.$$

Notice the similarity between the technique for evaluating the integral in the last example and the technique for evaluating $\int 2x\sin(x)dx$.

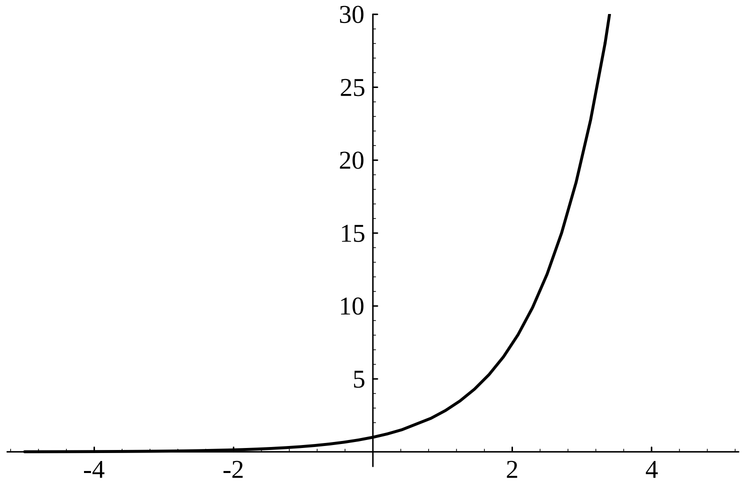


Figure 6.1.1 Graph of $y = e^x$

Example The integral $\int e^x \sin(x)dx$ may also be handled by integration by parts, although with a little more work than in the previous example. Here we will let

$$\begin{array}{cc} u = \sin(x) & dv = e^x dx \\ du = \cos(x)dx & v = e^x. \end{array}$$

Then

$$\int e^x \sin(x)dx = e^x \sin(x) - \int e^x \cos(x)dx.$$

We now perform another integration by parts by choosing

$$\begin{array}{cc} u = \cos(x) & dv = e^x dx \\ du = -\sin(x)dx & v = e^x. \end{array}$$

Then

$$\begin{aligned} \int e^x \sin(x) &= e^x \sin(x) - \left(e^x \cos(x) + \int e^x \sin(x)dx \right) \\ &= e^x \sin(x) - e^x \cos(x) - \int e^x \sin(x)dx. \end{aligned}$$

At first glance it may seem that we are back to where we started; however, all we need to do now is solve for $\int e^x \sin(x)dx$. That is, we have

$$2\int e^x \sin(x)dx = e^x \sin(x) - e^x \cos(x) = e^x(\sin(x) - \cos(x)),$$

so

$$\int e^x \sin(x)dx = \frac{1}{2}e^x(\sin(x) - \cos(x)) + c.$$

Note that we have added an arbitrary constant c since we are seeking the general antiderivative.

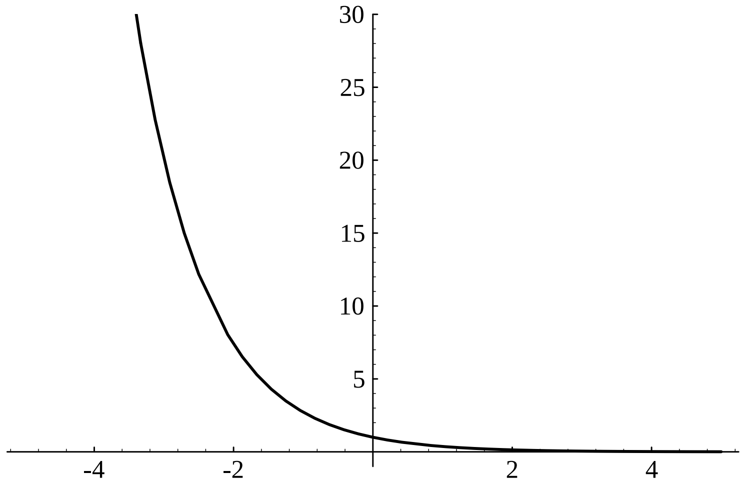


Figure 6.1.2 Graph of $y = e^{-x}$

We now have sufficient information about the exponential function to understand the geometry of its graph. Since $e > 0$, we know that $e^x > 0$ for all rational values of x, and hence, by continuity, for all values of x. Since $e > 1$, it follows that

$$\lim_{x \to \infty} e^x = \infty \tag{6.1.28}$$

and

$$\lim_{x \to -\infty} e^x = \lim_{u \to \infty} e^{-u} = \lim_{u \to \infty} \frac{1}{e^u} = 0. \tag{6.1.29}$$

Moreover, since

$$\frac{d}{dx} e^x = e^x > 0 \tag{6.1.30}$$

and

$$\frac{d^2}{dx^2} e^x = e^x > 0 \tag{6.1.31}$$

for all x, the graph of $y = e^x$ is always increasing and always concave up. Moreover, (6.1.30) and (6.1.31) indicate that as x increases, the graph is not only increasing, but its slope is increasing at the same rate that y is increasing. Thus we should expect y to grow at a very rapid rate, as we see in Figure 6.1.1. This rate of growth is characterized as *exponential growth*. Figure 6.1.2 shows the graph of $y = e^{-x}$, which is the graph of $y = e^x$ reflected about the y-axis. In this case y decreases asymptotically toward 0 as x increases; this is known as *exponential decay*

We will close this section with an application to the problem of uninhibited population growth, a problem we first considered in Section 1.4.

Uninhibited population growth

Recall from Section 1.4 that if x_n represents the size of a population after n units of time and the population grows at a constant rate of $\alpha 100\%$ per unit of time, then the sequence $\{x_n\}$ must satisfy the linear difference equation

$$x_{n+1} - x_n = \alpha x_n \tag{6.1.32}$$

for $n = 0, 1, 2, \ldots$. At that time we saw that the solution of this equation is given by

$$x_n = (1+\alpha)^n x_0. \tag{6.1.33}$$

The crucial aspect of (6.1.32) is the statement that amount of change in the size of the population over any unit of time is proportional to the current size of the population. Hence if $x(t)$ represents the size of a population at time t, where the population can change continuously over time, then the continuous time analog of (6.1.32) is the *differential equation*

$$\dot{x}(t) = \alpha x(t) \tag{6.1.34}$$

for all time t. If x_0 is the size of the population at time $t = 0$, then we know from our work in this section that the only solution to this equation is the function

$$x(t) = x_0 e^{\alpha t}. \tag{6.1.35}$$

Hence if the size of a population is growing at a rate which is proportional to itself, an assumption which, as we noted in Section 1.4, is often reasonable over short periods of time, then the population will grow exponentially. As in Section 1.4, we refer to such growth as *uninhibited population growth.*

Example In 1970 the population of the United States was 203.3 million and in 1980 the population was 226.5 million. Assuming an uninhibited growth model and letting $x(t)$ represent the population t years after 1970, by (6.1.35) we should have

$$x(t) = 203.3e^{\alpha t}$$

for some constant α. Since $x(10) = 226.5$, we can find α by solving

$$226.5 = 203.3e^{10\alpha}.$$

That is, we need to find a value for α such that

$$e^{10\alpha} = \frac{226.5}{203.3} = 1.114.$$

Unfortunately, solving this equation exactly requires being able to reverse the process of applying the exponential function. In other words, we need an inverse for the exponential function. We shall take up that problem in the next section; for now we may use a numerical approximation. You should verify that $\alpha = 0.0108$ satisfies the equation. Thus this model would predict the population of the United States t years after 1970 to be

$$x(t) = 203.3e^{0.0108t}.$$

For example, this model would predict a 1990 population of

$$x(20) = 203.3e^{(0.0108)(20)} = 252.3$$

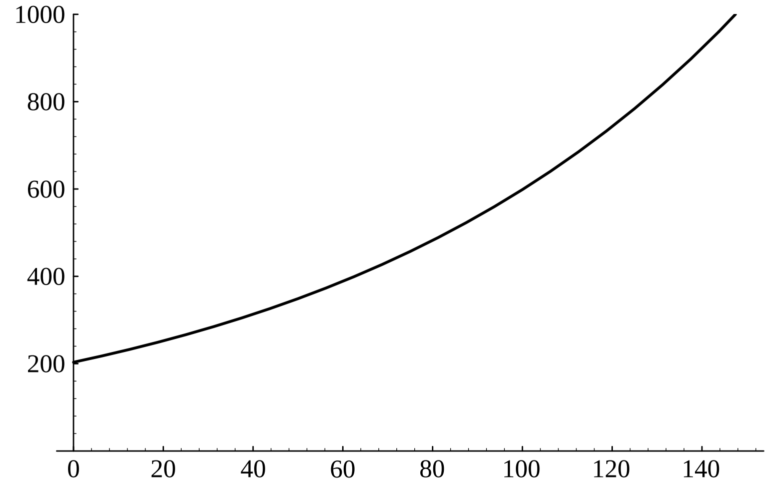


Figure 6.1.3 Uninhibited growth model for the United States (1970-2120)

and a population in the year 2000 of

$$x(30) = 203.3e^{(0.0108)(30)} = 281.1.$$

While the prediction for 1990 is fairly accurate (the actual population was approximately 249.6 million), the second prediction differs significantly from the Census Bureau's own prediction of a population of 268.3 million for the year 2000. As we discussed in Sections 1.4 and 1.5, an uninhibited growth model is a simple model which cannot be expected to be accurate for predictions too far into the future.

We shall have more to say about population models in Section 6.3, where we will also see another example of a differential equation. We will have a much fuller discussion of differential equations in Chapter 8.

Problems

1. Find the derivative of each of the following functions.

   (a) $f(x) = 3e^{2x}$

   (b) $g(t) = 4t^2e^{3t}$

   (c) $h(z) = (3z^2 - 6)e^{5z^3}$

   (d) $f(x) = e^{3x}\sin(2x)$

   (e) $g(x) = \dfrac{3x}{2e^x}$

   (f) $h(t) = e^{-6t}\cos(4t)$

   (g) $f(s) = \dfrac{3s-1}{e^{-2s}+2}$

   (h) $g(\theta) = 5\theta e^{6\theta}\sin(2\theta)$

2. Evaluate each of the following integrals.

   (a) $\displaystyle\int 3e^{2x}dx$

   (b) $\displaystyle\int 4xe^{3x^2}dx$

   (c) $\displaystyle\int 4te^{3t}dt$

   (d) $\displaystyle\int 5ye^{-y}dy$

(e) $\int z^2 e^z dz$

(f) $\int x^3 e^{-2x} dx$

(g) $\int e^x \cos(x) dx$

(h) $\int e^{-2t} \sin(3t) dt$

3. Find the maximum value of $f(x) = x^2 e^{-x}$ on the interval $(0, \infty)$.

4. (a) Use the Taylor series for e^{-x} to show that $e^{-1} > \frac{1}{3}$. Hence conclude that $e < 3$.
   (b) Show that if P_n is the nth order Taylor polynomial for e^x at 0, then
$$|e^x - P_n(x)| \leq \frac{3^x}{(n-1)!}|x|^{n+1}$$
   for any value of x.
   (c) Use (b) to find an approximation for e with an error of less than 0.000005.

5. Find the following limits.

(a) $\lim_{x \to \infty} xe^{-x}$

(b) $\lim_{x \to 0} \frac{e^x - 1}{x}$

(c) $\lim_{t \to 0} \frac{e^{-t} - 1}{t}$

(d) $\lim_{x \to \infty} x^2 e^{-2x}$

6. (a) Show that $\lim_{x \to \infty} x^n e^{-x} = 0$ for any positive integer n.
   (b) Use (a) to show that if p is any polynomial, then $\lim_{x \to \infty} p(x) e^{-x} = 0$. This shows that e^x grows faster as $x \to \infty$ than any polynomial function.

7. Graph the following functions on the specified intervals.

(a) $f(t) = 3e^{2t}$ on $[-3, 3]$

(b) $g(x) = 4x^2 e^{-2x}$ on $[0, 5]$

(c) $g(t) = \frac{e^t - e^{-t}}{e^t + e^{-t}}$ on $[-10, 10]$

(d) $f(t) = e^{-\frac{t}{4}} \sin(3t)$ on $[0, 10]$

8. Evaluate the following improper integrals.

(a) $\int_0^\infty e^{-x} dx$

(b) $\int_0^\infty 3e^{-2x} dx$

(c) $\int_0^\infty xe^{-x} dx$

(d) $\int_0^\infty 3xe^{-2x} dx$

(e) $\int_0^\infty x^2 e^{-x} x dx$

(f) $\int_0^\infty xe^{-x^2} dx$

9. Use the integral test to show that the infinite series $\sum_{n=0}^{\infty} e^{-n}$ converges.

10. (a) Find the Taylor series for $f(x) = e^{-x^2}$.
    (b) Use (a) to find the the Taylor series for
$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt,$$

known as the *error function*.

(c) Use your result in (b) to approximate erf(1) with an error less than 0.0001.

11. Suppose $x(t)$ is the population of a certain country t years after 1985, $x(0) = 23.4$ million, and

$$\dot{x}(t) = 0.008x(t).$$

(a) What will the population of the country be in the year 2000?

(b) In what year will the population be twice what it was in 1985?

12. Let

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

(a) Graph f on the interval $[-5, 5]$.

(b) Show that $f'(0) = 0$.

(c) Show that $f^{(n)}(0) = 0$ for $n = 0, 1, 2, \ldots$.

(d) Show that f is C^∞ on $(-\infty, \infty)$.

(e) Note that the Taylor series for f about 0 converges for all x in $(-\infty, \infty)$, but does not converge to $f(x)$ except at 0. Thus f is C^∞ on $(-\infty, \infty)$, but not analytic at 0.

Difference Equations to **D**ifferential Equations

Section 6.2

The Natural Logarithm Function

In the last example of Section 6.1 we saw the need for solving an equation of the form

$$e^x = b.$$

for x in terms of b. In general, for a given function f, a function g defined on the range of f is called the *inverse* of f if

$$g(f(x)) = x \tag{6.2.1}$$

for all x in the domain of f and

$$f(g(x)) = x \tag{6.2.2}$$

for all x in the domain of g. That is, if $f(x) = y$, then $g(y) = x$ and if $g(x) = y$, then $f(y) = x$. In order for a function f to have an inverse function g, for every point y in the range of f there must exist a unique point x in the domain of f such that $f(x) = y$, in which case $g(y) = x$. In other words, for any two points x_1 and x_2 in the domain of f, we must have $f(x_1) \neq f(x_2)$. Now this will be the case if f is increasing on its domain, since, for such an f, $x_1 < x_2$ implies $f(x_1) < f(x_2)$. In particular, a function f with domain (a, b) will have an inverse if $f'(x) > 0$ for all x in (a, b). Hence, since

$$\frac{d}{dx}e^x = e^x > 0$$

for all x in $(-\infty, \infty)$, the function $f(x) = e^x$ must have an inverse defined for every point in its range, namely, $(0, \infty)$. We call this inverse function the *natural logarithm function*.

Definition The inverse of the exponential function is called the *natural logarithm function*. The value of the natural logarithm function at a point x is denoted $\log(x)$.

Thus, by definition,

$$\log(e^x) = x \tag{6.2.3}$$

and

$$e^{\log(x)} = x. \tag{6.2.4}$$

Equivalently,

$$y = e^x \text{ if and only if } \log(y) = x. \tag{6.2.5}$$

Another common notation for $\log(x)$ is $\ln(x)$. In fact, most calculators use $\ln(x)$ for the natural logarithm of x and $\log(x)$ for the base 10 logarithm of x. However, since the natural logarithm function is the fundamental logarithm function of interest to us, we will denote it by $\log(x)$ and often refer to it simply as the logarithm function.

Example In the last example of Section 6.1 we needed to solve the equation

$$e^{10\alpha} = 1.114.$$

Taking the natural logarithm of both sides of this equation gives us

$$\log(e^{10\alpha}) = \log(1.114),$$

from which we obtain

$$10\alpha = \log(1.114),$$

and hence

$$\alpha = \frac{\log(1.114)}{10}.$$

Using a calculator and rounding to 4 decimal places, we find that $\alpha = 0.0108$.

Being the inverse of the exponential function, the logarithm function has domain $(0, \infty)$ (which is the range of the exponential function) and range $(-\infty, \infty)$ (which is the domain of the exponential function). Also, since $e^0 = 1$ and $e^1 = e$, it follows that $\log(1) = 0$ and $\log(e) = 1$.

Several basic algebraic properties of the logarithm function follow immediately from the algebraic properties of the exponential function. For example, since, for any positive numbers a and b,

$$e^{\log(a)+\log(b)} = e^{\log(a)}e^{\log(b)} = ab,$$

it follows, after taking the logarithm of both sides, that

$$\log(ab) = \log(a) + \log(b). \tag{6.2.6}$$

Similarly, since, for any positive numbers a and b,

$$e^{\log(a)-\log(b)} = e^{\log(a)}e^{-\log(b)} = \frac{e^{\log(a)}}{e^{\log(b)}} = \frac{a}{b},$$

we have

$$\log\left(\frac{a}{b}\right) = \log(a) - \log(b). \tag{6.2.7}$$

In particular,

$$\log\left(\frac{1}{b}\right) = \log(1) - \log(b) = -\log(b)$$

for any $b > 0$. Finally, if $a > 0$ and b is a rational number, then

$$e^{b\log(a)} = \left(e^{\log(a)}\right)^b = a^b$$

implies that

$$\log(a^b) = b\log(a). \tag{6.2.8}$$

We have restricted b to rational values here because we have not defined a^b for irrational values of b, except in the single case when $a = e$. However, the expression

$$e^{b\log(a)}$$

is defined for any value of b, rational or irrational. Hence the following definition provides a natural extension to the meaning of raising a number $a > 0$ to a power.

Definition If $a > 0$ and b is an irrational number, then we define

$$a^b = e^{b\log(a)}. \tag{6.2.9}$$

With this definition, we have

$$\log(a^b) = b\log(a). \tag{6.2.10}$$

for $a > 0$ and any value of b.

Example We now see that

$$2^\pi = e^{\pi\log(2)},$$

which, using a calculator, is 8.8250 to 4 decimal places.

The derivative of the logarithm function may be found using our knowledge of the derivative of the exponential function. Specifically, if $y = \log(x)$, then $e^y = x$. Thus, differentiating both sides of this expression with respect to x,

$$\frac{d}{dx}e^y = \frac{d}{dx}x,$$

from which we obtain

$$e^y\frac{dy}{dx} = 1.$$

Hence

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}.$$

Since we started with $y = \log(x)$, this gives us the following proposition.

Proposition

$$\frac{d}{dx}\log(x) = \frac{1}{x}. \tag{6.2.11}$$

Example Combining the chain rule with the previous proposition, we have

$$\frac{d}{dx}\log(x^2+1) = \left(\frac{1}{x^2+1}\right)(2x) = \frac{2x}{x^2+1}.$$

Example It is worth noting that, in general, for any differentiable function f,

$$\frac{d}{dx}\log(f(x)) = \left(\frac{1}{f(x)}\right) f'(x) = \frac{f'(x)}{f(x)}. \tag{6.2.12}$$

Thus, for another example,

$$\frac{d}{dx}\log(3x^4+8) = \frac{12x^3}{3x^4+8}.$$

Example In some circumstances it is useful to use the properties of the logarithm function before attempting to differentiate. For example,

$$\begin{aligned}\frac{d}{dx}\log\left(3x\sqrt{4x^2+2}\right) &= \frac{d}{dx}\left(\log(3x)+\log(4x^2+2)^{\frac{1}{2}}\right)\\ &= \frac{d}{dx}\left(\log(3)+\log(x)+\frac{1}{2}\log(4x^2+2)\right)\\ &= \frac{1}{x}+\frac{1}{2}\frac{8x}{4x^2+2}\\ &= \frac{1}{x}+\frac{2x}{2x^2+1}.\end{aligned}$$

Turning to integrals, we note that

$$\frac{d}{dx}\log(x)) = \frac{1}{x}$$

implies that

$$\int \frac{1}{x}\,dx = \log(x)+c,$$

provided x is in the domain of the logarithm function, that is, $x > 0$. For $x < 0$, we have

$$\frac{d}{dx}\log|x| = \frac{d}{dx}\log(-x) = \frac{1}{-x}(-1) = \frac{1}{x},$$

showing that

$$\int \frac{1}{x}\,dx = \log|x|+c$$

for $x < 0$. Since $|x| = x$ when $x > 0$, we can combine the above results into one statement.

Proposition

$$\int \frac{1}{x}\,dx = \log|x|+c. \tag{6.2.13}$$

Example To evaluate $\displaystyle\int \frac{x}{x^2+1}\,dx$ we make the substitution

$$\begin{aligned}u &= x^2+1\\ du &= 2xdx.\end{aligned}$$

Thus $\frac{1}{2}\,du = xdx$, so

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\log|u| + c = \frac{1}{2}\log(x^2+1) + c,$$

where we have removed the absolute value sign since $x^2 + 1 > 0$ for all x.

Example To evaluate

$$\int \tan(x)dx = \int \frac{\sin(x)}{\cos(x)}\,dx,$$

we make the substitution

$$\begin{aligned} u &= \cos(x) \\ du &= -\sin(x)dx \end{aligned}.$$

Thus $-du = \sin(x)dx$, so

$$\int \tan(x)dx = -\int \frac{1}{u}\,du = -\log|u| + c = -\log|\cos(x)| + c.$$

Example To evaluate $\int \sec(x)dx$, we first multiply $\sec(x)$ by

$$\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}$$

to obtain

$$\int \sec(x)dx = \int \frac{\sec^2(x)+\sec(x)\tan(x)}{\sec(x)+\tan(x)}\,dx.$$

Then we make the substitution

$$\begin{aligned} u &= \sec(x) + \tan(x) \\ du &= (\sec(x)\tan(x) + \sec^2(x))dx. \end{aligned}$$

Hence

$$\int \sec(x)dx = \int \frac{1}{u}\,du = \log|u| + c = \log|\sec(x) + \tan(x)| + c.$$

Note that this example does not illustrate a general technique for evaluating integrals, but rather a nice trick that works in this specific case. In fact, it is just as easy to remember the value of the integral as it is to remember the trick that was used to find it.

Example We may evaluate $\int \log(x)dx$ using integration by parts. To do so, we choose

$$\begin{aligned} u &= \log(x) & dv &= dx \\ du &= \frac{1}{x}\,dx & v &= x. \end{aligned}$$

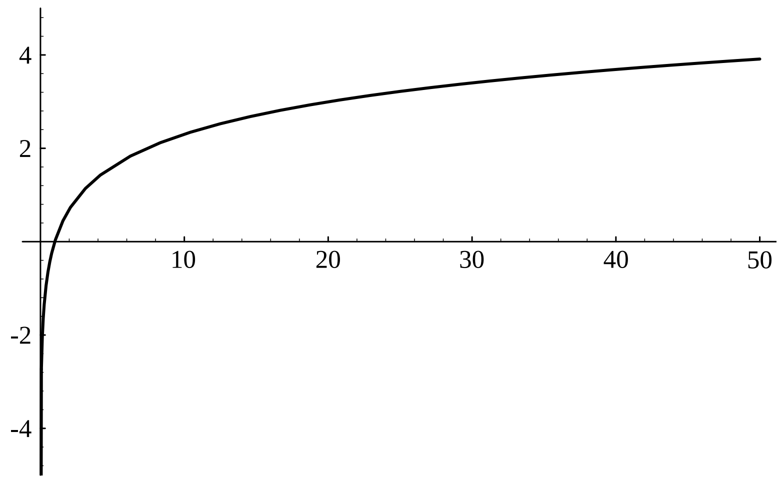


Figure 6.2.1 Graph of $y = \log(x)$

Then

$$\int \log(x)dx = x\log(x) - \int dx = x\log(x) + x + c.$$

We can now put together enough information to obtain a geometric understanding of the graph of $y = \log(x)$. Since

$$\frac{d}{dx}\log(x) = \frac{1}{x} > 0$$

for all x in $(0, \infty)$, the graph of $y = \log(x)$ is increasing on $(0, \infty)$. Note however that the slope of the graph decreases toward 0 as x increases; although the graph is always increasing, the rate of increase diminishes as x increases. This is also seen in the fact that

$$\frac{d^2}{dx^2}\log(x) = -\frac{1}{x^2} < 0$$

for all $x > 0$. As a consequence, the graph is concave down on $(0, \infty)$. Since the logarithm function is the inverse of the exponential function, it follows from

$$\lim_{x\to\infty} e^x = \infty$$

that

$$\lim_{x\to\infty} \log(x) = \infty \tag{6.2.14}$$

and from

$$\lim_{x\to-\infty} e^x = 0$$

that

$$\lim_{x\to 0^+} \log(x) = -\infty. \tag{6.2.15}$$

From (6.2.14) we see that, even though the slope of $y = \log(x)$ decreases toward 0 as x increases, y will continue to grow without any bound. From (6.2.15) we see that the y-axis is a vertical asymptote for the graph. Using this geometric information, we can understand why the graph of $y = \log(x)$ looks like it does in Figure 6.2.1. You should compare this graph with the graph of $y = e^x$ in Figure 6.1.1

We will use the relationship

$$\log(x) = \int_1^x \frac{1}{t}\,dt \tag{6.2.16}$$

to find the Taylor series representation for the logarithm function. Since, as we saw in Section 5.8,

$$\frac{1}{t} = \frac{1}{1-(1-t)} = \sum_{n=0}^{\infty}(1-t)^n = \sum_{n=0}^{\infty}(-1)^n(t-1)^n$$

for $0 < t < 2$, it follows that

$$\begin{aligned}\log(x) &= \int_1^x \frac{1}{t}\,dt \\ &= \sum_{n=0}^{\infty}(-1)^n \int_1^x (t-1)^n dt \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n(x-1)^{n+1}}{n+1} \\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x-1)^n}{n} \\ &= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots\end{aligned}$$

for $0 < x < 2$. Hence

$$\log(x) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x-1)^n}{n} = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$$

is the Taylor series representation of $\log(x)$ on $(0, 2)$. Notice that at $x = 0$, this series becomes a multiple of the harmonic series, and so does not converge, while at $x = 2$, it is the alternating harmonic series, which does converge. Thus we would suspect that

$$\log(2) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots.$$

This is in fact true, and may be verified using Taylor's theorem (see Problem 6).

We will end this section by extending an old result. In Chapter 3 we saw that for any rational number $n \neq 0$,

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Now that we have defined x^n for irrational n (provided $x > 0$), we see that

$$\frac{d}{dx}x^n = \frac{d}{dx}e^{n\log(x)} = \frac{n}{x}e^{n\log(x)} = \frac{nx^n}{x} = nx^{n-1}$$

for any real number $n \neq 0$.

Proposition For any real number $n \neq 0$,

$$\frac{d}{dx}x^n = nx^{n-1}. \tag{6.2.17}$$

Example Note that

$$\frac{d}{dx}x^\pi = \pi x^{\pi-1},$$

while

$$\frac{d}{dx}\pi^x = \frac{d}{dx}e^{x\log(\pi)} = \log(\pi)e^{x\log(\pi)} = \log(\pi)\pi^x.$$

In Section 6.3 we will consider some applications of the exponential and logarithm functions.

Problems

1. Let $a = \log(2)$ and $b = \log(3)$. Find the following in terms of a and b.
   (a) $\log(6)$
   (b) $\log(1.5)$
   (c) $\log(9)$
   (d) $\log(12)$
2. Find the derivative of each of the following functions.
   (a) $f(x) = \log(3x^2)$
   (b) $g(t) = t^3\log(3t+4)$
   (c) $g(x) = \log\left(4x^2\sqrt{x^2+5}\right)$
   (d) $h(t) = \log\left(\dfrac{13t^2+1}{5t+3}\right)$
   (e) $f(x) = e^{2x}\log(5x)$
   (f) $g(z) = 3z\log(4z+5)$
   (g) $h(x) = \log(\log(x))$
   (h) $f(x) = 2^x$
   (i) $f(x) = x^e$
   (j) $f(t) = \log\sqrt{\dfrac{4t^2+3}{t^4+1}}$
3. Evaluate each of the following integrals.
   (a) $\displaystyle\int \frac{1}{2x}\,dx$
   (b) $\displaystyle\int \frac{3x}{x^2+2}\,dx$
   (c) $\displaystyle\int \frac{5x}{3x^2+1}\,dx$
   (d) $\displaystyle\int \frac{3x^2}{4x^3+15}\,dx$
   (e) $\displaystyle\int \tan(3x)dx$
   (f) $\displaystyle\int \cot(x)dx$
   (g) $\displaystyle\int \csc(x)dx$
   (h) $\displaystyle\int \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx$
4. Evaluate each of the following integrals.
   (a) $\displaystyle\int \log(3x)dx$
   (b) $\displaystyle\int x\log(x)dx$

(c) $\displaystyle\int \frac{\log(x)}{x}\,dx$

(d) $\displaystyle\int 3x^2\log(x)dx$

(e) $\displaystyle\int \log(x+1)dx$

(f) $\displaystyle\int (x^2+3)\log(x)dx$

(g) $\displaystyle\int \frac{1}{x\log(x)}\,dx$

(h) $\int 2^x dx$

5. (a) Show that
$$\lim_{x\to\infty}\frac{\log(x)}{x}=0.$$
What does this say about the rate of growth of $\log(x)$ as x increases?

   (b) Show that for any real number $p>0$,
$$\lim_{x\to\infty}\frac{\log(x)}{x^p}=0.$$
What does this say about the rate of growth of $\log(x)$ as x increases?

6. (a) Use Taylor's theorem to show that the alternating harmonic series converges to $\log(2)$. That is, if $r_n(x)$ is the error in approximating $\log(x)$ by the nth order Taylor polynomial at x, show that $\lim\limits_{n\to\infty} r_n(2)=0$.

   (b) Use the Taylor series for $\log(x)$ about 1 to approximate $\log(2)$ with an error of less than 0.005.

   (c) Use the Taylor series for $\log(x)$ about 1 to approximate $\log(1.5)$ with an error of less than 0.001.

7. Graph each of the following on the given interval.

   (a) $y=\log(3x)$ on $(0,20]$

   (b) $y=\log|x|$ on $[20,20]$

   (c) $x=\dfrac{\log(t)}{t}$ on $(0,10]$

   (d) $x=t^2\log(t)$ on $(0,3]$

   (e) $y=\sin(\log(\theta))$ on $(0,2]$

   (f) $y=\log(x^2)$ on $(0,50]$

8. Compare the graphs of $y=2^x$ and $y=\left(\dfrac{1}{2}\right)^x$.

9. Use the integral test to show that the harmonic series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

10. (a) Find $\lim\limits_{x\to\infty}\log(\log(x))$.

   (b) Show that
$$\lim_{x\to\infty}\frac{\log(\log(x))}{\log(x)}=0.$$
What does this say about the rate of growth of $\log(\log(x))$ as x increases?

   (c) Graph $y=\log(\log(x))$.

(d) Find the value of x such that $\log(x) = 20$.

(e) Find the value of x such that $\log(\log(x)) = 20$.

11. Find the length of the curve $y = \log(x)$ over the interval $[1, 10]$.

12. Suppose x is a function with $\dot{x}(t) = \alpha x(t)$, $x(0) = 100$, and $x(5) = 200$. Find $x(t)$.

13. Given that g is the inverse function of f, show that

$$g'(x) = \frac{1}{f'(g(x))}.$$

Use this result to show that $\dfrac{d}{dx}\log(x)) = \dfrac{1}{x}$.

Difference Equations to Differential Equations

Section 6.3

Models of Growth and Decay

In this section we will look at several applications of the exponential and logarithm functions to problems involving growth and decay, including compound interest, radioactive decay, and population growth.

Compound interest

Suppose a principal of P dollars is deposited in a bank which pays $100i\%$ interest compounded n times a year. That is, each year is divided into n units and after each unit of time the bank pays $\frac{100i}{n}\%$ interest on all money currently in the account, including money that was earned as interest at an earlier time. Thus if x_m represents the amount of money in the account after m units of time, x_m must satisfy the difference equation

$$x_{m+1} - x_m = \frac{i}{n} x_m, \tag{6.3.1}$$

$m = 0, 1, 2, \ldots$, with initial condition $x_0 = P$. Hence the sequence $\{x_m\}$ satisfies the linear difference equation

$$x_{m+1} = \left(1 + \frac{i}{n}\right) x_m, \tag{6.3.2}$$

and so, from our work in Section 1.4, we know that

$$x_m = \left(1 + \frac{i}{n}\right)^m x_0 = \left(1 + \frac{i}{n}\right)^m P \tag{6.3.3}$$

for $m = 0, 1, 2, \ldots$. If we let $A(t)$ be the amount in the account after t years, then, since there are nt compounding periods in t years,

$$A(t) = x_{nt} = \left(1 + \frac{i}{n}\right)^{nt} P \tag{6.3.4}$$

Example Suppose \$1,000 is deposited at 5% interest which is compounded quarterly. If $A(t)$ is the amount in the account after t years, then, for example,

$$A(5) = 1000\left(1 + \frac{0.05}{4}\right)^{20} = 1,282.04,$$

rounded to the nearest cent. If the interest were compounded monthly instead, then we would have

$$A(5) = 1000\left(1 + \frac{0.05}{12}\right)^{60} = 1,283.36.$$

Of course, the more frequent the compounding, the faster the amount in the account will grow. At the same time, there is no limit to how often the bank could compound. However, is there some limit to how fast the account can grow? That is, for a fixed value of t, is $A(t)$ bounded as n grows? To answer this question, we need to consider

$$\lim_{n\to\infty}\left(1 + \frac{i}{n}\right)^{nt}.$$

To evaluate this limit, first consider the limit

$$\lim_{x\to\infty}\left(1 + \frac{k}{x}\right)^{x},$$

where k is a constant. If we let

$$y = \left(1 + \frac{k}{x}\right)^{x},$$

then

$$\log(y) = x\log\left(1 + \frac{k}{x}\right).$$

Using l'Hôpital's rule, we have

$$\begin{aligned}
\lim_{x\to\infty}\log(y) &= \lim_{x\to\infty} x\log\left(1 + \frac{k}{x}\right)\\
&= \lim_{x\to\infty}\frac{\log\left(1 + \frac{k}{x}\right)}{\frac{1}{x}}\\
&= \lim_{x\to\infty}\frac{\frac{d}{dx}\log\left(1 + \frac{k}{x}\right)}{\frac{d}{dx}\left(\frac{1}{x}\right)}\\
&= \lim_{x\to\infty}\frac{\frac{1}{1+\frac{k}{x}}\left(-\frac{k}{x^2}\right)}{-\frac{1}{x^2}}\\
&= \lim_{x\to\infty}\frac{k}{1 + \frac{k}{x}}\\
&= k.
\end{aligned}$$

It now follows that

$$\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\log(y)} = e^{k}.$$

That is, we have the following proposition.

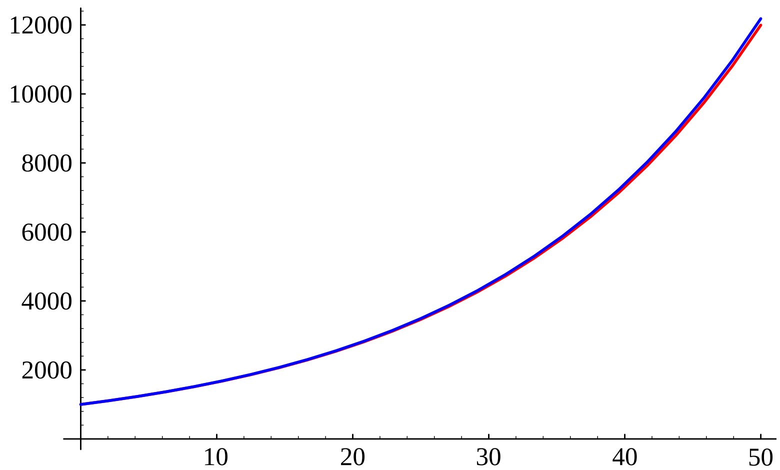


Figure 6.3.1 Compounding quarterly versus compounding continuously (5% interest)

Proposition For any constant k,

$$\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x = e^k. \tag{6.3.5}$$

It now follows that

$$\lim_{n\to\infty}\left(1+\frac{i}{n}\right)^{nt} = \left(\lim_{n\to\infty}\left(1+\frac{i}{n}\right)^n\right)^t = (e^i)^t = e^{it}. \tag{6.3.6}$$

Hence

$$\lim_{n\to\infty} A(t) = \lim_{n\to\infty} P\left(1+\frac{i}{n}\right)^{nt} = Pe^{it}.$$

Thus no matter how many times interest is compounded per year, the amount after t years will never exceed Pe^{it}. We think of Pe^{it} as the amount that would be in the account if interest were *compounded continuously.*

Example In the previous example, with $P = \$1,000$ and $i = 0.05$, the amount after five years of interest compounded continuously would be

$$1000e^{(0.05)(5)} = 1,284.03.$$

In other words, assuming a 5% interest rate, no matter how many times per year the bank compounds the interest, the amount in the account after five years can never exceed \$1,284.03. As Figure 6.3.1 shows, in this case there is only a slight difference between compounding quarterly and compounding continuously over a period of 50 years.

Growth and decay

We saw in Chapter 1 that the linear difference equation

$$x_{n+1} - x_n = \alpha x_n \tag{6.3.7}$$

may be used as as simple model for the growth of a population when $\alpha > 0$ or as a model for radioactive decay when $\alpha < 0$. As we discussed in Section 6.1, the continuous time version of this model is the differential equation

$$\dot{x}(t) = \alpha x(t). \tag{6.3.8}$$

At that time we saw that the solution of this equation is given by

$$x(t) = x_0 e^{\alpha t}, \tag{6.3.9}$$

where $x_0 = x(0)$. As before, when $\alpha > 0$ this is a model for uninhibited, also called natural, population growth, while when $\alpha < 0$ it is a model for radioactive decay. More generally, this model is applicable whenever a quantity is known to change at a rate proportional to itself, as expressed by (6.3.8).

Example Suppose the population of a certain country was 23 million in 1990 and 27 million in 1995. Assuming an uninhibited population growth model, if $x(t)$ represents the size of the population, in millions, t years after 1990, then

$$x(t) = 23e^{\alpha t}$$

for some value of α. To find α, we note that

$$27 = x(5) = 23e^{5\alpha}.$$

Hence

$$e^{5\alpha} = \frac{27}{23},$$

from which we obtain

$$5\alpha = \log\left(\frac{27}{23}\right).$$

Thus

$$\alpha = \frac{1}{5}\log\left(\frac{27}{23}\right) = 0.0321,$$

where we have rounded to four decimal places. Hence

$$x(t) = 23e^{0.0321t}.$$

For example, this model would predict a population in 2000 of

$$x(10) = 23e^{(0.0321)(10)} = 31.7 \text{ million}.$$

Also, assuming this model continues to be valid, we could compute how many years it would take for the population to reach any given size. For example, if T is the number of years until the population doubles, then we would have

$$46 = x(T) = 23e^{0.0321T}.$$

Thus

$$e^{0.0321T} = 2,$$

so

$$0.0321T = \log(2)$$

and

$$T = \frac{\log(2)}{0.0321} = 21.6,$$

to one decimal place. Hence a population growing at this rate will double in size in less than 22 years.

Example A common method for dating fossilized remains of animal and plant life is to compare the amount of carbon-14 to the amount of carbon-12 in the fossil. For example, the bones of a living animal contain approximately equal amounts of these two elements, but after death the carbon-14 begins to decay, whereas the carbon-12, not being radioactive, remains at a constant level. Hence it is possible to determine the age of the fossil from the amount of carbon-14 that remains. In particular, if $x(t)$ is the amount of carbon-14 in the fossil t years after the animal died, then

$$\dot{x}(t) = \alpha x(t)$$

for some constant α, and so

$$x(t) = x_0 e^{\alpha t},$$

where x_0 is the initial amount of carbon-14. Since it is known that the half-life of carbon-14 is 5,730 years (that is, one-half of any initial amount of carbon-14 will decay over a period of 5,730 years), we can find the value of α. Namely, we know that

$$\frac{1}{2}x_0 = x(5730) = x_0 e^{5730\alpha},$$

so

$$e^{5730\alpha} = \frac{1}{2}.$$

Hence

$$5730\alpha = -\log(2),$$

so

$$\alpha = -\frac{\log(2)}{5730}.$$

For example, suppose a fossilized bone is found which has 10% of its original carbon-14. If T is the time since the death of the animal, we must have

$$\frac{1}{10}x_0 = x(T) = x_0 e^{\alpha T}.$$

Thus

$$e^{\alpha T} = \frac{1}{10},$$

so

$$\alpha T = -\log(10)$$

and

$$T = -\frac{\log(10)}{\alpha} = 5730\frac{\log(10)}{\log(2)} = 19,035 \text{ years},$$

rounding to the nearest year. Hence the fossil is from an animal that died more than 19,000 years ago.

Inhibited growth models

In Section 1.5 we discussed a modification of the uninhibited growth model which took into account the limits placed on growth by environmental factors. In this model, which we called the inhibited growth model, if x_n is the size of the population after n units of time, α is the natural growth rate of the population (that is, the rate of growth the population would experience if it were not for the limiting factors), and M is the maximum population which is sustainable in the given environment, then

$$x_{n+1} - x_n = \alpha x_n \left(\frac{M - x_n}{M}\right) \tag{6.3.10}$$

for $n = 0, 1, 2, \ldots$. Hence this model modifies the natural rate of growth by the factor

$$\frac{M - x_n}{M}, \tag{6.3.11}$$

representing the proportion of room which is left for future growth. As a result, when x_n is small, (6.3.11) is close to 1 and the population grows at a rate close to its natural rate; however, as x_n increases toward M, (6.3.11) decreases, causing the rate of the growth of the population to decrease toward 0.

Now (6.3.10) says that the amount of increase in the population during one unit of time is jointly proportional to the size of the population and the proportion of room left for growth. Thus for a continuous time model, if $x(t)$ is the size of the population at time t, then the rate of change of $x(t)$ should be jointly proportional to $x(t)$ and $\dfrac{M - x(t)}{M}$. That is, $x(t)$ should satisfy the differential equation

$$\dot{x}(t) = \alpha x(t)\left(\frac{M - x(t)}{M}\right) = \frac{\alpha}{M}x(t)(M - x(t)). \tag{6.3.12}$$

This equation is called the *logistic differential equation*. It has many applications other than population growth; for example, it is frequently used as a model for the spread of an infectious disease, where $x(t)$ represents the number of people who have contracted the disease by time t, M is the total size of the population that could potentially be infected, and α is a parameter controlling the rate at which the disease spreads.

To solve the logistic differential equation, we begin by rewriting (6.3.12) as

$$\frac{\dot{x}(t)}{x(t)(M - x(t))} = \frac{\alpha}{M}. \tag{6.3.13}$$

Since this is an equation involving the derivative of the function we are trying to find, we might try integrating as a step toward finding $x(t)$. That is, if we replace t by s in (6.3.13) and then integrate from 0 to t, we obtain

$$\int_0^t \frac{x(s)}{x(s)(M - x(s))}\, ds = \int_0^t \frac{\alpha}{M} = \frac{\alpha}{M}t. \tag{6.3.14}$$

To evaluate the remaining integral in (6.3.14), we first make the substitution

$$\begin{aligned} u &= x(s) \\ du &= \dot{x}(s)ds. \end{aligned}$$

Then, letting $x_0 = x(0)$, which we assume to be less than M, we have

$$\int_0^t \frac{\dot{x}(s)}{x(s)(M - x(s))}\, ds = \int_{x_0}^{x(t)} \frac{1}{u(M-u)}\, du. \tag{6.3.15}$$

To evaluate this integral, we use the algebraic fact, known as *partial fraction decomposition*, that there exist constants A and B such that

$$\frac{1}{u(M-u)} = \frac{A}{u} + \frac{B}{M-u}. \tag{6.3.16}$$

Once we find the values for A and B, the integration will follow easily. Now (6.3.16) implies that

$$\frac{1}{u(M-u)} = \frac{A(M-u) + Bu}{u(M-u)}.$$

Since two rational functions with equal denominators are equal only if their numerators are also equal, it follows that

$$1 = A(M-u) + Bu.$$

This final equality must hold for all values of u, so, in particular, when $u = 0$ we obtain

$$1 = AM$$

and when $u = M$ we have

$$1 = BM.$$

It follows that

$$A = \frac{1}{M}$$

and

$$B = \frac{1}{M}.$$

Hence

$$\frac{1}{u(M-u)} = \frac{1}{M}\frac{1}{u} + \frac{1}{M}\frac{1}{M-u},$$

and so

$$\begin{aligned}\int_{x_0}^{x(t)} \frac{1}{u(M-u)}\,du &= \frac{1}{M}\int_{x_0}^{x(t)} \frac{1}{u}\,du + \frac{1}{M}\int_{x_0}^{x(t)} \frac{1}{M-u}\,du \\ &= \left(\frac{1}{M}\log(u) - \frac{1}{M}\log(M-u)\right)\Bigg|_{x_0}^{x(t)} \\ &= \frac{1}{M}\log\left(\frac{u}{M-u}\right)\Bigg|_{x_0}^{x(t)} \\ &= \frac{1}{M}\log\left(\frac{x(t)}{M-x(t)}\right) - \frac{1}{M}\log\left(\frac{x_0}{M-x_0}\right) \\ &= \frac{1}{M}\log\left(\left(\frac{x(t)}{M-x(t)}\right)\left(\frac{M-x_0}{x_0}\right)\right).\end{aligned}$$

Here we have used the fact that $x(t) > 0$ for all t and the assumption that we are working with values of t for which $x(t) < M$ to avoid the need for absolute values. We will see below that in fact the latter assumption holds for all t. Combining with (6.3.14), we have

$$\frac{1}{M}\log\left(\left(\frac{x(t)}{M-x(t)}\right)\left(\frac{M-x_0}{x_0}\right)\right) = \frac{1}{M}t.$$

Multiply both sides by M and the applying the exponential function gives us

$$\left(\frac{x(t)}{M-x(t)}\right)\left(\frac{M-x_0}{x_0}\right) = e^{\alpha t}.$$

Letting

$$\beta = \frac{M-x_0}{x_0}, \tag{6.3.17}$$

we have

$$\beta x(t) = e^{\alpha t}(M - x(t)).$$

Hence

$$\beta x(t) + x(t)e^{\alpha t} = Me^{\alpha t},$$

so

$$(\beta + e^{\alpha t})x(t) = Me^{\alpha t}.$$

This gives us

$$x(t) = \frac{Me^{\alpha t}}{\beta + e^{\alpha t}},$$

or, after dividing through by $e^{\alpha t}$,

$$x(t) = \frac{M}{1+\beta e^{-\alpha t}}. \tag{6.3.18}$$

Note that since $1+\beta e^{-\alpha t}>1$ for all t, we have, as we assumed above, $x(t)<M$ for all t. If we substitute back in the value for β, we have, finally,

$$x(t)=\frac{x_0M}{x_0+(M-x_0)e^{-\alpha t}}. \tag{6.3.19}$$

Note that

$$\lim_{t\to\infty}x(t)=\lim_{t\to\infty}\frac{x_0M}{x_0+(M-x_0)e^{-\alpha t}}=\frac{x_0M}{x_0}=M, \tag{6.3.20}$$

showing that the population, although never exceeding M, will nevertheless approach M asymptotically.

Example The population of the United States was 179.3 million in 1960, 203.3 million in 1970, and 226.5 million in 1980. Let $x(t)$ represent the population, in millions, of the United States t years after 1960. To fit the logistic model to this data, we need to find constants α and M so that

$$x(t)=\frac{179.3M}{179.3+(M-179.3)e^{-\alpha t}}$$

for $t=10$ and $t=20$ (note that we already have $x(0)=179.3$). That is, we need to solve the equations

$$\begin{aligned}203.3=x(10)&=\frac{179.3M}{179.3+(M-179.3)e^{-10\alpha}}\\226.5=x(20)&=\frac{179.3M}{179.3+(M-179.3)e^{-20\alpha}}\end{aligned}$$

for α and M. Working with the first equation, we have

$$(203.3)(179.3)+(203.3)(M-179.3)e^{-10\alpha}=179.3M,$$

which gives us

$$(203.3)(M-179.3)e^{-10\alpha}=179.3(M-203.3).$$

Thus

$$e^{-10\alpha}=\frac{179.3(M-203.3)}{203.3(M-179.3)}. \tag{6.3.21}$$

Similarly, the second equation gives us

$$e^{-20\alpha}=\frac{179.3(M-226.5)}{226.5(M-179.3)}.$$

Now

$$e^{-20\alpha}=\left(e^{-10\alpha}\right)^2,$$

so we have

$$\frac{179.3(M-226.5)}{226.5(M-179.3)}=\left(\frac{179.3(M-203.3)}{203.3(M-179.3)}\right)^2.$$

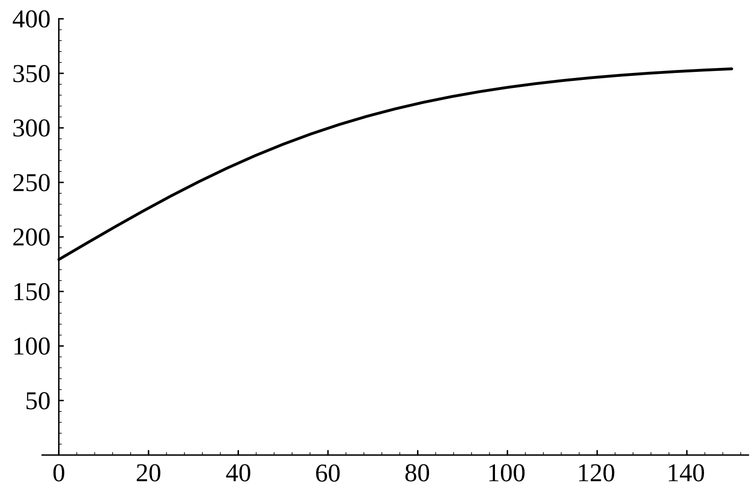


Figure 6.3.2 Inhibited growth model for the United States (1960-2110)

Thus
$$(203.3)^2(M-226.5)(M-179.3)=(179.3)(226.5)(M-203.3)^2,$$
which, when expanded, gives us
$$(203.3)^2(M^2-405.8M+(179.3)(226.5))=(179.3)(226.5)(M^2-406.6M+(203.3)^2).$$
Hence
$$719.44M^2-259,459.59M=0.$$
Since $M \neq 0$, the desired solution must be
$$M=\frac{259,459.59}{719.44}=360.6,$$
rounded to the first decimal place. Substituting this value for M into (6.3.21), we have
$$e^{-10\alpha}=\frac{179.3(360.6-203.3)}{203.3(360.6-179.3)},$$
and so
$$\alpha=-\frac{1}{10}\log\left(\frac{179.3(360.6-203.3)}{203.3(360.6-179.3)}\right)=0.02676,$$
rounded to five decimal places. Thus we have
$$x(t)=\frac{(179.3)(360.6)}{179.3+(360.6-179.3)e^{-0.02676t}}=\frac{64,655.6}{179.3+181.3e^{-0.02676t}}.$$
For example, this model would predict a population in 1990 of
$$x(30)=\frac{64,655.6}{179.3+181.3e^{-(0.02676)(30)}}=248.2 \text{ million}$$
and a population in 2000 of
$$x(40)=\frac{64,655.6}{179.3+181.3e^{-(0.02676)(40)}}=267.8 \text{ million}.$$

The 1990 prediction is very close to the actual population in 1990, which was approximately 249.6 million, and the prediction for the year 2000 is very close to the Census Bureau's prediction of 268.3 million. Recall that the uninhibited growth model for the United States, based on population data for 1970 and 1980, predicted a population of 281.1 million for the year 2000. To see how different the two models are, you should compare the graph for the uninhibited growth model, shown in Figure 6.1.3, with the graph of the inhibited growth model, shown in Figure 6.3.2.

Problems

1. Evaluate the following limits.

   (a) $\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ (b) $\displaystyle\lim_{n\to\infty}\left(1+\frac{5}{n}\right)^n$

   (c) $\displaystyle\lim_{n\to\infty}\left(1-\frac{2}{n}\right)^n$ (d) $\displaystyle\lim_{n\to\infty}\left(1-\frac{3}{n}\right)^n$

   (e) $\displaystyle\lim_{n\to\infty}\left(1+\frac{2}{n^2}\right)^n$ (f) $\displaystyle\lim_{n\to\infty}\left(1-\frac{4}{n}+\frac{1}{n^2}\right)^n$

2. Suppose \$1500 is deposited in a bank account paying 5.5% interest. Find the amount in the account after 5 years if the interest is compounded (a) quarterly, (b) monthly, (c) weekly, (d) daily, and (e) continuously.

3. Suppose \$4500 is deposited in a bank account paying 6.25% interest. Find the amount in the account after 7 years if the interest is compounded (a) quarterly, (b) monthly, (c) weekly, (d) daily, and (e) continuously.

4. A customer deposits P dollars in a bank account. Which is more advantageous to the bank customer: 5% interest compounded continuously, 5.25% interest compounded monthly, or 5.5% interest compounded quarterly?

5. Let $A(x)$ be amount in a bank account after one year if \$1000 is deposited at 5% interest compounded x times per year.

   (a) Plot $A(x)$ on the interval $[1, 100]$.

   (b) Show that $A(x)$ is an increasing function on $(1, \infty)$.

6. A bone fossil is determined to have 5% of its original carbon-14 remaining. How old is the fossil?

7. Suppose an analysis of a bone fossil shows that it has between 4% and 6% percent of its original carbon-14. Find upper and lower bounds for the age of the fossil.

8. Carbon-11 has a half-life of 20 minutes. Given an initial amount x_0, find $x(t)$, the amount of carbon-11 remaining after t minutes. How long will it take before there is only 10% left? How long until only 5% remains?

9. Plutonium-239, the fuel for nuclear reactors, has a half-life of 24,000 years. Given an initial amount x_0, find $x(t)$, the amount of plutonium-239 remaining after t years. How

many years will it take before there is only 10% left? How many years until only 5% remains?

10. If 1% of a certain radioactive element decays in one year, what is the half-life of the element?

11. (a) In 1960 the population of the United States was 179.3 million and in 1970 it was 203.3 million. If $y(t)$ represents the size of the population of the United States t years after 1960, find an expression for $y(t)$ using an uninhibited growth model.
    (b) Use $y(t)$ from part (a) to predict the population of the United States in 1980, 1990, and 2000. How accurate are these predictions?
    (c) Let $x(t)$ be the population of the United States t years after 1960 as given by the inhibited growth model used in the last example in the section. Compare $y(t)$ to $x(t)$ by graphing them together over the interval $[0, 200]$.

12. The population of the United States was 3,929,214 in 1790, 5,308,483 in 1800, and 7,239,881 in 1810.
    (a) Let $y(t)$ be the population of the United States t years after 1790 as predicted by an uninhibited growth model using the data from 1790 and 1800. Graph $y(t)$ over the interval $[0, 100]$ and find the predicted population for 1810, 1820, 1840, 1870, 1900, and 1990. How accurate are these predictions?
    (b) Let $x(t)$ be the population of the United States t years after 1790 as predicted by an inhibited growth model using the data from 1790, 1800, and 1810. Graph $x(t)$ over the interval $[0, 200]$ and find the predicted population for 1820, 1840, 1870, 1900, and 1990. How accurate are these predictions? How do they compare with your results in a part (a)? What does this model predict for the eventual limiting population of the United States?

13. The population of the United States was 75,994,575 in 1900, 91,972,266 in 1910, and 105,710,620 in 1920.
    (a) Let $y(t)$ be the population of the United States t years after 1900 as predicted by an uninhibited growth model using the data from 1900 and 1910. Graph $y(t)$ over the interval $[0, 100]$ and find the predicted population for 1920, 1930, 1950, 1970, 1990, and 2000. How accurate are these predictions? Using this model, in what year will the population be twice what it was in 1900?
    (b) Let $x(t)$ be the population of the United States t years after 1900 as predicted by an inhibited growth model using the data from 1900, 1910, and 1920. Graph $x(t)$ over the interval $[0, 200]$ and find the predicted population for 1930, 1950, 1970, 1990, and 2000. How accurate are these predictions? How do they compare with your results in a part (a)? What does this model predict for the eventual limiting population of the United States? Using this model, in what year will the population be twice what it was in 1900?

14. Show that the graph of a solution to the logistic differential equation

$$\dot{x}(t) = \frac{\alpha}{M} x(t)(M - x(t)),$$

with $0 < x(0) < M$, is concave up when

$$x(t) > \frac{M}{2}$$

and concave down when

$$x(t) > \frac{M}{2}.$$

What does this say about the rate of growth of the population?

Difference Equations to **D**ifferential Equations

Section 6.4

Integration of Rational Functions

In this section we will take a more detailed look at the use of partial fraction decompositions in evaluating integrals of rational functions, a technique we first encountered in the inhibited growth model example in the previous section. However, we will not be able to complete the story until after the introduction of the inverse tangent function in Section 6.5.

We begin with a few examples to illustrate how some integration problems involving rational functions may be simplified either by a long division or by a simple substitution.

Example To evaluate $\int \frac{x^2}{x+1}\, dx$, we first perform a long division of $x+1$ into x^2 to obtain

$$\frac{x^2}{x+1} = x - 1 + \frac{1}{x+1}.$$

Then

$$\int \frac{x^2}{x+1}\, dx = \int \left(x - 1 + \frac{1}{x+1} \right) dx = \frac{1}{2}x^2 - x + \log|x+1| + c.$$

Example To evaluate $\int \frac{2x+1}{x^2+x}\, dx$, we make the substitution

$$\begin{aligned} u &= x^2 + x \\ du &= (2x+1)dx. \end{aligned}$$

Then

$$\int \frac{2x+1}{x^2+x}\, dx = \int \frac{1}{u}\, du = \log|u| + c = \log|x^2+x| + c.$$

Example To evaluate $\int \frac{x}{x+1}\, dx$, we perform a long division of $x+1$ into x to obtain

$$\frac{x}{x+1} = 1 - \frac{1}{x+1}.$$

Then

$$\int \frac{x}{x+1}\, dx = \int \left(1 - \frac{1}{x+1} \right) dx = x - \log|x+1| + c.$$

Alternatively, we could evaluate this integral with the substitution

$$\begin{aligned} u &= x + 1 \\ du &= dx. \end{aligned}$$

With this substitution, $x = u - 1$, so we have

$$\begin{aligned} \int \frac{x}{x+1}\,dx &= \int \frac{u-1}{u}\,du \\ &= \int \left(1 - \frac{1}{u}\right)\,du \\ &= u - \log|u| + c \\ &= x + 1 - \log|x+1| + c. \end{aligned}$$

Note that this is the same answer we obtained above, although with a different constant of integration.

Partial fraction decomposition: Distinct linear factors

Now we consider the general problem of evaluating

$$\int \frac{f(x)}{g(x)}\,dx$$

where both f and g are polynomials. We will assume that the degree of g is less than the degree of f. As illustrated in the first and third examples above, if this is not the case, we can first perform a long division to simplify the quotient into the form of a polynomial plus a remainder term which is a rational function with numerator of degree less than the denominator. To begin we will suppose that g factors completely into n distinct linear factors. That is, suppose there are constants $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ such that

$$g(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_nx + b_n), \tag{6.4.1}$$

where the factors on the right are all distinct. From a theorem of linear algebra, which we will not attempt to prove here, there exist constants $A_1, A_2, \ldots, A_n$ such that

$$\frac{f(x)}{g(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_n}{a_nx + b_n}. \tag{6.4.2}$$

The expression on the right of (6.4.2) is called the *partial fraction decomposition* of $\dfrac{f(x)}{g(x)}$. Once the constants $A_1, A_2, \ldots, A_n$ are determined, the evaluation of

$$\int \frac{f(x)}{g(x)}\,dx$$

becomes a routine problem. The next examples will illustrate one method for finding these constants.

Example To evaluate $\int \frac{1}{(x-2)(x-3)}\,dx$, we need to find constants A and B such that

$$\frac{1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}.$$

Combining the terms on the right, we have

$$\frac{1}{(x-2)(x-3)} = \frac{A(x-3)+B(x-2)}{(x-2)(x-3)}.$$

Now two rational functions with equal denominators are equal only if their numerators are also equal; hence we must have

$$1 = A(x-3) + B(x-2)$$

for all values of x. In particular, for $x = 2$ we obtain

$$1 = -A,$$

from which it follows that $A = -1$, and for $x = 3$ we have

$$1 = B.$$

Thus

$$\frac{1}{(x-2)(x-3)} = -\frac{1}{x-2} + \frac{1}{x-3},$$

so

$$\begin{aligned}\int \frac{1}{(x-2)(x-3)}\,dx &= -\int \frac{1}{x-2}\,dx + \int \frac{1}{x-3}\,dx \\ &= -\log|x-2| + \log|x-3| + c.\end{aligned}$$

Example To evaluate $\int \frac{3x}{(x+5)(2x-1)}\,dx$, we need to find constants A and B such that

$$\frac{3x}{(x+5)(2x-1)} = \frac{A}{x+5} + \frac{B}{2x-1}.$$

Combining the terms on the right, we have

$$\frac{3x}{(x+5)(2x-1)} = \frac{A(2x-1)+B(x+5)}{(x+5)(2x-1)}.$$

As before, it follows that

$$3x = A(2x-1) + B(x+5)$$

for all values of x. In particular, for $x = -5$ we obtain

$$-15 = -11A,$$

from which it follows that

$$A = \frac{15}{11},$$

and for $x = \frac{1}{2}$ we have

$$\frac{3}{2} = \frac{11}{2}B,$$

from which it follows that

$$B = \frac{3}{11}.$$

Hence

$$\frac{3x}{(x+5)(2x-1)} = \frac{15}{11}\frac{1}{x+5} + \frac{3}{11}\frac{1}{2x-1},$$

so

$$\begin{aligned}\int \frac{1}{(x+5)(2x-1)}\,dx &= \frac{15}{11}\int \frac{1}{x+5}\,dx + \frac{3}{11}\frac{1}{2x-1}\,dx \\ &= \frac{15}{11}\log|x+5| + \frac{3}{22}\log|2x-1| + c.\end{aligned}$$

Partial fraction decomposition: Repeated linear factors

Returning to the general problem of evaluating

$$\int \frac{f(x)}{g(x)}\,dx,$$

where f and g are both polynomials and the degree of f is less than the degree of g, we will now consider the case where g factors completely into linear factors, allowing for the possibility that one or more of these factors may be repeated. Specifically, suppose the factor $ax + b$ occurs n times in the factorization of g. Then the partial fraction decomposition of $\frac{f(x)}{g(x)}$ must contain a sum of terms of the form

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}, \tag{6.4.3}$$

for some constants $A_1, A_2, \ldots, A_n$, in addition to similar terms for every other factor of g. This is best illustrated in an example.

Example To evaluate $\frac{x+1}{(x-1)^3(x-2)}\,dx$, we need to find constants A, B, C, and D such that

$$\frac{x+1}{(x-1)^3(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{x-2}. \tag{6.4.4}$$

That is, this partial fraction decomposition contains three terms corresponding to the factor $x-1$, since it is repeated three times, and only one term corresponding to the factor $x-2$, since it occurs only once. Moreover, the degrees of the denominators of the terms for $x-1$ increase from 1 to 3. Now combining the terms on the right of (6.4.4), we have

$$\frac{x+1}{(x-1)^3(x-2)} = \frac{A(x-1)^2(x-2)+B(x-1)(x-2)+C(x-2)+D(x-1)^3}{(x-1)^3(x-2)}.$$

Again, it follows that

$$x+1 = A(x-1)^2(x-2)+B(x-1)(x-2)+C(x-2)+D(x-1)^3 \tag{6.4.5}$$

for all values of x. However, because of the repeated factors, we cannot choose values for x which will isolate each of the constants one at a time as we did in the previous examples. Instead, we will illustrate another technique for finding the constants. By multiplying out (6.4.5) and collecting terms, we obtain

$$\begin{aligned} x+1 &= A(x^3-4x^2+5x-2)+B(x^2-3x+2)+C(x-2)+D(x^3-3x^2+3x-1) \\ &= (A+D)x^3+(-4A+B-3D)x^2+(5A-3B+C+3D)x-2A+2B-2C-D \end{aligned}$$

for all values of x. Since two polynomials are equal only if they have equal coefficients, we can equate the coefficients of $x+1$ with the coefficients of the polynomial on the right to obtain the four equations

$$\begin{aligned} A+D&=0 \\ -4A+B-3D&=0 \\ 5A-3B+C+3D&=1 \\ -2A+2B-2C-D&=1. \end{aligned} \tag{6.4.6}$$

From the first equation we learn that

$$D=-A.$$

Substituting this into the second equation gives us

$$B=A.$$

Substituting both of these values into the third equation results in

$$C=A+1.$$

Finally, substituting for D, B, and C in the fourth equation gives us

$$-2A+2A-2(A+1)+A=1,$$

which gives us $A = -3$. Hence $B = -3$, $C = -2$, and $D = 3$. Thus

$$\begin{aligned}\int \frac{x+1}{(x-1)^3(x-2)}\,dx &= -\int \frac{3}{(x-1)}\,dx - \int \frac{3}{(x-1)^2}\,dx \\ &\qquad - \int \frac{2}{(x-1)^3}\,dx + \int \frac{3}{x-2}\,dx \\ &= -3\log|x-1| + \frac{3}{x-1} + \frac{1}{(x-1)^2} + 3\log|x-2| + c.\end{aligned}$$

Note that in solving for A, B, C, and D, we could have first substituted $x = 1$ and $x = 2$ into (6.4.5) to obtain values for C and D, respectively. These values could have then been used to simplify (6.4.6) before solving for A and B.

The Fundamental Theorem of Algebra states that every polynomial factors into a product of linear factors and irreducible quadratic factors; hence, to complete the story of integrating rational functions, we need to consider the case where the factorization of the denominator includes irreducible quadratic factors. However, we will learn in Section 6.5 that for an irreducible quadratic polynomial g,

$$\int \frac{1}{g(x)}\,dx$$

involves the inverse tangent function. Thus we need to discuss the inverse trigonometric functions before continuing the story of integrating rational functions.

Problems

1. Evaluate each of the following integrals.

   (a) $\displaystyle\int \frac{x-1}{x}\,dx$
   (b) $\displaystyle\int \frac{x}{x-1}\,dx$
   (c) $\displaystyle\int \frac{3x^2}{x-2}\,dx$
   (d) $\displaystyle\int \frac{x^3+1}{x+2}\,dx$
   (e) $\displaystyle\int \frac{4x+1}{2x^2+x-3}\,dx$
   (f) $\displaystyle\int \frac{x+2}{x^2+4x+1}\,dx$

2. Evaluate each of the following integrals.

   (a) $\displaystyle\int \frac{1}{(x+2)(x-4)}\,dx$
   (b) $\displaystyle\int \frac{3}{(x-3)(x-7)}\,dx$
   (c) $\displaystyle\int \frac{3x}{(2x+3)(x+1)}\,dx$
   (d) $\displaystyle\int \frac{3x+1}{(x-2)(x+3)}\,dx$
   (e) $\displaystyle\int \frac{x}{x^2+x-6}\,dx$
   (f) $\displaystyle\int \frac{3x}{(x+2)(x-3)(x+1)}\,dx$
   (g) $\displaystyle\int \frac{3}{x^2+5x+6}\,dx$
   (h) $\displaystyle\int \frac{3x+2}{(x^2-4)(x^2-9)}\,dx$

3. Evaluate each of the following integrals.

   (a) $\displaystyle\int \frac{1}{(x-1)^2}\,dx$ (b) $\displaystyle\frac{1}{(x-1)^2(x+2)}\,dx$

   (c) $\displaystyle\int \frac{x}{x^2+2x+1}\,dx$ (d) $\displaystyle\int \frac{3x+1}{(x+2)^3(x-1)}\,dx$

   (e) $\displaystyle\int \frac{5}{(x+2)^3}\,dx$ (f) $\displaystyle\int \frac{4}{(x^2-4)^2}\,dx$

   (g) $\displaystyle\int \frac{3x^2}{(x+1)^2(x-3)}\,dx$ (h) $\displaystyle\int \frac{5x-1}{(2x+1)^2(x+2)}\,dx$

4. Evaluate each of the following integrals.

   (a) $\displaystyle\int \frac{1}{(3x+2)^2}\,dx$ (b) $\displaystyle\int \frac{3}{x^2+7x+10}\,dx$

   (c) $\displaystyle\int \frac{9x^2-4x}{3x^3-2x^2+5}\,dx$ (d) $\displaystyle\int \frac{2x}{(x^2-1)(x^2-4)}\,dx$

   (e) $\displaystyle\int_{-1}^{1} \frac{1}{x^2-4}\,dx$ (f) $\displaystyle\int_{0}^{1} \frac{1}{x^2-x-6}\,dx$

   (g) $\displaystyle\int \frac{4x+5}{(x-2)^2(x+5)}\,dx$ (h) $\displaystyle\int \frac{x^3}{x^2-1}\,dx$

5. Solve the differential equation

$$\dot{x}(t) = (x(t)-1)(x(t)+1)$$

   using the method used to solve the logistic differential equation in Section 6.3. Assume $x(0) = 0$ and $-1 < x(t) < 1$ for all t.

Difference Equations to **D**ifferential Equations

Section 6.5

Inverse Trigonometric Functions

In this section we will introduce the inverse trigonometric functions. We will begin with the inverse tangent function since, as indicated in Section 6.4, we need it to complete the story of the integration of rational functions.

Strictly speaking, the tangent function does not have an inverse. Recall that in order for a function f to have an inverse function, for every y in the range of f there must be exactly one x in the domain of f such that $f(x) = y$. This is false for the tangent function since, for example, both $\tan(0) = 0$ and $\tan(\pi) = 0$. In fact, since the tangent function is periodic with period π, if $\tan(x) = y$, then $\tan(x + n\pi) = y$ for any integer n. However, the tangent function is increasing on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, taking on every value in its range $(-\infty, \infty)$ exactly once. Hence we may define an inverse for the tangent function if we consider it with the restricted domain $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. That is, we will define an inverse tangent function so that it takes on only values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Definition The *arc tangent function*, with value at x denoted by either $\arctan(x)$ or $\tan^{-1}(x)$, is the inverse of the tangent function with restricted domain $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

In other words, for $-\frac{\pi}{2} < y < \frac{\pi}{2}$,

$$y = \tan^{-1}(x) \text{ if and only if } \tan(y) = x. \tag{6.5.1}$$

For example, $\tan^{-1}(0) = 0$, $\tan^{-1}(1) = \frac{\pi}{4}$, and $\tan^{-1}(-1) = -\frac{\pi}{4}$. In particular, note that even though $\tan(\pi) = 0$, $\tan^{-1}(0) = 0$ since 0 is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, but π is not between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

The domain of the arc tangent function is $(-\infty, \infty)$, the range of the tangent function, and the range of the arc tangent function is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the domain of the restricted tangent function. Moreover, since

$$\lim_{x \to \frac{\pi}{2}^-} \tan(x) = \infty$$

and

$$\lim_{x \to -\frac{\pi}{2}^+} \tan(x) = -\infty,$$

we have

$$\lim_{x \to \infty} \tan^{-1}(x) = \frac{\pi}{2} \tag{6.5.2}$$

and

$$\lim_{x \to -\infty} \tan^{-1}(x) = -\frac{\pi}{2}. \tag{6.5.3}$$

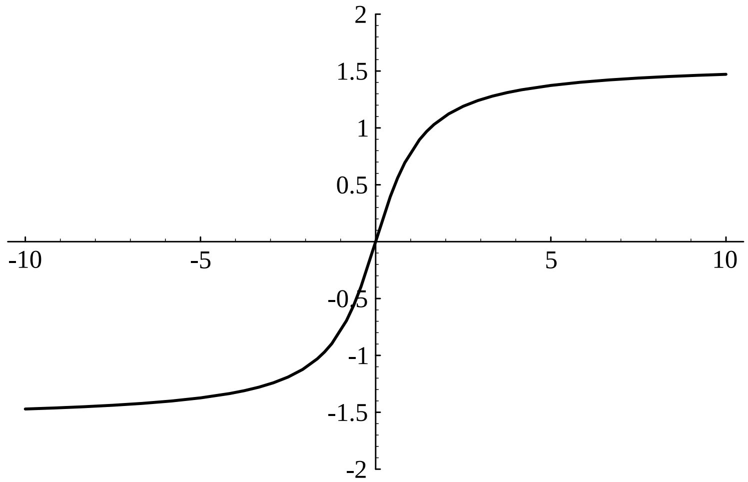


Figure 6.5.1 Graph of $y = \tan^{-1}(x)$

Hence $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$ are horizontal asymptotes for the graph of $y = \tan^{-1}(x)$, as shown in Figure 6.5.1.

To differentiate the arc tangent function we imitate the method we used to differentiate the logarithm function. Namely, if $y = \tan^{-1}(x)$, then $\tan(y) = x$, so

$$\frac{d}{dx}\tan(y) = \frac{d}{dx}x.$$

Hence

$$\sec^2(y)\frac{dy}{dx} = 1,$$

from which it follows that

$$\frac{dy}{dx} = \frac{1}{\sec^2(y)}.$$

Now

$$\sec^2(y) = 1 + \tan^2(y) = 1 + x^2,$$

so we have

$$\frac{dy}{dx} = \frac{1}{1 + x^2}.$$

Hence we have demonstrated the following proposition.

Proposition

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1 + x^2}. \tag{6.5.4}$$

As a consequence of the proposition, we also have

$$\int \frac{1}{1 + x^2}\, dx = \tan^{-1}(x) + c. \tag{6.5.5}$$

Note that $1+x^2$ is an irreducible quadratic polynomial. We will see more examples of this type in the following examples.

Example Using the chain rule, we have

$$\frac{d}{dx}\tan^{-1}(4x^2) = \frac{8x}{1+16x^4}.$$

Example Evaluating $\int \tan^{-1}(x)dx$ is similar to evaluating $\int \log(x)dx$. That is, we will use integration by parts with

$$\begin{aligned} u &= \tan^{-1}(x) & dv &= dx \\ du &= \frac{1}{1+x^2}\,dx & v &= x. \end{aligned}$$

Then

$$\int \tan^{-1}(x)dx = x\tan^{-1}(x) - \int \frac{x}{1+x^2}\,dx.$$

Using the substitution

$$\begin{aligned} u &= 1+x^2 \\ du &= 2xdx, \end{aligned}$$

we have $\frac{1}{2}du = xdx$, from which it follows that

$$\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\log|u| + c = \frac{1}{2}\log(1+x^2) + c.$$

Thus

$$\int \tan^{-1}(x)dx = x\tan^{-1}(x) - \frac{1}{2}\log(1+x^2) + c.$$

Example To evaluate $\int \frac{1}{1+4x^2}\,dx$, we make the substitution

$$\begin{aligned} u &= 2x \\ du &= 2dx. \end{aligned}$$

Then $\frac{1}{2}du = dx$, so

$$\int \frac{1}{1+4x^2}\,dx = \frac{1}{2}\int \frac{1}{1+u^2}\,du = \frac{1}{2}\tan^{-1}(u) + c = \frac{1}{2}\tan^{-1}(2x) + c.$$

Example To evaluate $\int \frac{1}{x^2+x+1}\,dx$, we first note that x^2+x+1 does not factor, that is, is irreducible, and so we cannot use a partial fraction decomposition. In general, a quadratic polynomial $ax^2 + bx + c$ is irreducible if $b^2 - 4ac < 0$ since, in that case, the quadratic formula yields complex solutions for the equation $ax^2+bx+c = 0$. For x^2+x+1 we have $b^2 - 4ac = -3$. In this case it is helpful to simplify the function algebraically by completing the square of the denominator, thus making the problem similar to the previous example. That is, since

$$x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4},$$

we have

$$\int \frac{1}{x^2 + x + 1}\, dx = \int \frac{1}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}}\, dx = \frac{4}{3} \int \frac{1}{\frac{4}{3}\left(x + \frac{1}{2}\right)^2 + 1}\, dx.$$

Now we can make the substitution

$$\begin{aligned} u &= \sqrt{\frac{4}{3}}\left(x + \frac{1}{2}\right) \\ du &= \sqrt{\frac{4}{3}}\, dx. \end{aligned}$$

Then $\sqrt{\frac{3}{4}}\, du = dx$, so

$$\begin{aligned} \int \frac{1}{x^2 + x + 1}\, dx &= \frac{4}{3}\sqrt{\frac{3}{4}} \int \frac{1}{u^2 + 1}\, du \\ &= \sqrt{\frac{4}{3}} \tan^{-1}(u) + c \\ &= \sqrt{\frac{4}{3}} \tan^{-1}\left(\sqrt{\frac{4}{3}}\left(x + \frac{1}{2}\right)\right) + c. \end{aligned}$$

Partial fraction decomposition: Irreducible quadratic factors

The last two examples illustrate techniques that we may use to evaluate the integral of a rational function with an irreducible quadratic polynomial in the denominator. With this we are now in a position to consider the final case of partial fraction decomposition. Specifically, suppose we want to evaluate

$$\int \frac{f(x)}{g(x)}\, dx,$$

where f and g are both polynomials and the degree of f is less than the degree of g. Moreover, suppose that $(ax^2 + bx + c)^n$ is a factor of g, where n is a positive integer and

ax^2+bx+c is irreducible. Then the partial fraction decomposition of $\dfrac{f(x)}{g(x)}$ must contain a sum of terms of the form

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_nx+B_n}{(ax^2+bx+c)^n}, \tag{6.5.6}$$

where $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ are constants. Note that the terms in the partial fraction decomposition corresponding to an irreducible quadratic factor differ from the terms for a linear factor in that the numerators of the terms in (6.5.6) need not be constants, but may be first degree polynomials themselves. As before, this is best illustrated with an example.

Example To evaluate $\displaystyle\int \frac{1+x}{x(1+x^2)}\,dx$ we need to find constants A, B, and C such that

$$\frac{1+x}{x(1+x^2)}=\frac{A}{x}+\frac{Bx+C}{1+x^2}.$$

Combining the terms on the right, we have

$$\frac{1+x}{x(1+x^2)}=\frac{A(1+x^2)+(Bx+C)x}{x(1+x^2)}.$$

Hence

$$1+x=A(1+x^2)+(Bx+C)x=(A+B)x^2+Cx+A.$$

Equating the coefficients of the polynomials on the left and right gives us the system of equations

$$\begin{aligned} A+B&=0,\\ C&=1,\\ A&=1. \end{aligned}$$

Thus $B=-1$ and

$$\frac{1+x}{x(1+x^2)}=\frac{1}{x}+\frac{1-x}{1+x^2}=\frac{1}{x}+\frac{1}{1+x^2}-\frac{x}{1+x^2}.$$

Hence

$$\begin{aligned}\int \frac{1+x}{x(1+x^2)}\,dx &= \int \frac{1}{x}\,dx+\int\frac{1}{1+x^2}\,dx-\int\frac{x}{1+x^2}\,dx\\ &=\log|x|+\tan^{-1}(x)-\frac{1}{2}\log(1+x^2)+c.\end{aligned}$$

where the final integral follows from the substitution $u=1+x^2$ as in an earlier example.

If, unlike this example, the partial fraction decomposition of $\dfrac{f(x)}{g(x)}$ results in a term of the form

$$\frac{Ax+B}{(ax^2+bx+c)^n},$$

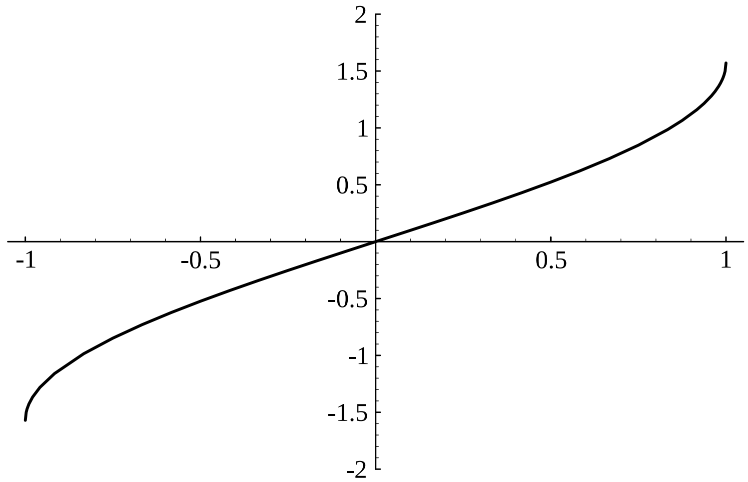


Figure 6.5.2 Graph of $y = \sin^{-1}(x)$

where $n > 1$ and $ax^2 + bx + c$ is irreducible, then the integration may still be difficult to carry out, perhaps even requiring some of the ideas of trigonometric substitutions that we will discuss in the next section. However, there is a limit to what should be done without the aid of a computer, or at least a table of integrals. There is a point after which some integrations become so complicated and time-consuming that in practice they should be given to a computer algebra system.

The inverse sine function

The remaining trigonometric functions all have inverses when their domains are restricted to appropriate intervals. Since the sine function is increasing on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, taking on every value in its range $[1, 1]$ exactly once, we obtain an inverse for the sine function by restricting its domain to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Definition The *arc sine function*, with value at x denoted by either $\arcsin(x)$ or $\sin^{-1}(x)$, is the inverse of the sine function with restricted domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

In other words, for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$,

$$y = \sin^{-1}(x) \text{ if and only if } \sin(y) = x. \tag{6.5.7}$$

For example, $\sin^{-1}(0) = 0$, $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$, $\sin^{-1}(1) = \frac{\pi}{2}$, and $\sin^{-1}(-1) = -\frac{\pi}{2}$. Note that the domain of the arc sine function is $[-1, 1]$, the range of the sine function, and the range of the arc sine function is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the domain of the restricted sine function.The graph of $y = \sin^{-1}(x)$ is shown in Figure 6.5.2.

To find the derivative of the arc sine function, let $y = \sin^{-1}(x)$. Then $\sin(y) = x$, so

$$\frac{d}{dx}\sin(y) = \frac{d}{dx}x.$$

Hence

$$\cos(y)\frac{dy}{dx} = 1,$$

and so

$$\frac{dy}{dx} = \frac{1}{\cos(y)}.$$

Now

$$\cos^2(y) = 1 - \sin^2(y) = 1 + x^2,$$

so $\cos(y) = \pm\sqrt{1 - x^2}$. Since $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, $\cos(y) \geq 0$. Thus $\cos(y) = \sqrt{1 - x^2}$, and

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}.$$

Proposition

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1 - x^2}}. \tag{6.5.8}$$

As a consequence of this proposition, we also have

$$\int \frac{1}{\sqrt{1 - x^2}}\, dx = \sin^{-1}(x) + c. \tag{6.5.9}$$

Example Using the product and chain rules,

$$\frac{d}{dx}\left(x \sin^{-1}(2x)\right) = \frac{2x}{\sqrt{1 - 4x^2}} + \sin^{-1}(2x).$$

Example To evaluate $\displaystyle\int \frac{1}{\sqrt{4 - x^2}}\, dx$, we first note that

$$\frac{1}{4 - x^2} = \frac{1}{\sqrt{4\left(1 - \frac{x^2}{4}\right)}} = \frac{1}{2}\frac{1}{\sqrt{1 - \frac{x^2}{4}}}.$$

Then the substitution

$$\begin{aligned} u &= \frac{x}{2} \\ du &= \frac{1}{2}\, dx \end{aligned}$$

gives us

$$\int \frac{1}{\sqrt{4 - x^2}}\, dx = \int \frac{1}{\sqrt{1 - u^2}}\, du = \sin^{-1}(u) + c = \sin^{-1}\left(\frac{x}{2}\right) + c.$$

The inverse secant function

Defining an inverse for the secant function is slightly more complicated than defining the arc tangent or arc sine functions. On the interval $\left[0, \frac{\pi}{2}\right)$, the secant function is increasing

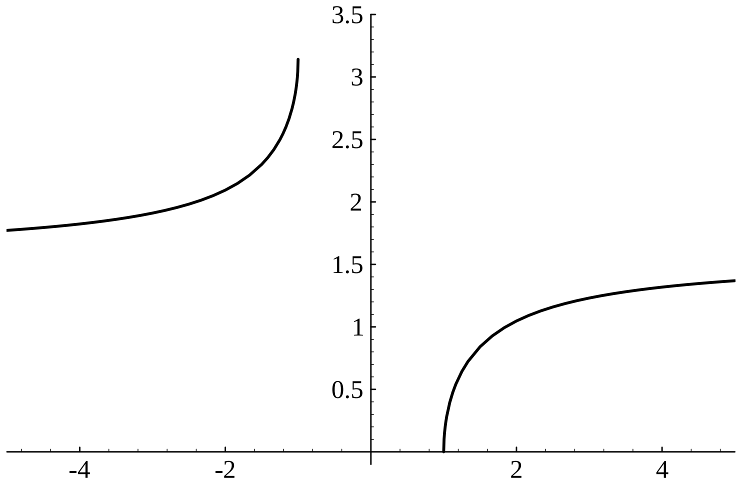


Figure 6.5.3 Graph of $y = \sec^{-1}(x)$

and takes on all values in the interval $[1, \infty)$; on the interval $\left(\frac{\pi}{2}, \pi\right]$, the secant function is also increasing, taking on all values in the interval $(-\infty, 1]$. Hence between these two intervals the secant function takes on every value in its range exactly once. From these considerations we obtain the following definition.

Definition The *arc secant function*, with value at x denoted by either $\operatorname{arcsec}(x)$ or $\sec^{-1}(x)$, is the inverse of the secant function with domain restricted to the intervals $\left[0, \frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2}, \pi\right]$.

Thus for $0 \leq y < \frac{\pi}{2}$ or $\frac{\pi}{2} < y \leq \pi$,

$$y = \sec^{-1}(x) \text{ if and only if } \sec(y) = x. \tag{6.5.10}$$

For example, $\sec^{-1}(2) = \frac{\pi}{3}$, $\sec^{-1}(1) = 0$, $\sec^{-1}(-2) = \frac{2\pi}{3}$, and $\sec^{-1}(-1) = \pi$. Note that the domain of the arc secant function consists of the two intervals $(-\infty, -1]$ and $[1, \infty)$, the range of the secant function, and the range is composed of the two intervals $\left[0, \frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2}, \pi\right]$, the domain of the restricted secant function.

Since

$$\lim_{x \to \frac{\pi}{2}^-} \sec(x) = \infty$$

and

$$\lim_{x \to \frac{\pi}{2}^+} \sec(x) = -\infty,$$

it follows that

$$\lim_{x \to \infty} \sec^{-1}(x) = \frac{\pi}{2} \tag{6.5.11}$$

and

$$\lim_{x \to -\infty} \sec^{-1}(x) = \frac{\pi}{2}. \tag{6.5.12}$$

Thus the line $y = \frac{\pi}{2}$ is a horizontal asymptote for the graph of $y = \sec^{-1}(x)$ both as x goes to ∞ and as x goes to $-\infty$, as shown in Figure 6.5.3.

To find the derivative of the arc secant function, let $y = \sec^{-1}(x)$. Then $\sec(y) = x$, so

$$\frac{d}{dx}\sec(y) = \frac{d}{dx}x.$$

Hence

$$\sec(y)\tan(y)\frac{dy}{dx} = 1,$$

and so

$$\frac{dy}{dx} = \frac{1}{\sec(y)\tan(y)}.$$

Now $\sec(y) = x$ and

$$\tan^2(y) = \sec^2(y) - 1 = x^2 - 1.$$

Hence $\tan(y) = \pm\sqrt{x^2-1}$. If x is in $[1,\infty)$, then $0 \leq y < \frac{\pi}{2}$ and $\tan(y) \geq 0$; if x is in $(-\infty,-1]$, then $\frac{\pi}{2} < y \leq \pi$ and $\tan(y) \leq 0$. Thus

$$\sec(y)\tan(y) = \begin{cases} x\sqrt{x^2-1}, & \text{if } x \geq 1 \\ -x\sqrt{x^2-1}, & \text{if } x \leq -1. \end{cases}$$

Since $|x| = x$ when $x \geq 1$ and $|x| = -x$ when $x \leq -1$, it follows that

$$\sec(y)\tan(y) = |x|\sqrt{x^2-1}.$$

Hence

$$\frac{dy}{dx} = \frac{1}{|x|\sqrt{x^2-1}}.$$

Proposition

$$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}}. \tag{6.5.13}$$

Example Using the chain rule, we have

$$\frac{d}{dx}\sec^{-1}(3x) = \frac{3}{|3x|\sqrt{9x^2-1}} = \frac{1}{|x|\sqrt{9x^2-1}}.$$

We will leave the definition of inverse functions for the cotangent, cosine, and cosecant functions for the problems at the end of the section. In the next section we will see how the arc tangent, arc sine, and arc secant functions are useful in evaluating certain integrals; the arc cotangent, arc cosine, and arc cosecant functions could be used in similar roles, but, wherever they are used, we could just as well use arc tangent, arc sine, or arc secant. Hence the former, although useful in other situations, will not be as important for our present study as the latter.

Problems

1. Find the derivatives of each of the following functions.

   (a) $f(x) = x\tan^{-1}(x)$ (b) $g(t) = \tan^{-1}(3t^2)$

   (c) $g(x) = \dfrac{\sin^{-1}(3x)}{x}$ (d) $f(x) = 3x\sec^{-1}(5x)$

2. Evaluate each of the following.

   (a) $\tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$ (b) $\tan^{-1}(-\sqrt{3})$

   (c) $\sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$ (d) $\sec^{-1}\left(-\dfrac{2}{\sqrt{3}}\right)$

   (e) $\sin^{-1}\left(\sin\left(\dfrac{3\pi}{4}\right)\right)$ (f) $\sin\left(\sin^{-1}\left(-\dfrac{1}{\sqrt{2}}\right)\right)$

3. Evaluate the following integrals.

   (a) $\displaystyle\int \frac{1}{1+2x^2}\,dx$ (b) $\displaystyle\int \frac{4x}{2+x^2}\,dx$

   (c) $\displaystyle\int \frac{3}{x^2+4}\,dx$ (d) $\displaystyle\int \frac{5}{x^2+2x+3}\,dx$

   (e) $\displaystyle\int \frac{x}{x^2+4x+5}\,dx$ (f) $\displaystyle\int \frac{x+1}{x^2+2x+6}\,dx$

   (g) $\displaystyle\int_{-1}^{1} \frac{1}{1+x^2}\,dx$ (h) $\displaystyle\int_{0}^{\frac{1}{3}} \frac{1}{1+9x^2}\,dx$

4. Evaluate the following integrals.

   (a) $\displaystyle\int \frac{1}{x^3+x}\,dx$ (b) $\displaystyle\int \frac{2+x}{x(4x^2+1)}\,dx$

   (c) $\displaystyle\int \frac{1}{x^2(x^2+1)}\,dx$ (d) $\displaystyle\int \frac{1}{(x+1)(x^2+2)}\,dx$

   (e) $\displaystyle\int \sin^{-1}(x)dx$ (f) $\displaystyle\int \tan^{-1}(3x)dx$

   (g) $\displaystyle\int \frac{5}{\sqrt{1-9x^2}}\,dx$ (h) $\displaystyle\int \frac{1}{\sqrt{4-8x^2}}\,dx$

   (i) $\displaystyle\int \frac{3x}{\sqrt{1-x^2}}\,dx$ (j) $\displaystyle\int_{-2}^{2} \frac{1}{\sqrt{16-x^2}}\,dx$

5. The cosine function has an inverse, called the *arc cosine function*, if its domain is restricted to $[0, \pi]$. That is, for $0 \le y \le \pi$,

$$y = \cos^{-1}(x) \text{ if and only if } \cos(y) = x.$$

(a) Show that $\dfrac{d}{dx}\cos^{-1}(x) = -\dfrac{1}{\sqrt{1-x^2}}$.

(b) Show that $\sec^{-1}(x) = \cos^{-1}\left(\dfrac{1}{x}\right)$.

(c) Use the result from (b) to find $\dfrac{d}{dx}\sec^{-1}(x)$.

(d) Use the fact that

$$\frac{d}{dx}\sin^{-1}(x) = \frac{d}{dx}(-\cos^{-1}(x))$$

to show that

$$\sin^{-1}(x) + \cos^{-1}(x) = \frac{\pi}{2}$$

for all x in $[-1, 1]$.

6. The cotangent function has an inverse, called the *arc cotangent function*, if its domain is restricted to $(0, \pi)$. That is, for $0 < y < \pi$,

$$y = \cot^{-1}(x) \text{ if and only if } \cot(y) = x.$$

Show that $\dfrac{d}{dx}\cot^{-1}(x) = -\dfrac{1}{1+x^2}$.

7. The cosecant function has an inverse, called the *arc cosecant function*, if its domain is restricted to the intervals $\left[-\frac{\pi}{2}, 0\right)$ and $\left(0, \frac{\pi}{2}\right]$. That is, for $-\frac{\pi}{2} \le y < 0$ or $0 < y \le \frac{\pi}{2}$,

$$y = \csc^{-1}(x) \text{ if and only if } \csc(y) = x.$$

Show that $\dfrac{d}{dx}\csc^{-1}(x) = -\dfrac{1}{|x|\sqrt{x^2-1}}$.

8. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$.

9. (a) Use the fact that

$$\tan^{-1}(x) = \int_0^x \frac{1}{1+t^2}\, dt$$

to find the Taylor series expansion for $\tan^{-1}(x)$ about 0. On what interval does this series converge?

(b) Use your result in (a) and the fact that $\pi = 4\tan^{-1}(1)$ to approximate π with an error of no more than 0.001.

10. (a) Show that

$$\frac{d}{dx}\tan^{-1}\left(\frac{1}{x}\right) = -\frac{1}{1+x^2}.$$

(b) Use the result from (a) to show that

$$\tan^{-1}(x) + \tan^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2}$$

for all $x > 0$.

(c) Find a result similar to (b) for $x < 0$.

Difference Equations to **D**ifferential Equations

Section 6.6

Trigonometric Substitutions

In the last section we saw that

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + c.$$

However, we arrived at this result as a consequence of our differentiation of the arc sine function, not as the outcome of the application of some systematic approach to the evaluation of integrals of this type. In this section we will explore how substitutions based on the arc sine, arc tangent, and arc secant functions provide a systematic method for evaluating integrals similar to this one.

Sine substitutions

To begin, consider evaluating

$$\int \frac{1}{\sqrt{1-x^2}}\,dx$$

by using the substitution $u = \sin^{-1}(x)$. The motivation for such a substitution stems from the fact that, for $-\frac{\pi}{2} \leq u \leq \frac{\pi}{2}$, $u = \sin^{-1}(x)$ if and only if $x = \sin(u)$. In the latter form, we see that

$$dx = \cos(u)du$$

and

$$\sqrt{1-x^2} = \sqrt{1-\sin^2(u)} = \sqrt{\cos^2(u)} = |\cos(u)|. \tag{6.6.1}$$

Since $\cos(u) \geq 0$ when $\frac{\pi}{2} \leq u \leq \frac{\pi}{2}$, (6.6.1) becomes

$$\sqrt{1-x^2} = \cos(u).$$

Thus

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \int \frac{\cos(u)}{\cos(u)}\,du = \int du = u + c = \sin^{-1}(x) + c.$$

Of course, there is nothing new in the result itself; it is the technique, which we may generalize to other integrals of a similar type, which is of interest. Specifically, for an integral with a factor of the form

$$\sqrt{a^2-x^2}$$

or

$$\frac{1}{\sqrt{a^2-x^2}},$$

where $a > 0$, the substitution

$$x = a\sin(u), \frac{\pi}{2} \leq u \leq \frac{\pi}{2}, \tag{6.6.2}$$

may prove to be useful because of the simplification

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - ax^2\sin^2(x)} = \sqrt{a^2(1 - \sin^2(u))} = a\sqrt{\cos^2(u)} = a\cos(u). \tag{6.6.3}$$

Although this substitution is equivalent to the substitution $u = \sin\left(\frac{x}{a}\right)$, we will see that it is more convenient to work with it in the form $x = a\sin(u)$.

Example To evaluate the integral $\int \frac{1}{\sqrt{9 - x^2}}\, dx$, we make the substitution

$$\begin{aligned} x &= 3\sin(u), -\frac{\pi}{2} < u < \frac{\pi}{2}, \\ dx &= 3\cos(u)du. \end{aligned}$$

Note that we omit both $u = -\frac{\pi}{2}$ and $u = \frac{\pi}{2}$ since the function being integrated is not defined at either $x = -3$ or $x = 3$. Then

$$\begin{aligned} \int \frac{1}{\sqrt{9 - x^2}}\, dx &= \int \frac{3\cos(u)}{\sqrt{9 - 9\sin^2(u)}}\, du \\ &= \int \frac{3\cos(u)}{3\sqrt{1 - sin^2(u)}}\, du \\ &= \int \frac{\cos(u)}{\sqrt{\cos^2(u)}}\, du \\ &= \int \frac{\cos(u)}{\cos(u)}\, du \\ &= \int du \\ &= u + c. \end{aligned}$$

Now $x = 3\sin(u)$ implies that $u = \sin^{-1}\left(\frac{x}{3}\right)$, so we have

$$\int \frac{1}{\sqrt{9 - x^2}}\, du = \sin^{-1}\left(\frac{x}{3}\right) + c.$$

Example To evaluate $\int \sqrt{4 - x^2}\, dx$, we make the substitution

$$\begin{aligned} x &= 2\sin(u), -\frac{\pi}{2} \leq u \leq \frac{\pi}{2}, \\ dx &= 2\cos(u)du. \end{aligned}$$

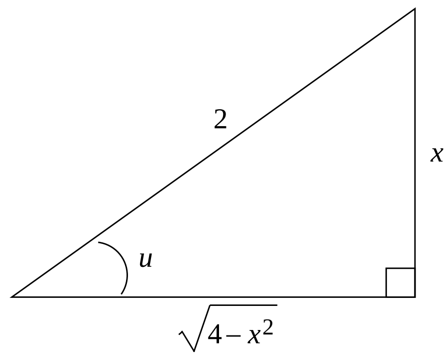


Figure 6.6.1 Right triangle with $\sin(u) = \frac{x}{2}$

Then

$$\begin{aligned}\int \sqrt{4 - x^2}\, dx &= 2 \int \sqrt{4 - 4\sin^2(u)}\cos(u)du \\ &= 4 \int \sqrt{1 - \sin^2(u)}\cos(u)du \\ &= 4 \int \sqrt{\cos^2(u)}\cos(u)du \\ &= 4 \int \cos^2(u)du \\ &= 4 \int \frac{1 + \cos(2u)}{2}\, du \\ &= 2 \int (1 + \cos(2u))du \\ &= 2u + \sin(2u) + c \\ &= 2u + 2\sin(u)\cos(u) + c.\end{aligned}$$

Since $x = 2\sin(u)$, $\sin(u) = \frac{x}{2}$ and $u = \sin^{-1}\left(\frac{x}{2}\right)$. Moreover,

$$\cos^2(u) = 1 - \sin^2(u) = 1 - \frac{x^2}{4} = \frac{4 - x^2}{4},$$

so

$$\cos(u) = \frac{1}{2}\sqrt{4 - x^2},$$

where we have, once again, used the fact that $\cos(u) \geq 0$ since $-\frac{\pi}{2} \leq u \leq \frac{\pi}{2}$. Note that this expression for $\cos(u)$ may also be deduced from Figure 6.6.1, where we have a right triangle with an acute angle of size u such that $\sin(u) = \frac{x}{2}$. Putting everything together, we have

$$\int \sqrt{4 - x^2}\, dx = 2\sin^{-1}\left(\frac{x}{2}\right) + \frac{1}{2}x\sqrt{4 - x^2} + c.$$

Notice that a considerable amount of the work in the previous example involved expressing the answer in terms of x once it had been found in terms of u. The next example illustrates how this work is unnecessary when evaluating definite integrals since we can change the limits of integration and, from that point on, do all our work in terms of u.

Example Recall that, for $r > 0$, the graph of $y = \sqrt{r^2 - x^2}$ is the upper half of a circle of radius r centered at the origin. Hence we should have

$$2\int_{-r}^{r} \sqrt{r^2 - x^2}\, dr = \pi r^2.$$

We are now able to verify this. Let

$$\begin{aligned} x &= r\sin(u), -\frac{\pi}{2} \leq u \leq \frac{\pi}{2}, \\ dx &= r\cos(u)du. \end{aligned}$$

Then $u = \sin^{-1}\left(\frac{x}{r}\right)$, so when $x = -r$,

$$u = \sin^{-1}(-1) = -\frac{\pi}{2},$$

and when $x = r$,

$$u = \sin^{-1}(1) = \frac{\pi}{2}.$$

Thus

$$\begin{aligned} 2\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx &= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{r^2 - r^2\sin^2(u)}\; r\cos(u)du \\ &= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^2\sqrt{\cos^2(u)}\cos(u)du \\ &= 2r^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(u)du \\ &= 2r^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos(2u)}{2}\, du \\ &= r^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\cos(2u))du \\ &= r^2\left(u + \frac{\sin(2u)}{2}\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= r^2\left(\left(\frac{\pi}{2}+0\right) - \left(-\frac{\pi}{2}+0\right)\right) \\ &= \pi r^2. \end{aligned}$$

Tangent substitutions

In a similar fashion, the substitution

$$x = a\tan(u), -\frac{\pi}{2} < u < \frac{\pi}{2}, \tag{6.6.4}$$

may be useful for integrals which have a factor of the form

$$\sqrt{a^2+x^2},$$

$$\frac{1}{\sqrt{a^2+x^2}},$$

or

$$\frac{1}{a^2+x^2}$$

because of the simplification

$$a^2+x^2 = a^2 + a^2\tan^2(u) = a^2(1+\tan^2(u)) = a^2\sec^2(u). \tag{6.6.5}$$

Note that with our restriction on u, this substitution is equivalent to the substitution $u = \tan^{-1}\left(\frac{x}{a}\right)$.

Example To evaluate the integral $\displaystyle\int \frac{1}{4+x^2}\,dx$, we make the substitution

$$\begin{aligned} x &= 2\tan(u), -\frac{\pi}{2} < u < \frac{\pi}{2}, \\ dx &= 2\sec^2(u)du. \end{aligned}$$

Then

$$\begin{aligned} \int \frac{1}{4+x^2}\,dx &= \int \frac{2sec^2(u)}{4+4\tan^2(u)}\,du \\ &= \frac{1}{2}\int \frac{\sec^2(u)}{1+\tan^2(u)}\,du \\ &= \frac{1}{2}\int \frac{\sec^2(u)}{\sec^2(u)}\,du \\ &= \frac{1}{2}\int du \\ &= \frac{1}{2}u + c. \end{aligned}$$

Since $x = 2\tan(u)$, $u = \tan^{-1}\left(\frac{x}{2}\right)$. Thus

$$\int \frac{1}{4+x^2}\,dx = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c.$$

Example To evaluate $\displaystyle\int \frac{1}{\sqrt{1+x^2}}\,dx$, we make the substitution

$$\begin{aligned} x &= \tan(u), -\frac{\pi}{2} < u < \frac{\pi}{2}, \\ dx &= \sec^2(u)du. \end{aligned}$$

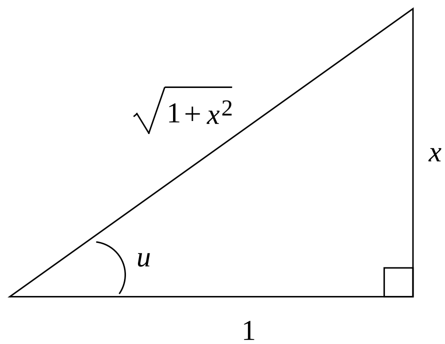


Figure 6.6.2 Right triangle with $\tan(u) = x$

Then

$$\sqrt{1+x^2} = \sqrt{1+\tan^2(u)} = \sqrt{\sec^2(u)} = |\sec(u)|.$$

Since $-\frac{\pi}{2} < u < \frac{\pi}{2}$, $\sec(u) > 0$, so $|\sec(u)| = \sec(u)$. Hence

$$\begin{aligned} \int \frac{1}{1+x^2}\, dx &= \int \frac{\sec^2(u)}{\sec(u)}\, du \\ &= \int \sec(u) du \\ &= \log|\sec(u) + \tan(u)| + c, \end{aligned}$$

where the final integral follows from an example in Section 6.2. Now $\tan(u) = x$, so

$$\sec^2(u) = 1 + \tan^2(u) = 1 + x^2.$$

Since $\sec(u) > 0$, it follows that

$$\sec(u) = \sqrt{1+x^2}.$$

Note that this expression for $\sec(u)$ may also be deduced from Figure 6.6.2, where we have a right triangle with an acute angle of size u such that $\tan(u) = x$. Thus

$$\int \frac{1}{\sqrt{1+x^2}}\, dx = \log\left|\sqrt{1+x^2} + x\right| + c.$$

Secant substitutions

For integrals involving a factor of the form

$$\sqrt{x^2 - a^2}$$

or

$$\frac{1}{\sqrt{x^2-a^2}},$$

where $a > 0$, the substitution

$$x = a\sec(u) \tag{6.6.6}$$

may be useful. With this substitution,

$$\sqrt{x^2 - a^2} = \sqrt{a^2\sec^2(u) - a^2} = a\sqrt{\sec^2(u) - 1} = a\sqrt{\tan^2(u)} = a|\tan(u)|. \tag{6.6.7}$$

Now $\sqrt{x^2 - a^2}$ is meaningful only if either $x \geq a$ or $x \leq -a$. Since $x = a\sec(u)$, the former case corresponds to u in the interval $[0, \frac{\pi}{2})$ and the latter to u in the interval $(\frac{\pi}{2}, \pi]$. For $0 \leq u < \frac{\pi}{2}$, $\tan(u) \geq 0$, so

$$\sqrt{x^2 - a^2} = a\tan(u);$$

for $\frac{\pi}{2} < u \leq \pi$, $\tan(u) \leq 0$, so

$$\sqrt{x^2 - a^2} = -a\tan(u).$$

Hence it is important when evaluating integrals of this type to be careful about which values of x are of interest.

Example To evaluate $\displaystyle\int \frac{1}{\sqrt{x^2 - 9}}\,dx$ for $x > 3$, we make the substitution

$$\begin{aligned} x &= 3\sec(u), 0 < u < \frac{\pi}{2}, \\ dx &= 3\sec(u)\tan(u)du. \end{aligned}$$

Then

$$\begin{aligned} \int \frac{1}{\sqrt{x^2 - 9}}\,dx &= \int \frac{3\sec(u)\tan(u)}{\sqrt{9\sec^2(u) - 9}}\,du \\ &= \int \frac{3\sec(u)\tan(u)}{3\sqrt{\sec^2(u) - 1}}\,du \\ &= \int \frac{\sec(u)\tan(u)}{\sqrt{\tan^2(u)}}\,du \\ &= \int \frac{\sec(u)\tan(u)}{\tan(u)}\,du \\ &= \int \sec(u)du \\ &= \log|\sec(u) + \tan(u)| + c. \end{aligned}$$

Now $\sec(u) = \frac{x}{3}$, so

$$\tan^2(u) = \sec^2(u) - 1 = \frac{x^2}{9} - 1 = \frac{x^2 - 9}{9}.$$

Hence

$$\tan(u) = \frac{1}{3}\sqrt{x^2 - 9},$$

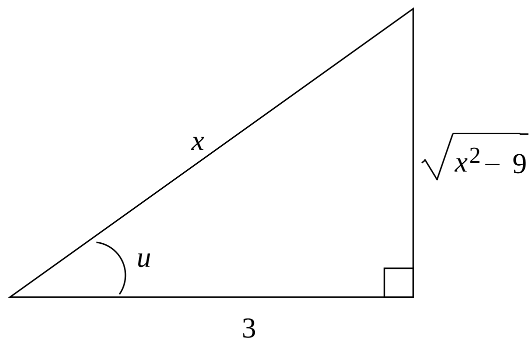


Figure 6.6.3 Right triangle with $\sec(u) = \frac{x}{3}$

where we have used the fact that $\tan(u) > 0$ since $0 < u < \frac{\pi}{2}$. Note that this expression for $\sec(u)$ may also be deduced from Figure 6.6.3, where we have a right triangle with an acute angle of size u such that $\sec(u) = \frac{x}{3}$. Thus

$$\int \frac{1}{\sqrt{x^2-9}}\,dx = \log\left|\frac{x}{3} + \frac{1}{3}\sqrt{x^2-9}\right| = \log|x + \sqrt{x^2-9}| - \log(3) + c.$$

Since $\log(3)$ is a constant, we may combine it with the arbitrary constant of integration. Moreover, since we are assuming $x > 3$, we may remove the absolute value and write

$$\int \frac{1}{\sqrt{x^2-9}}\,dx = \log(x + \sqrt{x^2-9}) + c.$$

Problems

1. Evaluate the following integrals.

   (a) $\displaystyle\int \frac{3}{\sqrt{16-x^2}}\,dx$

   (b) $\displaystyle\int \frac{x}{\sqrt{4-x^2}}\,dx$

   (c) $\displaystyle\int \sqrt{5-z^2}\,dz$

   (d) $\displaystyle\int \frac{5}{6+x^2}\,dx$

   (e) $\displaystyle\int \frac{1}{\sqrt{4+x^2}}\,dx$

   (f) $\displaystyle\int \frac{4x}{\sqrt{1+x^2}}\,dx$

2. Evaluate the following integrals.

   (a) $\displaystyle\int z\sqrt{1-z^2}\,dz$

   (b) $\displaystyle\int \frac{1}{(1-x^2)^{\frac{3}{2}}}dx$

   (c) $\displaystyle\int \frac{1}{\sqrt{x^2-4}}\,dx,\ x > 2$

   (d) $\displaystyle\int \frac{1}{\sqrt{x^2-4}}\,dx,\ x < -2$

   (e) $\displaystyle\int \frac{4}{\sqrt{3-2x^2}}\,dx$

   (f) $\displaystyle\int \frac{3}{5+2x^2}\,dx$

(g) $\int \sqrt{4-t^2}\,dt$ (h) $\int \sqrt{1-4x^2}\,dx$

3. Evaluate the following integrals.

   (a) $\int_0^3 \sqrt{9-t^2}\,dt$ (b) $\int_{-1}^1 \sqrt{1+x^2}\,dx$

   (c) $\int_0^1 \frac{1}{(1+x^2)^{\frac{3}{2}}}\,dx$ (d) $\int_0^1 x^2\sqrt{1-x^2}\,dx$

   (e) $\int_{5\sqrt{2}}^{10} \frac{1}{\sqrt{x^2-25}}\,dx$ (f) $\int_{\sqrt{2}}^{2} \frac{3}{x^2\sqrt{x^2-1}}\,dx$

4. Evaluate
$$\int \frac{1}{1-x^2}\,dx$$
using (a) partial fractions and (b) the substitution $x=\sin(u)$. How do the two methods compare?

5. Evaluate
$$\int \frac{1}{\sqrt{1-x^2}}\,dx$$
using the substitution $x=\cos(u)$ with $0<u<\pi$.

6. (a) Evaluate
$$\int \frac{1}{\sqrt{x^2-9}}\,dx$$
for $x<-3$.

   (b) Show that
$$\int \frac{1}{\sqrt{x^2-9}}\,dx = \log|x+\sqrt{x^2-9}|+c,$$
for both $x>3$ and $x<-3$.

Difference Equations
to
Differential Equations

Section 6.7

Hyperbolic Functions

The final class of functions we will consider are the hyperbolic functions. In a sense these functions are not new to us since they may all be expressed in terms of the exponential function and its inverse, the natural logarithm function. However, we will see that they have many interesting and useful properties.

Definition For any real number x, the *hyperbolic sine* of x, denoted $\sinh(x)$, is defined by

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x}) \tag{6.7.1}$$

and the *hyperbolic cosine* of x, denoted $\cosh(x)$, is defined by

$$\cosh(x) = \frac{1}{2}(e^x + e^{-x}). \tag{6.7.2}$$

Note that, for any real number t,

$$\begin{aligned}\cosh^2(t) - \sinh^2(t) &= \frac{1}{4}(e^t + e^{-t})^2 - \frac{1}{4}(e^t - e^{-t})^2 \\ &= \frac{1}{4}(e^{2t} + 2e^te^{-t} + e^{-2t}) - \frac{1}{4}(e^{2t} - 2e^te^{-t} + e^{-2t}) \\ &= \frac{1}{4}(2+2) \\ &= 1.\end{aligned}$$

Thus we have the useful identity

$$\cosh^2(t) - \sinh^2(t) = 1 \tag{6.7.3}$$

for any real number t. Put another way, $(\cosh(t), \sinh(t))$ is a point on the hyperbola $x^2 - y^2 = 1$. Hence we see an analogy between the hyperbolic cosine and sine functions and the cosine and sine functions: Whereas $(\cos(t), \sin(t))$ is a point on the circle $x^2 + y^2 = 1$, $(\cosh(t), \sinh(t))$ is a point on the hyperbola $x^2 - y^2 = 1$. In fact, the cosine and sine functions are sometimes referred to as the *circular* cosine and sine functions. We shall see many more similarities between the hyperbolic trigonometric functions and their circular counterparts as we proceed with our discussion.

To understand the graphs of the hyperbolic sine and cosine functions, we first note that, for any value of x,

$$\sinh(-x) = \frac{1}{2}(e^{-x} - e^x) = -\sinh(x), \tag{6.7.4}$$

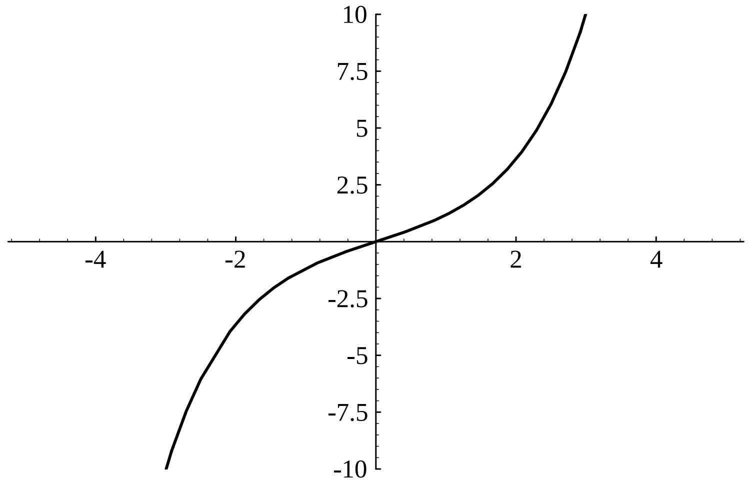


Figure 6.7.1 Graph of $y = \sinh(x)$

and

$$\cosh(-x) = \frac{1}{2}(e^{-x} + e^{x}) = \cosh(x). \tag{6.7.5}$$

Now for large values of x, $e^{-x} \approx 0$, in which case

$$\sinh(x) = \frac{1}{2}(e^{x} - e^{-x}) \approx= \frac{1}{2}e^{x}$$

and

$$\sinh(-x) = -\sinh(x) \approx -\frac{1}{2}e^{x}.$$

Thus the graph of $y = \sinh(x)$ appears as in Figure 6.7.1. Similarly, for large values of x,

$$\cosh(x) = \frac{1}{2}(e^{x} + e^{-x}) \approx \frac{1}{2}e^{x}$$

and

$$\cosh(-x) = \cosh(x) \approx \frac{1}{2}e^{x}.$$

The graph of $y = \cosh(x)$ is shown in Figure 6.7.2.

The derivatives of the hyperbolic sine and cosine functions follow immediately from their definitions. Namely,

$$\frac{d}{dx}\sinh(x) = \frac{d}{dx}\frac{1}{2}(e^{x} - e^{-x}) = \frac{1}{2}(e^{x} + e^{-x}) = \cosh(x)$$

and

$$\frac{d}{dx}\cosh(x) = \frac{d}{dx}\frac{1}{2}(e^{x} + e^{-x}) = \frac{1}{2}(e^{x} - e^{-x}) = \sinh(x).$$

Here again we see similarities between the circular and hyperbolic sine and cosine functions.

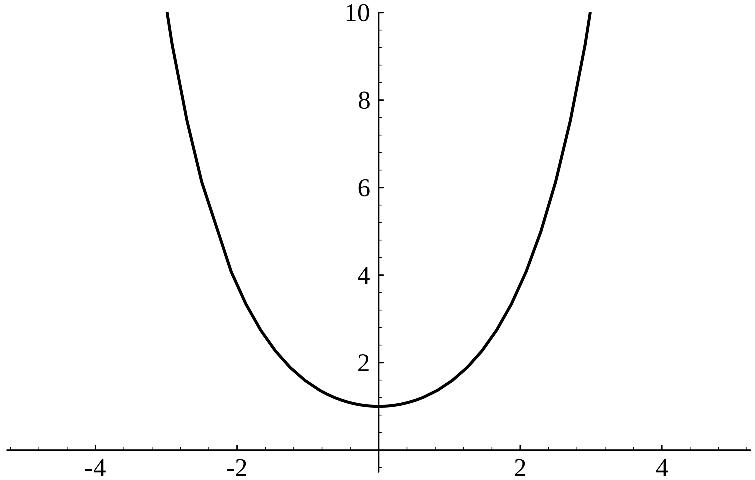


Figure 6.7.2 Graph of $y = \cosh(x)$

Proposition

$$\frac{d}{dx}\sinh(x) = \cosh(x). \tag{6.7.6}$$

$$\frac{d}{dx}\cosh(x) = \sinh(x). \tag{6.7.7}$$

As a consequence of this proposition, we also have

$$\int \sinh(x)dx = \cosh(x) + c \tag{6.7.8}$$

and

$$\int \cosh(x)dx = \sinh(x) + c. \tag{6.7.9}$$

Example Using the chain rule, we have

$$\frac{d}{dx}\sinh^2(3x) = 2\sinh(3x)\frac{d}{dx}\sinh(3x) = 6\sinh(3x)\cosh(3x).$$

Example Using the chain and product rules, we have

$$\begin{aligned}\frac{d}{dx}\sinh(2x)\cosh(2x) &= \sinh(2x)(2\sinh(2x)) + \cosh(2x)(2\cosh(2x))\\ &= 2\sinh^2(2x) + 2\cosh^2(2x).\end{aligned}$$

Example Analogous to

$$\int \sin(3x)dx = -\frac{1}{3}\cos(3x) + c,$$

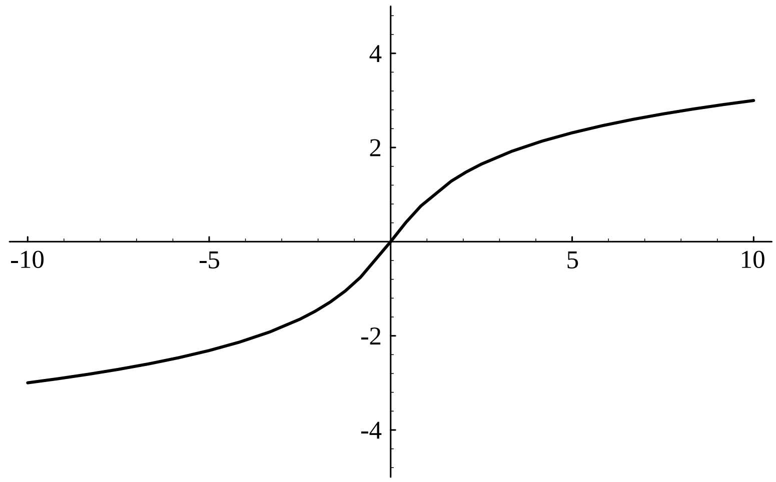


Figure 6.7.3 Graph of $y = \sinh^{-1}(x)$

we have

$$\int \sinh(3x)dx = \frac{1}{3}\cosh(3x) + c.$$

Example It is tempting to evaluate

$$\int e^{-x}\sinh(x)dx$$

using integration by parts in the same manner that we would evaluate

$$\int e^{-x}\sin(x)dx.$$

However, this integral is much easier if we notice that

$$e^{-x}\sinh(x) = e^{-x}\left(\frac{1}{2}(e^{x} - e^{-x})\right) = \frac{1}{2}(1 - e^{-2x}).$$

Hence

$$\int e^{-x}\sinh(x)dx = \frac{1}{2}\int (1 - e^{-2x})dx = \frac{x}{2} + \frac{1}{4}e^{-2x} + c.$$

Since $\frac{d}{dx}\sinh(x) = \cosh(x) > 0$ for all x, the hyperbolic sine function is increasing on the interval $(-\infty, \infty)$. Thus it has an inverse function, called the *inverse hyperbolic sine function*, with value at x denoted by $\sinh^{-1}(x)$. Since the domain and range of the hyperbolic sine function are both $(-\infty, \infty)$, the domain and range of the inverse hyperbolic sine function are also both $(-\infty, \infty)$. As usual with inverse functions,

$$y = \sinh^{-1}(x) \text{ if and only if } \sinh(y) = x. \tag{6.7.10}$$

The graph of $y = \sinh^{-1}(x)$ is shown in Figure 6.7.3.

Example The hyperbolic sine function and its inverse provide an alternative method for evaluating

$$\int \frac{1}{\sqrt{1+x^2}}\, dx.$$

Namely, if we make the substitution

$$\begin{aligned} x &= \sinh(u), -\infty < u < \infty, \\ dx &= \cosh(u) du, \end{aligned}$$

then

$$\sqrt{1+x^2} = \sqrt{1+\sinh^2(u)} = \sqrt{\cosh^2(u)} = \cosh(u),$$

where the second equality follows from the identity $\cosh^2(u) - \sinh^2(u) = 1$ and the last equality from the fact that $\cosh(u) > 0$ for all u. Hence

$$\int \frac{1}{\sqrt{1+x^2}}\, dx = \int \frac{\cosh(u)}{\cosh(u)}\, du = \int du = u + c = \sinh^{-1}(x) + c.$$

The following proposition is a consequence of the previous example.

Proposition

$$\frac{d}{dx} \sinh^{-1}(x) = \frac{1}{\sqrt{1+x^2}}. \tag{6.7.11}$$

In Section 6.6 we saw, using the substitution $x = \tan(u)$, $-\frac{\pi}{2} < u < \frac{\pi}{2}$, that

$$\int \frac{1}{\sqrt{1+x^2}}\, dx = \log\left|x + \sqrt{1+x^2}\right| + c.$$

Since two antiderivatives of a function can differ at most by a constant, there must exist a constant k such that

$$\sinh^{-1}(x) = \log\left|x + \sqrt{1+x^2}\right| + k$$

for all x. Evaluating both sides of this equality at $x = 0$, we have

$$0 = \sinh^{-1}(0) = \log(1) + k = k.$$

Thus $k = 0$ and

$$\sinh^{-1}(x) = \log\left|x + \sqrt{1+x^2}\right| \tag{6.7.12}$$

for all x. Since the hyperbolic sine function is defined in terms of the exponential function, we should not find it surprising that the inverse hyperbolic sine function may be expressed in terms of the natural logarithm function.

Similarly, since $\frac{d}{dx}\cosh(x) = \sinh(x) > 0$ for all $x > 0$, the hyperbolic cosine function is increasing on the interval $[0, \infty)$, and so has an inverse if we restrict its domain to $[0, \infty)$. That is, we define the *inverse hyperbolic cosine* function by the relationship

$$y = \cosh^{-1}(x) \text{ if and only if } x = \cosh(y), \tag{6.7.13}$$

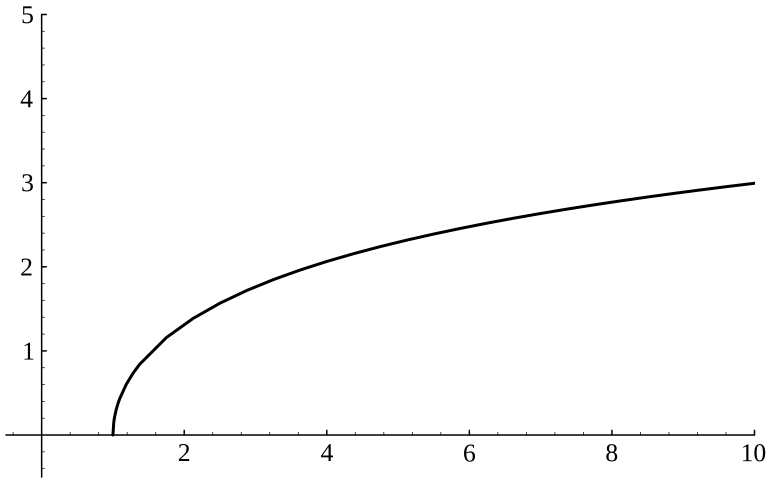


Figure 6.7.4 Graph of $y = \cosh^{-1}(x)$

where we require $y \geq 0$. Note that since $\cosh(x) \geq 1$ for all x, the domain of of the inverse hyperbolic cosine function is $[1, \infty)$. The graph of $y = \cosh^{-1}(x)$ is shown in Figure 6.7.4.

In Problem 3 at the end of this section you are asked to show that

$$\int \frac{1}{\sqrt{x^2 - 1}}\, dx = \cosh^{-1}(x) + c$$

for $x > 1$, from which the following proposition follows.

Proposition

$$\frac{d}{dx}\cosh^{-1}(x) = \frac{1}{\sqrt{x^2 - 1}}. \tag{6.7.14}$$

In the same problem you are asked to show that, for $x \geq 1$,

$$\cosh^{-1}(x) = \log\left|x + \sqrt{x^2 - 1}\right| \tag{6.7.15}$$

Example In Section 6.6 we evaluated the integral

$$\int \frac{1}{\sqrt{x^2 - 9}}\, dx,$$

for $x > 3$, using the substitution $x = 3\sec(u)$, $0 < u < \frac{\pi}{2}$. The substitution

$$\begin{aligned} x &= 3\cosh(u), u > 0, \\ dx &= 3\sinh(u)du \end{aligned}$$

provides a somewhat simpler approach. Namely,

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2-9}}\,dx &= \int \frac{3\sinh(u)}{\sqrt{9\cosh^2(u)-9}}\,du \\
&= \int \frac{3\sinh(u)}{3\sqrt{\cosh^2(u)-1}}\,du \\
&= \int \frac{\sinh(u)}{\sqrt{\sinh^2(u)}}\,du \\
&= \int \frac{\sinh(u)}{\sinh(u)}\,du \\
&= \int du \\
&= u + c \\
&= \cosh^{-1}\left(\frac{x}{3}\right) + c,
\end{aligned}$$

where we have used the fact that $\sinh(u) > 0$ when $u > 0$.

Having defined the hyperbolic sine and cosine functions, it is possible to define four more hyperbolic trigonometric functions in analogy with the circular trigonometric functions. Namely, the *hyperbolic tangent function* is given by

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}, \tag{6.7.16}$$

where $-\infty < x < \infty$; the *hyperbolic cotangent function* by

$$\coth(x) = \frac{\cosh(x)}{\sinh(x)}, \tag{6.7.17}$$

where $x \neq 0$; the *hyperbolic secant function* by

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)}, \tag{6.7.18}$$

where $-\infty < x < \infty$; and the *hyperbolic cosecant function* by

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)}, \tag{6.7.19}$$

where $x \neq 0$. In Problem 5 at the end of this section you are asked to verify the following results.

Proposition

$$\frac{d}{dx}\tanh(x) = \operatorname{sech}^2(x). \tag{6.7.20}$$

Proposition

$$\frac{d}{dx}\coth(x) = -\text{csch}^2(x) \tag{6.7.21}$$

Proposition

$$\frac{d}{dx}\text{sech}(x) = -\text{sech}(x)\tanh(x). \tag{6.7.22}$$

Proposition

$$\frac{d}{dx}\text{csch}(x) = -\text{csch}(x)\coth(x). \tag{6.7.23}$$

Since

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}},$$

we have

$$\begin{aligned}\lim_{x\to\infty}\tanh(x) &= \lim_{x\to\infty}\frac{e^x - e^{-x}}{e^x + e^{-x}}\\ &= \lim_{x\to\infty}\frac{e^x(1 - e^{-2x})}{e^x(1 + e^{-2x})}\\ &= \lim_{x\to\infty}\frac{1 - e^{-2x}}{1 + e^{-2x}}\\ &= 1\end{aligned}$$

and

$$\begin{aligned}\lim_{x\to-\infty}\tanh(x) &= \lim_{x\to-\infty}\frac{e^x - e^{-x}}{e^x + e^{-x}}\\ &= \lim_{x\to-\infty}\frac{e^{-x}(e^{2x} - 1)}{e^{-x}(e^{2x} + 1)}\\ &= \lim_{x\to-\infty}\frac{e^{2x} - 1}{e^{2x} + 1}\\ &= -1.\end{aligned}$$

Hence $y = 1$ and $y = -1$ are both horizontal asymptotes for the graph of $y = \tanh(x)$. Combining this information with $\tanh(0) = 0$ and

$$\frac{d}{dx}\tanh(x) = \text{sech}^2(x) > 0$$

for all x, we can see why the graph of $y = \tanh(x)$ looks as it does in Figure 6.7.5.

Since the hyperbolic tangent function is increasing on $(-\infty, \infty)$, it has an inverse, called the *inverse hyperbolic tangent function*, with value at x denoted by $\tanh^{-1}(x)$. That is, as usual,

$$y = \tanh^{-1}(x) \text{ if and only if } \tanh(y) = x. \tag{6.7.24}$$

The domain of the inverse hyperbolic tangent function is $(-1, 1)$ the range of the hyperbolic tangent function, and its range is $(-\infty, \infty)$, the domain of the hyperbolic tangent

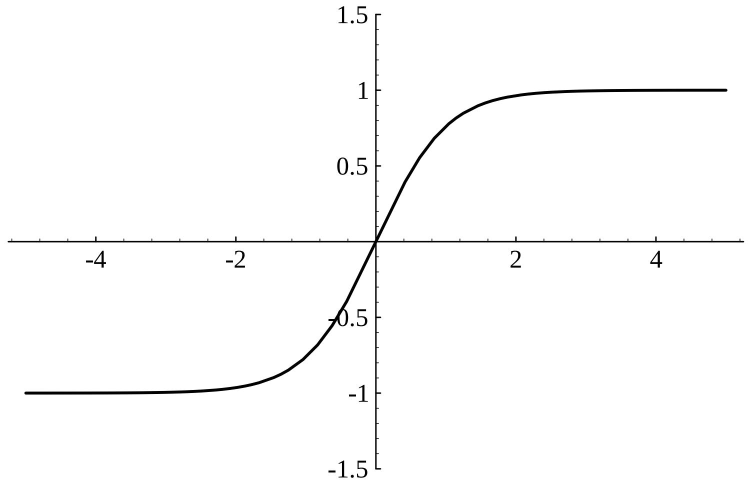


Figure 6.7.5 Graph of $y = \tanh(x)$

function. Corresponding to the horizontal asymptotes of the graph of the hyperbolic tangent function, the graph of the inverse hyperbolic tangent function has vertical asymptotes $x = -1$ and $x = 1$, as shown in Figure 6.7.6.

Example As an alternative to using partial fractions, we may evaluate the integral

$$\int \frac{1}{1-x^2}\, dx$$

for $-1 < x < 1$ using the substitution

$$\begin{aligned} x &= \tanh(u), -\infty < u < \infty, \\ dx &= \operatorname{sech}^2(u) du. \end{aligned}$$

Then

$$\int \frac{1}{1-x^2}\, dx = \int \frac{\operatorname{sech}^2(u)}{1-\tanh^2(u)}\, du.$$

Now from the identity

$$\cosh^2(x) - \sinh^2(x) = 1$$

we obtain

$$\frac{\cosh^2(x)}{\cosh^2(x)} - \frac{\sinh^2(x)}{\cosh^2(x)} = \frac{1}{\cosh^2(x)}.$$

In other words,

$$1 - \tanh^2(x) = \operatorname{sech}^2(x). \tag{6.7.25}$$

Hence

$$\int \frac{1}{1-x^2}\, dx = \int \frac{\operatorname{sech}^2(u)}{\operatorname{sech}^2(u)}\, du = \int du = u + c = \tanh^{-1}(x) + c.$$

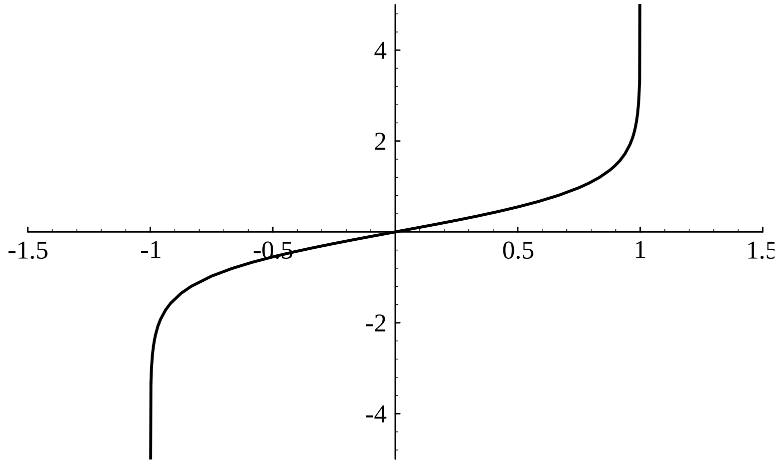


Figure 6.7.6 Graph of $y = \tanh^{-1}(x)$

Note that (6.7.25) gives us the useful identity

$$\tanh^2(x) + \operatorname{sech}^2(x) = 1 \tag{6.7.26}$$

for all x. Moreover, we have the following proposition as a consequence of this example.

Proposition

$$\frac{d}{dx}\tanh^{-1}(x) = \frac{1}{1-x^2}. \tag{6.7.27}$$

If we were to use partial fractions to evaluate the integral of the previous example, we would obtain, for $-1 < x < 1$,

$$\int \frac{1}{1-x^2}\,dx = \frac{1}{2}\log(1-x) - \frac{1}{2}\log(1+x) + c = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right) + c.$$

It follows that

$$\tanh^{-1}(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right) + k$$

for some constant k. Evaluating at 0, we have

$$0 = 0 + k.$$

Thus $k = 0$ and we have

$$\tanh^{-1}(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right) \tag{6.7.28}$$

for $1 < x < 1$.

Problems

1. Differentiate each of the following functions.

   (a) $f(x) = \sinh(3x)$ (b) $g(t) = 3t\cosh(4t)$

   (c) $f(t) = 3t\sinh(t)\cosh(2t)$ (d) $g(x) = 4x\sinh(3x^2 - 1)$

   (e) $y(t) = 5t^2\cosh^2(4t)$ (f) $f(t) = 3\cosh^2(2t) - 13\sinh(3t^2)$

2. Evaluate each of the following integrals.

   (a) $\displaystyle\int \sinh(3x)dx$ (b) $\displaystyle\int \cosh(4t-3)dt$

   (c) $\displaystyle\int \sinh(z)\cosh(z)dz$ (d) $\displaystyle\int 3x\sinh(2x)dx$

   (e) $\displaystyle\int e^{-2t}\cosh(2t)dt$ (f) $\displaystyle\int \cosh^2(x)\sinh(x)dx$

   (g) $\displaystyle\int 5t^2\cosh(2t)dt$ (h) $\displaystyle\int \frac{\sinh(t)}{\cosh^2(t)}\,dt$

3. (a) Use the substitution $x = \cosh(u), u > 0$, to show that

   $$\int \frac{1}{\sqrt{x^2-1}}\,dx = \cosh^{-1}(x) + c$$

   for $x > 1$.

   (b) Use the substitution $x = \sec(u), 0 < u < \frac{\pi}{2}$, to show that

   $$\int \frac{1}{\sqrt{x^2-1}}\,dx = \log\left|x + \sqrt{x^2-1}\right| + c$$

   for $x > 1$.

   (c) Using (a) and (b), show that

   $$\cosh^{-1}(x) = \log\left|x + \sqrt{x^2-1}\right|$$

   for $x > 1$.

4. Evaluate the following integrals.

   (a) $\displaystyle\int \frac{1}{\sqrt{4+x^2}}\,dx$ (b) $\displaystyle\int \frac{1}{\sqrt{x^2-4}}\,dx, x > 2$

   (c) $\displaystyle\int \frac{3}{\sqrt{9+3t^2}}\,dt$ (d) $\displaystyle\int \frac{1}{\sqrt{x^2-1}}\,dx, x < -1$

5. Verify the following derivatives.

   (a) $\displaystyle\frac{d}{dx}\tanh(x) = \mathrm{sech}^2(x)$ (b) $\displaystyle\frac{d}{dx}\coth(x) = -\mathrm{csch}^2(x)$

(c) $\dfrac{d}{dx}\text{sech}(x) = -\text{sech}(x)\tanh(x)$

(d) $\dfrac{d}{dx}\text{csch}(x) = -\text{csch}(x)\coth(x)$

6. Differentiate each of the following functions.

   (a) $f(x) = 3x\tanh(4x)$

   (b) $g(t) = \text{sech}^2(3t)$

   (c) $h(\theta) = 4\tanh^2(\theta)\text{sech}(\theta)$

   (d) $f(x) = 5x\text{sech}(4x) - 21\tanh^3(4x)$

7. Evaluate each of the following integrals.

   (a) $\displaystyle\int \tanh(x)dx$

   (b) $\displaystyle\int \tanh(2x)\text{sech}(2x)dx$

   (c) $\displaystyle\int \frac{1}{4-x^2}\,dx$

   (d) $\displaystyle\int \frac{5}{9-3t^2}\,dt$

8. Graph each of the following functions on an appropriate interval.

   (a) $y = \text{sech}(x)$

   (b) $y = \coth(x)$

   (c) $y = \text{csch}(x)$

   (d) $y = 3\tanh(4x)$

Difference Equations to Differential Equations

Section 7.1

The Algebra of Complex Numbers

At this point we have considered only real-valued functions of a real variable. That is, all of our work has centered on functions of the form $f : \mathbb{R} \to \mathbb{R}$, functions which take a real number to a real number. In this chapter we will discuss complex numbers and the calculus of associated functions. In particular, if we let $\mathbb{C}$ represent the set of all complex numbers, then we will be interested in functions of the form $f : \mathbb{R} \to \mathbb{C}$ and $f : \mathbb{C} \to \mathbb{C}$. We will begin the story in this section with a discussion of what complex numbers are and how we work with them.

Perhaps because of their name, it is sometimes thought that complex numbers are in some way more mysterious than real numbers, that a number such as $i = \sqrt{-1}$ is not as "real" as a number like 2 or -351.127 or even π. However, all of these numbers are equally meaningful, they are all useful mathematical abstractions. Although complex numbers are a relatively recent invention of mathematics, dating back just over 200 years in their current form, it is also the case that negative numbers, which were once called fictitious numbers to indicate that they were less "real" than positive numbers, have only been accepted for about the same period of time, and we have only started to understand the nature of real numbers during the past 150 years or so. In fact, if you think about their underlying meaning, π is a far more "complex" number than i.

Although complex numbers originate with attempts to solve certain algebraic equations, such as

$$x^2 + 1 = 0,$$

we will give a geometric definition which identifies complex numbers with points in the plane. This definition not only gives complex numbers a concrete geometrical meaning, but also provides us with a powerful algebraic tool for working with points in the plane.

Definition A *complex number* is an ordered pair of real numbers with addition defined by

$$(a, b) + (c, d) = (a + c, b + d) \tag{7.1.1}$$

and multiplication defined by

$$(a, b) \times (c, d) = (ac - bd, ad + bc), \tag{7.1.2}$$

where a, b, c, and d are any real numbers.

We will let i denote the complex number $(0, 1)$. Then, by our definition of multiplication,

$$i^2 = (0, 1) \times (0, 1) = (0 - 1, 0 + 0) = (-1, 0). \tag{7.1.3}$$

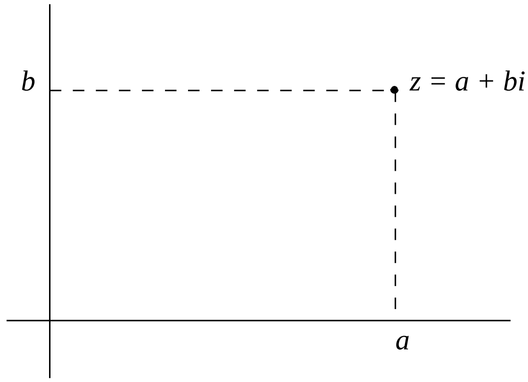


Figure 7.1.1 Geometric representation of a complex number

If we identify the real number a with the complex number $(a, 0)$, then we have

$$ai = (a, 0) \times (0, 1) = (0 - 0, a + 0) = (0, a).$$

Then for any two real numbers, we have

$$(a, b) = (a, 0) + (0, b) = a + bi. \tag{7.1.4}$$

That is, $a + bi$ is another way to write the complex number (a, b). In particular, with this convention, (7.1.3) becomes

$$i^2 = -1, \tag{7.1.5}$$

that is,

$$i = \sqrt{-1}. \tag{7.1.6}$$

Moreover, we may write (7.1.1) as

$$(a + bi) + (c + di) = (a + c) + (b + d)i \tag{7.1.7}$$

and (7.1.2) as

$$(a + bi) \times (c + di) = (ac - bd) + (ad + bc)i. \tag{7.1.8}$$

In fact, we may view the latter as a consequence of the ordinary algebraic expansion of the product

$$(a + bi)(c + di)$$

combined with the equality $i^2 = -1$. That is,

$$(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i.$$

It also follows from this formulation that if r is a real number, which we identify with $r + 0i$, and $z = a + bi$ is a complex number, then

$$rz = r(a + bi) = (r + 0i)(a + bi) = ra + rbi. \tag{7.1.9}$$

As indicated above, we let $\mathbb{C}$ denote the set of all complex numbers. Because of our identification of $\mathbb{C}$ with the plane, we usually refer to $\mathbb{C}$ as the *complex plane*. Since the description of complex numbers as points in the plane is often associated with the work of Carl Friedrich Gauss (1777-1855) (although appearing first in the work of Caspar Wessel (1745-1818)), $\mathbb{C}$ is also referred to as the *Gaussian plane*.

Example $(3 + 4i) + (5 - 6i) = 8 - 2i$.

Example $(2 + i)(3 - 2i) = 6 - 4i + 3i - 2i^2 = 8 - i$.

Example $-3(4 + 2i) = -12 - 6i$.

We have yet to define subtraction and division for complex numbers. If z and w are complex numbers, we may define

$$z - w = z + (-1)w. \tag{7.1.10}$$

It follows that if $z = a + bi$ and $w = c + di$, then

$$z - w = a + bi + (-c - di) = (a - c) + (b - d)i. \tag{7.1.11}$$

As a first step toward defining division, note that if $z = a + bi$ with either $a \neq 0$ or $b \neq 0$, then

$$(a + bi)\left(\frac{a - bi}{a^2 + b^2}\right) = \frac{a^2 - b^2 i^2}{a^2 + b^2} = \frac{a^2 + b^2}{a^2 + b^2} = 1.$$

In other words,

$$\frac{a - bi}{a^2 + b^2}$$

is the multiplicative inverse, or reciprocal, of $z = a + bi$. Hence we will write

$$z^{-1} = \frac{1}{z} = \frac{a - bi}{a^2 + b^2}. \tag{7.1.12}$$

Given another complex number w, we may define w divided by z by

$$\frac{w}{z} = wz^{-1}. \tag{7.1.13}$$

Definition Given a complex number $z = a + bi$, the number $a - bi$ is called the *conjugate* of z and is denoted $\bar{z}$.

Note that if $z = a + bi$, then

$$z\bar{z} = (a + bi)(a - bi) = a^2 + b^2. \tag{7.1.14}$$

Hence

$$\sqrt{z\bar{z}} = \sqrt{a^2 + b^2}, \tag{7.1.15}$$

which is the distance in the complex plane from z to the origin.

Definition Given a complex number z, the *magnitude* of z, denoted $|z|$, is defined by

$$|z| = \sqrt{z\bar{z}}. \tag{7.1.16}$$

The magnitude of a complex number generalizes the idea of the absolute value of a real number, and in fact reduces to the absolute value when z is a real number. Moreover, note that if $z = a + bi$, with either $a \neq 0$ or $b \neq 0$, we may now write

$$z^{-1} = \frac{\bar{z}}{|z|^2}. \tag{7.1.17}$$

Although (7.1.12) and (7.1.17) are useful expressions, in most situations the easiest way to simplify a quotient of two complex numbers is to multiply numerator and denominator by the conjugate of the denominator.

Example $\dfrac{2+i}{1+i} = \dfrac{2+i}{1+i}\dfrac{1-i}{1-i} = \dfrac{2-2i+i+1}{1+1} = \dfrac{3-i}{2} = \dfrac{3}{2} - \dfrac{1}{2}i.$

Definition Given a complex number $z = a + bi$, we call a the *real part* of z, denoted $\Re(z)$, and b the *imaginary part* of z, denoted $\Im(z)$.

Because of this definition, we call the horizontal axis of the complex plane the *real axis* and the vertical axis the *imaginary axis*. In this way we may identify the real number line with the the real axis of the complex plane. A complex number of the form bi, where b is a real number, lies on the imaginary axis of the complex plane and is said to be *purely imaginary*. However, we should be careful with our interpretation of this terminology: a purely imaginary number is just as "real", in the ordinary sense of real, as a real number, in the same way that an irrational number is just as "rational", in the sense of reasonable, as a rational number. Note that with this terminology, the conjugate $\bar{z}$ of a complex number z is the point in the complex plane obtained by reflecting z about the real axis.

Example If $z = 3 + 6i$, then $\Re(z) = 3$, $\Im(z) = 6$, $\bar{z} = 3 - 6i$, and

$$|z| = \sqrt{9+36} = \sqrt{45} = 3\sqrt{5}.$$

With the above definitions, we may work with the arithmetic and algebra of complex numbers in the same way we work with real numbers. For example, for any complex numbers z and w,

$$z + w = w + z$$

and

$$zw = wz.$$

You will be asked to verify these and other standard properties of the complex numbers in Problem 7 at the end of this section.

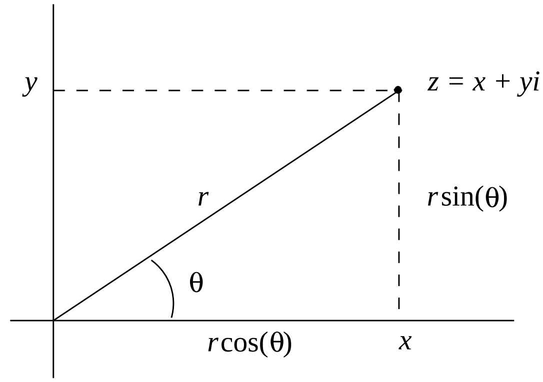


Figure 7.1.2 Polar coordinates for a complex number

Polar notation

When we write a complex number z in the form $z = x + yi$, we refer to x and y as the *rectangular* or *Cartesian coordinates* of z. We now consider another method of representing complex numbers. Let us begin with a complex number $z = x + yi$ written in rectangular form. Assume for the moment that x and y are not both 0. If we let θ be the angle between the real axis and the line segment from $(0, 0)$ to (x, y), measured in the counterclockwise direction, then z is completely determined by the two numbers z and θ. We call θ the *argument* of z and denote it by $\arg(z)$. Geometrically, if we are given $|z|$ and θ, we can locate z in the complex plane by taking the line segment of length $|z|$ lying on the positive real axis, with a fixed endpoint at the origin, and rotating it counterclockwise through an angle θ; the final resting point of the rotating endpoint is the location of z. Algebraically, if $z = x + yi$ is a complex number with $r = |z|$ and $\theta = \arg(z)$, then

$$x = r\cos(\theta) \tag{7.1.18}$$

and

$$y = r\sin(\theta). \tag{7.1.19}$$

Together, r and θ are called the *polar coordinates* of z. See Figure 7.1.2.

Example If $|z| = 2$ and $\arg(z) = \frac{\pi}{6}$, then

$$z = 2\cos\left(\frac{\pi}{6}\right) + 2\sin\left(\frac{\pi}{6}\right) = \sqrt{3} + i.$$

Example If $z = 1 - i$, then $|z| = \sqrt{2}$ and $\arg(z) = -\frac{\pi}{4}$.

Note that in the last example we could have taken $\arg(z) = \frac{7\pi}{4}$, or, in fact,

$$\arg(z) = -\frac{\pi}{4} + 2n\pi$$

for any integer n. In particular, there are an infinite number of possible values for $\arg(z)$ and we will let $\arg(z)$ stand for any one of these values. At the same time, it is often important to choose $\arg(z)$ in a consistent fashion; to this end, we call the value of $\arg(z)$ which lies in the interval $(-\pi, \pi]$ the *principal value* of $\arg(z)$ and denote it by $\mathrm{Arg}(z)$. For our example, $\mathrm{Arg}(z) = -\frac{\pi}{4}$.

In general, if we are given a complex number in rectangular coordinates, say $z = x + yi$, then, as we can see from Figure 7.1.2, the polar coordinates $r = |z|$ and $\theta = \mathrm{Arg}(z)$ are determined by

$$r = \sqrt{x^2 + y^2} \tag{7.1.20}$$

and

$$\tan(\theta) = \frac{y}{x}, \tag{7.1.21}$$

where the latter holds only if $x \neq 0$. If $x = 0$ and $y \neq 0$, then z is purely imaginary and hence lies on the imaginary axis of the complex plane. In that case, $\theta = \frac{\pi}{2}$ if $y > 0$ and $\theta = \frac{\pi}{2}$ if $y < 0$. If both $x = 0$ and $y = 0$, then z is completely specified by the condition $r = 0$ and θ may take on any value.

Note that, since the range of the arc tangent function is $(-\frac{\pi}{2}, \frac{\pi}{2})$, the condition

$$\tan(\theta) = \frac{y}{x}$$

only implies that

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

if $x > 0$, that is, if θ is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

Example Suppose $z = -1 - \sqrt{3}i$. Then

$$|z| = \sqrt{1 + 3} = 2$$

and, if $\theta = \mathrm{Arg}(z)$,

$$\tan(\theta) = \frac{-\sqrt{3}}{-1} = -\sqrt{3}.$$

Since z lies in the third quadrant, we have

$$\mathrm{Arg}(z) = -\frac{2\pi}{3}.$$

Now suppose z_1 and z_2 are two nonzero complex numbers with $|z_1| = r_1$, $|z_2| = r_2$, $\arg(z_1) = \theta_1$, and $\arg(z_2) = \theta_2$. Then

$$z_1 = r_1 \cos(\theta_1) + r_1 \sin(\theta_1)i = r_1(\cos(\theta_1) + \sin(\theta_1)i)$$

and

$$z_2 = r_2 \cos(\theta_2) + r_2 \sin(\theta_2)i = r_2(\cos(\theta_2) + \sin(\theta_2)i).$$

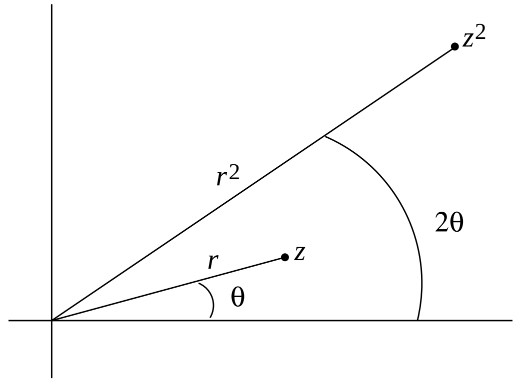


Figure 7.1.3 Geometry of z and z^2 in the complex plane

Hence

$$\begin{aligned} z_1 z_2 &= (r_1 \cos(\theta_1) + r_1 \sin(\theta_1) i)(r_2 \cos(\theta_2) + r_2 \sin(\theta_2) i) \\ &= r_1 r_2 (\cos(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2) i + \sin(\theta_1)\cos(\theta_2) i - \sin(\theta_1)\sin(\theta_2)) \\ &= r_1 r_2 (\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) + (\sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\cos(\theta_2)) i) \\ &= r_1 r_2 (\cos(\theta_1 + \theta_2) + \sin(\theta_1 + \theta_2) i). \end{aligned}$$

It follows that

$$|z_1 z_2| = r_1 r_2 = |z_1||z_2| \tag{7.1.22}$$

and

$$\arg(z_1 z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2). \tag{7.1.23}$$

In other words, the magnitude of the product of two complex numbers is the product of their respective magnitudes and the argument of the product of two complex numbers is the sum of their respective arguments.

In particular, for any complex number z, $|z^2| = |z|^2$ and $\arg(z^2) = 2\arg(z)$. More generally, for any positive integer n,

$$|z^n| = |z|^n \tag{7.1.24}$$

and

$$\arg(z^n) = n \arg(z). \tag{7.1.25}$$

See Figure 7.1.3.

If z is a complex number with $|z| = r$ and $\arg(z) = \theta$, then

$$z = r(\cos(\theta) + \sin(\theta))$$

and

$$\bar{z} = r(\cos(\theta) - \sin(\theta) i) = r(\cos(\theta) + \sin(-\theta) i). \tag{7.1.26}$$

Hence

$$|\bar{z}| = |z| \tag{7.1.27}$$

and

$$\arg(\bar{z}) = -\arg(z), \tag{7.1.28}$$

in agreement with our previous observation that $\bar{z}$ is obtained from z by reflection about the real axis.

If z_1 and z_2 are two nonzero complex numbers with $|z_1| = r_1$, $|z_2| = r_2$, $\arg(z_1) = \theta_1$, and $\arg(z_2) = \theta_2$, then

$$\begin{aligned}
\frac{z_1}{z_2} &= \frac{z_1\bar{z}_2}{z_2\bar{z}_2} \\
&= \frac{r_1r_2(\cos(\theta_1-\theta_2)+\sin(\theta_1-\theta_2)i)}{r_2^2} \\
&= \frac{r_1}{r_2}(\cos(\theta_1-\theta_2)+\sin(\theta_1-\theta_2)i).
\end{aligned}$$

Hence

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \tag{7.1.29}$$

and

$$\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2. \tag{7.1.30}$$

In other words, the magnitude of the quotient of two complex numbers is the quotient of their respective magnitudes and the argument of the quotient of two complex numbers is the difference of their respective arguments.

Example Let $z = 2(\cos(\frac{\pi}{12}) + \sin(\frac{\pi}{12})i)$ and $w = 3(\cos(\frac{\pi}{6}) + \sin(\frac{\pi}{6})i)$. Then

$$\begin{aligned}
zw &= 6\left(\cos\left(\frac{\pi}{12}+\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{12}+\frac{\pi}{6}\right)i\right) \\
&= 6\left(\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right)i\right) \\
&= 6\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) \\
&= 3\sqrt{2} + 3\sqrt{2}i.
\end{aligned}$$

Also,

$$\begin{aligned}
\frac{z}{w} &= 6\left(\cos\left(\frac{\pi}{12}-\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{12}-\frac{\pi}{6}\right)i\right) \\
&= 6\left(\cos\left(-\frac{\pi}{12}\right) + \sin\left(-\frac{\pi}{12}\right)i\right) \\
&= 6\left(\cos\left(\frac{\pi}{12}\right) - \sin\left(\frac{\pi}{12}\right)i\right) \\
&= 5.796 - 1.553i,
\end{aligned}$$

where we have rounded the real and imaginary parts to three decimal places.

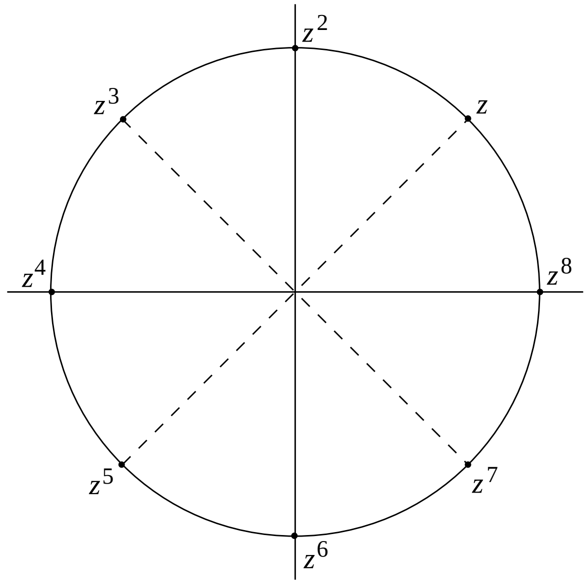


Figure 7.1.4 Powers of $z = \cos(\frac{\pi}{4}) + \sin(\frac{\pi}{4})i$

Example Let

$$z = \cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{4}\right)i = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i.$$

Since $|z| = 1$ and $\arg(z) = \frac{\pi}{4}$, z is a point on the unit circle centered at the origin, one-eighth of the way around the circle from $(1,0)$ (see Figure 7.1.4). Then

$$\begin{aligned}
z^2 &= \cos\left(2\cdot\frac{\pi}{4}\right) + \sin\left(2\cdot\frac{\pi}{4}\right)i = \cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right)i = i,\\
z^3 &= \cos\left(3\cdot\frac{\pi}{4}\right) + \sin\left(3\cdot\frac{\pi}{4}\right)i = \cos\left(\frac{3\pi}{4}\right) + \sin\left(\frac{3\pi}{4}\right)i = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i,\\
z^4 &= \cos\left(4\cdot\frac{\pi}{4}\right) + \sin\left(4\cdot\frac{\pi}{4}\right)i = \cos(\pi) + \sin(\pi)i = -1,\\
z^5 &= \cos\left(5\cdot\frac{\pi}{4}\right) + \sin\left(5\cdot\frac{\pi}{4}\right)i = \cos\left(\frac{5\pi}{4}\right) + \sin\left(\frac{5\pi}{4}\right)i = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\\
z^6 &= \cos\left(6\cdot\frac{\pi}{4}\right) + \sin\left(6\cdot\frac{\pi}{4}\right)i = \cos\left(\frac{3\pi}{2}\right) + \sin\left(\frac{3\pi}{2}\right)i = -i,\\
z^7 &= \cos\left(7\cdot\frac{\pi}{4}\right) + \sin\left(7\cdot\frac{\pi}{4}\right)i = \cos\left(\frac{7\pi}{4}\right) + \sin\left(\frac{7\pi}{4}\right)i = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i,\\
z^8 &= \cos\left(8\cdot\frac{\pi}{4}\right) + \sin\left(8\cdot\frac{\pi}{4}\right)i = \cos(2\pi) + \sin(2\pi)i = 1,
\end{aligned}$$

and

$$z^9 = zz^8 = (z)(1) = z.$$

Hence each successive power of z is obtained by rotating the previous power through an angle of $\frac{\pi}{4}$ on the unit circle centered at the origin; after eight rotations, the point has returned to where it started. See Figure 7.1.4. Notice in particular that z is a root of the polynomial

$$P(z) = z^8 - 1.$$

In fact, z^n is a solution of $z^8 - 1 = 0$ for any positive integer n since

$$(z^n)^8 - 1 = (z^8)^n - 1 = 1^n - 1 = 1 - 1 = 0.$$

Thus there are eight distinct roots of $P(z)$, namely, z, z^2, z^3, z^4, z^5, z^6, z^7, and z^8, only two of which, $z^4 = -1$ and $z^8 = 1$, are real numbers.

Problems

1. Evaluate the following if $w = 3 - 4i$ and $z = -2 + 7i$.
   (a) $w + z$ (b) $w - z$
   (c) $3w - 2z$ (d) $\bar{w}$
   (e) zw (f) $\dfrac{z}{w}$
   (g) $|z|$ (h) $\dfrac{z^2 - w}{z + w}$
   (i) $\Re(z - w)$ (j) $\Im(3z + w)$
2. Find the real and imaginary parts of each of the following.
   (a) $\dfrac{1}{i}$ (b) $\dfrac{3}{1 + 2i}$
   (c) $\dfrac{3 - 4i}{-2 + 3i}$ (d) $(1 + i)^3$
3. For each of the following, write the given z in rectangular coordinates and plot it in the complex plane.
   (a) $|z| = 3$, $\text{Arg}(z) = \dfrac{\pi}{2}$ (b) $|z| = 5$, $\text{Arg}(z) = \dfrac{2\pi}{3}$
   (c) $|z| = 0.5$, $\text{Arg}(z) = -\dfrac{3\pi}{4}$ (d) $|z| = 2$, $\text{Arg}(z) = \pi$
4. For each of the following, find $|z|$ and $\text{Arg}(z)$ and plot z in the complex plane.
   (a) $z = -i$ (b) $z = -5$
   (c) $z = 1 + i$ (d) $z = -1 - i$
   (e) $z = 2 + 2\sqrt{3}i$ (f) $z = \sqrt{3} - i$
5. Suppose w and z are complex numbers with $|w| = 3$, $\text{Arg}(w) = \frac{\pi}{6}$, $|z| = 2$, and $\text{Arg}(z) = -\frac{\pi}{3}$. Find both polar and rectangular coordinates for each of the following.
   (a) w^2 (b) z^3
   (c) wz (d) $\dfrac{w}{z}$

(e) $\dfrac{z}{w^2}$

(f) w^5

6. Find all the roots of the polynomial $P(z) = z^6 - 1$ and plot them in the complex plane.

7. Let $v = a_1 + b_1 i$, $w = a_2 + b_2 i$, and $z = a_3 + b_3 i$ be complex numbers. Verify each of the following.

   (a) $v + w = w + v$

   (b) $vw = wv$

   (c) $v(w + z) = vw + vz$

   (d) $(v + w) + z = v + (w + z)$

   (e) $v(wz) = (vw)z$

   (f) $(w + z)^2 = w^2 + 2wz + z^2$

8. Suppose z is a complex number with $|z| = r$ and $\arg(z) = \theta$.

   (a) Let w be a complex number with $|w| = \sqrt{r}$ and $\arg(w) = \frac{\theta}{2}$. Show that $w^2 = z$.

   (b) Let v be a complex number with $|v| = \sqrt{r}$ and $\arg(v) = \frac{\theta}{2} + \pi$. Show that $v^2 = z$.

   (c) From (a) and (b) we see that every nonzero complex number has two distinct square roots. Find the square roots, in rectangular form, of $1 + \sqrt{3}i$ and -9.

Difference Equations to **D**ifferential Equations

Section 7.2

The Calculus of Complex Functions

In this section we will discuss limits, continuity, differentiation, and Taylor series in the context of functions which take on complex values. Moreover, we will introduce complex extensions of a number of familiar functions. Since complex numbers behave algebraically like real numbers, most of our results and definitions will look like the analogous results for real-valued functions. We will avoid going into much detail; the complete story of the calculus of complex-valued functions is best left to a course in complex analysis. However, we will see enough of the story to enable us to make effective use of complex numbers in elementary calculations.

We begin with a definition of the limit of a sequence of complex numbers.

Definition We say that the *limit* of a sequence of complex numbers $\{z_n\}$ is L, and write

$$\lim_{n\to\infty} z_n = L,$$

if for every $\epsilon > 0$ there exists an integer N such that

$$|z_n - L| < \epsilon$$

whenever $n > N$.

Notice that the only difference between this definition and the definition of the limit of a sequence given in Section 1.2 is the use of the magnitude of a complex number in place of the absolute value of a real number. Even here, the notation is the same. The point is the same as it was in Chapter 1: the limit of the sequence $\{z_n\}$ is L if we can always ensure that the values of the sequence are within a desired distance of L by going far enough out in the sequence.

Now if $z_n = x_n + y_n i$ and $L = a + bi$, then $\lim_{n\to\infty} z_n = L$ if and only if

$$\lim_{n\to\infty} |z_n - L| = \lim_{n\to\infty} \sqrt{(x_n - a)^2 + (y_n - b)^2} = 0,$$

the latter of which occurs if and only if $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} y_n = b$. Hence we have the following useful result.

Proposition Let $z_n = x_n + y_n i$ and $L = a + bi$. Then

$$\lim_{n\to\infty} z_n = L$$

if and only if

$$\lim_{n\to\infty} x_n = a \text{ and } \lim_{n\to\infty} y_n = b.$$

Thus to determine the limiting behavior of a sequence $\{z_n\}$ of complex numbers, we need only consider the behavior of the two sequences of real numbers, $\{\Re(z_n)\}$ and $\{\Im(z_n)\}$.

Example Suppose

$$z_n = \frac{3n-1}{2n+2} + \frac{n+1}{n-1}i$$

for $n = 1, 2, 3, \ldots$. Then

$$\lim_{n\to\infty} \Re(z_n) = \lim_{n\to\infty} \frac{3n-1}{2n+2} = \lim_{n\to\infty} \frac{3 - \frac{1}{n}}{2 + \frac{2}{n}} = \frac{3}{2}$$

and

$$\lim_{n\to\infty} \Im(z_n) = \lim_{n\to\infty} \frac{n+1}{n-1} = \lim_{n\to\infty} \frac{1 + \frac{1}{n}}{1 - \frac{1}{n}} = 1,$$

so

$$\lim_{n\to\infty} z_n = \frac{3}{2} + i.$$

Example Suppose

$$z_n = \frac{1}{n}\left(\cos\left(\frac{n\pi}{3}\right) + \sin\left(\frac{n\pi}{3}\right)i\right)$$

for $n = 1, 2, 3, \ldots$. Then

$$\lim_{n\to\infty} \Re(z_n) = \lim_{n\to\infty} \frac{\cos\left(\frac{n\pi}{3}\right)}{n} = 0$$

and

$$\lim_{n\to\infty} \Im(z_n) = \lim_{n\to\infty} \frac{\sin\left(\frac{n\pi}{3}\right)}{n} = 0,$$

so

$$\lim_{n\to\infty} z_n = 0.$$

Geometrically, since $|z_n| = \frac{1}{n}$ and $\arg(z_n) = \frac{n\pi}{3}$, the points in this sequence are converging to 0 along a spiral path, as seen in Figure 7.2.1.

Having defined the limit of a sequence of complex numbers, we may define the limit of a complex-valued function, as in Section 2.3, and then define continuity, as in Section 2.4.

Definition Suppose $f : \mathbb{C} \to \mathbb{C}$, that is, f is a complex-valued function of a complex variable. We say the *limit* of $f(z)$ as z approaches a is L, written

$$\lim_{z\to a} f(z) = L,$$

if whenever $\{z_n\}$ is a sequence of points with $z_n \neq a$ for all n and $\lim\limits_{n\to\infty} z_n = a$, then

$$\lim_{n\to\infty} f(z_n) = L.$$

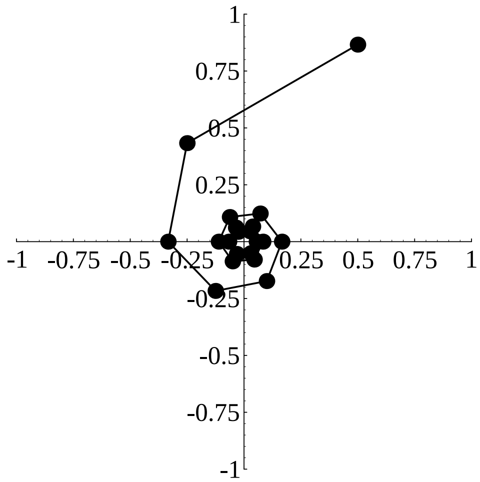


Figure 7.2.1 Plot of the points $z_n = \frac{1}{n}\left(\cos\left(\frac{n\pi}{3}\right) + \sin\left(\frac{n\pi}{3}\right)i\right)$, $n = 1, 2, 3, \ldots 20$

Definition We say the function $f : \mathbb{C} \to \mathbb{C}$ is *continuous* at a if $\lim\limits_{z \to a} f(z) = f(a)$.

As with real-valued functions of a real variable, it is easy to show that algebraic functions of a complex variable are continuous wherever they are defined. In particular, complex polynomials, that is, functions P of the form

$$P(z) = a_n z_n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0,$$

where n is a nonnegative integer and the coefficients $a_0, a_1, \ldots, a_n$ are complex numbers, are continuous at all points in the complex plane. Complex rational functions, that is, functions R of the form

$$R(z) = \frac{P(z)}{Q(z)},$$

where both P and Q are polynomials, are continuous at all points where they are defined.

Example Since $f(z) = 3z^2 - iz + 4 - 5i$ is a polynomial, it is continuous at all points in the complex plane. In particular,

$$\lim_{z \to i} f(z) = \lim_{z \to i}(3z^2 - iz + 4 - 5i) = 3i^2 - (i)(i) + 4 - 5i = 2 - 5i.$$

Example Algebraic simplification may be useful in evaluating limits here as it was in Section 2.3. For example,

$$\lim_{z \to i} \frac{z - i}{z^2 + 1} = \lim_{z \to i} \frac{z - i}{(z - i)(z + i)} = \lim_{z \to i} \frac{1}{z + i} = \frac{1}{2i} = \frac{1}{2i}\frac{i}{i} = -\frac{1}{2}i.$$

Although this is not the time to go into any detail about the geometric meaning of the derivative of a function $f : \mathbb{C} \to \mathbb{C}$, the algebraic definition and manipulation of derivatives follows the pattern of the results for real-valued functions in Chapter 3.

Definition If $f : \mathbb{C} \to \mathbb{C}$, then the *derivative* of f at a, denoted $f'(a)$, is given by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}, \tag{7.2.1}$$

provided the limit exists.

Note that h in this definition is, in general, a complex number, not just a real number. Since the algebraic properties of the complex numbers are very similar to the algebraic properties of the real numbers, much of what we learned about differentiation in Chapter 3 still holds true in our new situation. For example, if n is a nonzero rational number, then

$$\frac{d}{dz} z^n = n z^{n-1}. \tag{7.2.2}$$

Moreover, all the techniques we learned for computing derivatives in Sections 3.3 and 3.4, including the quotient, product, and chain rules, still hold.

Example If $f(z) = 3z^5 + iz^3 - (3+2i)z$, then

$$f'(z) = 15z^4 + 3iz^2 - 3 - 2i.$$

Example If

$$g(w) = \frac{(3+i)w^2}{2w-1},$$

then, using the quotient rule,

$$g'(w) = \frac{(2w-1)(6+2i)w - (3+i)w^2(2)}{(2w-1)^2} = \frac{(6+2i)(w^2-w)}{(2w-1)^2}.$$

From this point it is possible to follow the pattern of Chapter 5 and develop the theory of polynomial approximations using Taylor polynomials, defined in a manner analogous to the definition in Section 5.1, as well as the theory of power series and Taylor series. In particular, a power series

$$\sum_{n=0}^{\infty} a_n (z-a)^n, \tag{7.2.3}$$

where $a_0, a_1, a_2, \ldots$ and a are complex numbers, is said to *converge absolutely* at those points z for which the series

$$\sum_{n=0}^{\infty} |a_n||z-a|^n \tag{7.2.4}$$

converges. Since the latter series involves only real numbers, its convergence may be determined using the tests developed in Chapter 5. As before, absolute convergence implies convergence. Moreover, if the series (7.2.3) converges at points other than a, then there exists an R, either a positive real number or ∞, such that the series converges absolutely for all z such that $|z-a|<R$ and diverges for all z such that $|z-a|>R$. However, note that in this case the set of all points in the complex plane such that $|z-a|<R$ is a disk of radius R centered at a, not an interval as it was in the real number case.

Example Consider the power series

$$\sum_{n=0}^{\infty} z^n. \tag{7.2.5}$$

Since the series

$$\sum_{n=0}^{\infty} |z|^n$$

is a geometric series, it converges for all values of z for which $|z|<1$. Hence

$$\sum_{n=0}^{\infty} z^n$$

converges for all z for which $|z|<1$, that is, for all z inside the unit circle centered at the origin of the complex plane. Thus the radius of convergence of (7.2.5) is $R=1$. Using the same argument as we used in Section 1.3, we can show that

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

for all z with $|z|<1$. For example,

$$\sum_{n=0}^{\infty} \left(\frac{i}{2}\right)^n = \frac{1}{1-\frac{i}{2}} = \frac{2}{2-i} = \frac{2(2+i)}{(2-i)(2+i)} = \frac{4}{5} + \frac{2}{5}i.$$

Example Consider the power series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}. \tag{7.2.6}$$

To determine its radius of convergence, we apply the ratio test to the series

$$\sum_{n=0}^{\infty} \left|\frac{z^n}{n!}\right| = \sum_{n=0}^{\infty} \frac{|z|^n}{n!}, \tag{7.2.7}$$

obtaining

$$\rho = \lim_{n\to\infty} \frac{\dfrac{|z|^{n+1}}{(n+1)!}}{\dfrac{|z|^n}{n!}} = \lim_{n\to\infty} \frac{|z|}{n+1} = 0$$

for all values of z. Since $\rho = 0$ for any value of z, (7.2.7) converges for all z in the complex plane. That is, the radius of convergence of (7.2.6) is $R = \infty$. Of course, we also know that (7.2.7) converges for all z because, from our work in Section 6.1, it is equal to $e^{|z|}$.

The power series in the last example is the extension to complex numbers of the series we used to define the exponential function in Section 6.1. With it, we can define the complex exponential function.

Definition The *complex exponential function*, with value at z denoted by $\exp(z)$, is defined for all points in the complex plane by

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}. \tag{7.2.8}$$

Of course, this definition agrees with our old definition when z is real.

In Chapter 6 we used the exponential function to give meaning to exponents which were not rational numbers. Similarly, the complex exponential function may be used to define complex exponents. However, we will only consider the case of raising e to a complex power.

Definition If z is a complex number with $\Im(z) \neq 0$, then we define $e^z = \exp(z)$.

With this definition we now have $e^z = \exp(z)$ for all z in the complex plane, the case when $\Im(z) = 0$, that is, when z is real, having been treated in Section 6.1. Although we will not repeat them here, the arguments from Section 6.1 come over to establish the following proposition.

Proposition For any complex numbers w and z,

$$e^{w+z} = e^w e^z \tag{7.2.9}$$

and

$$e^{w-z} = \frac{e^w}{e^z}. \tag{7.2.10}$$

Also, as in Section 6.1, direct differentiation of (7.2.8) yields the following result.

Proposition

$$\frac{d}{dz} e^z = e^z. \tag{7.2.11}$$

Example Using the product and chain rules,

$$\frac{d}{dz}\left(z^2 e^{-z^2}\right) = z^2(-2z)e^{-z^2} + 2ze^{-z^2} = 2z(1 - z^2)e^{-z^2}.$$

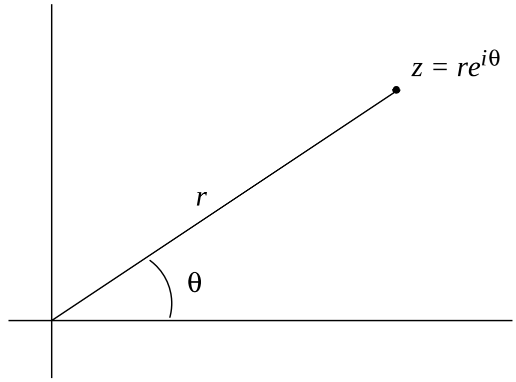


Figure 7.2.2 Plot of the point $re^{i\theta}$ in the complex plane

The exponential of a pure imaginary number is particularly interesting. To see why, let θ be a real number and consider

$$\begin{aligned} e^{i\theta} &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} \\ &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\ &= \cos(\theta) + i\sin(\theta). \end{aligned}$$

Proposition For any real number θ,

$$e^{i\theta} = \cos(\theta) + i\sin(\theta). \tag{7.2.12}$$

As a consequence, if θ is a real number, then $|e^{i\theta}| = 1$ and $\arg(e^{i\theta}) = \theta$. That is, $e^{i\theta}$ is a point in the complex plane on the unit circle centered at the origin, a distance of θ radians away, in a counterclockwise direction, along the circle from $(1, 0)$. Moreover, if z is a nonzero complex number with $|z| = r$ and $\arg(z) = \theta$, then

$$z = r(\cos(\theta) + i\sin(\theta)) = re^{i\theta}. \tag{7.2.13}$$

This exponential notation provides a compact way to display any nonzero complex number in polar form. See Figure 7.2.2.

Example If $z = 1 - i$, then $|z| = \sqrt{2}$ and $\mathrm{Arg}(z) = -\frac{\pi}{4}$, so

$$z = \sqrt{2}e^{-i\frac{\pi}{4}}.$$

Moreover,

$$\bar{z} = \sqrt{2}e^{i\frac{\pi}{4}}$$

and

$$z^2 = 2e^{-2i\frac{\pi}{4}} = 2e^{-i\frac{\pi}{2}} = -2i.$$

Example If $w = 3e^{i\frac{\pi}{3}}$ and $z = 5e^{i\frac{\pi}{8}}$, then

$$wz = \left(3e^{i\frac{\pi}{3}}\right)\left(5e^{i\frac{\pi}{8}}\right) = 15e^{i\left(\frac{\pi}{3}+\frac{\pi}{8}\right)} = 15e^{i\frac{11\pi}{24}}$$

and

$$\frac{w}{z} = \frac{3e^{i\frac{\pi}{3}}}{5e^{i\frac{\pi}{8}}} = \frac{3}{5}e^{i\left(\frac{\pi}{3}-\frac{\pi}{8}\right)} = \frac{3}{5}e^{i\frac{5\pi}{24}}.$$

Since for any real number θ,

$$e^{i\theta} = \cos(\theta) + i\sin(\theta),$$

it follows that

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos(\theta) - i\sin(\theta).$$

Hence

$$e^{i\theta} - e^{-i\theta} = \cos(\theta) + i\sin(\theta) - (\cos(\theta) - i\sin(\theta)) = 2i\sin(\theta). \tag{7.2.14}$$

Solving (7.2.14) for $\sin(\theta)$, we have

$$\sin(\theta) = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}).$$

Similarly,

$$e^{i\theta} + e^{-i\theta} = \cos(\theta) + i\sin(\theta) + \cos(\theta) - i\sin(\theta) = 2\cos(\theta), \tag{7.2.15}$$

from which we obtain

$$\cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta}).$$

Proposition For any real number θ,

$$\sin(\theta) = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) \tag{7.2.16}$$

and

$$\cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta}). \tag{7.2.17}$$

These formulas are very similar to the formulas we used to define the hyperbolic sine and cosine functions in Section 6.7. We will now use these formulas to define the complex sine and cosine functions; at the same time, we will extend the definitions of the hyperbolic sine and cosine functions. In doing so, we will see just how closely related the circular and hyperbolic trigonometric functions really are.

Definition The *complex sine function*, with value at z denoted by $\sin(z)$, and the *complex cosine function*, with value at z denoted by $\cos(z)$, are defined for all z in the complex plane by

$$\sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz}) \tag{7.2.18}$$

and

$$\cos(z) = \frac{1}{2}(e^{iz} + e^{-iz}). \tag{7.2.19}$$

The *complex hyperbolic sine function*, with value at z denoted by $\sinh(z)$, and the *complex hyperbolic cosine function*, with value at z denoted by $\cosh(z)$, are defined for all z in the complex plane by

$$\sinh(z) = \frac{1}{2}(e^{z} - e^{-z}) \tag{7.2.20}$$

and

$$\cosh(z) = \frac{1}{2}(e^{z} + e^{-z}). \tag{7.2.21}$$

Note that these functions are defined so that they agree with their original versions when evaluated at real numbers.

With these definitions it is a simple matter to prove that

$$\frac{d}{dz}\sin(z) = \cos(z), \tag{7.2.22}$$

$$\frac{d}{dz}\cos(z) = -\sin(z), \tag{7.2.23}$$

$$\frac{d}{dz}\sinh(z) = \cosh(z), \tag{7.2.24}$$

and

$$\frac{d}{dz}\cosh(z) = \sinh(z). \tag{7.2.25}$$

For example,

$$\begin{aligned}
\frac{d}{dz}\cos(z) &= \frac{d}{dz}\left(\frac{1}{2}(e^{iz} + e^{-iz})\right) \\
&= \frac{1}{2}(ie^{iz} - ie^{-iz}) \\
&= \frac{i}{2}(e^{iz} - e^{-iz}) \\
&= \frac{i^2}{2i}(e^{iz} - e^{-iz}) \\
&= -\frac{1}{2}(e^{iz} - e^{-iz}) \\
&= -\sin(z).
\end{aligned}$$

Example Note that

$$\begin{aligned}\sin(i) &= \frac{1}{2i}(e^{i^2} - e^{-i^2})\\ &= \frac{1}{2i}(e^{-1} - e^{1})\\ &= -\frac{1}{i}\sinh(1)\\ &= -\frac{i}{i^2}\sinh(1)\\ &= i\sinh(1).\end{aligned}$$

Example Using the product and chain rules, we have

$$\begin{aligned}\frac{d}{dz}\sin(2z)\cos(3z) &= \sin(2z)(-\sin(3z))(3) + \cos(3z)\cos(2z)(2)\\ &= 3\sin(2z)\sin(3z) + 2\cos(2z)\cos(3z).\end{aligned}$$

The final complex-valued function we will define is the complex logarithm function. Analogous to our other definitions in this section, we would like this function to share the basic characteristic properties of the ordinary logarithm function and to agree with that function when evaluated at a positive real number. In particular, if we let $\mathrm{Log}(z)$ denote the complex logarithm of a complex number z and $\log(r)$ denote the real logarithm of a positive real number r, then for a nonzero complex number z with $|z| = r$ and $\mathrm{Arg}(z) = \theta$ we would like to have

$$\mathrm{Log}(z) = \mathrm{Log}(re^{i\theta}) = \mathrm{Log}(r) + \mathrm{Log}(e^{i\theta}) = \log(r) + i\theta. \tag{7.2.26}$$

Moreover, using (7.2.26) to define the complex logarithm function will guarantee that our new function agrees with the ordinary logarithm function when evaluated at positive real numbers, for if z is a positive real number, then $|z| = z$ and $\mathrm{Arg}(z) = 0$, giving us $\mathrm{Log}(z) = log(z)$.

Definition The *complex logarithm function*, with value at z denoted by $\mathrm{Log}(z)$, is defined for all nonzero complex numbers z with $|z| = r$ and $\mathrm{Arg}(z) = \theta$ by

$$\mathrm{Log}(z) = \log(r) + i\theta, \tag{7.2.27}$$

where $\log(r)$ is the ordinary real-valued logarithm of r.

Note that we have used the principal value of $\arg(z)$, that is, $\mathrm{Arg}(z)$, in the definition of $\mathrm{Log}(z)$ in order to give $\mathrm{Log}(z)$ a unique value. Moreover, note that this definition gives meaning to the logarithm of a negative real number, although it still does not define the logarithm of 0.

Example Since $|2 - 2i| = \sqrt{8}$ and $\mathrm{Arg}(2 - 2i) = -\frac{\pi}{4}$, we have

$$\mathrm{Log}(2 - 2i) = \log(\sqrt{8}) - \frac{\pi}{4}i = \frac{1}{2}\log(8) - \frac{\pi}{4}i = \frac{3}{2}\log(2) - \frac{\pi}{4}i.$$

Example Since $|-4| = 4$ and $\operatorname{Arg}(-4) = \pi$, we have

$$\operatorname{Log}(-4) = \log(4) + \pi i = 2\log(2) + \pi i.$$

Problems

1. For each of the following, find $\lim_{n\to\infty} z_n$. Also, plot $z_1, z_2, z_3, \ldots, z_{15}$ in the complex plane.

   (a) $z_n = \dfrac{3-n}{n} + \dfrac{n+1}{2n+3}i$ (b) $\dfrac{2-n}{n^2} - \left(4 + \dfrac{6}{n}\right)i$

   (c) $z_n = 3e^{i\frac{\pi}{n}}$ (d) $z_n = e^{i\frac{\pi(n-1)}{n}}$

2. Evaluate each of the following limits.

   (a) $\lim_{z\to i}(4z^3 - 6z + 3)$ (b) $\lim_{z\to 1-i}(z^2 - 3z)$

   (c) $\lim_{w\to 3i} \dfrac{z^2+9}{z-3i}$ (d) $\lim_{z\to i} \dfrac{z^4-1}{z^2+1}$

3. Find the derivative of each of the following functions.

   (a) $f(z) = 3z^2 - 6z^5 + 18i$ (b) $g(w) = \dfrac{13w - 6i + 3}{w+i}$

   (c) $f(z) = (z-4i)e^{-z^2}$ (d) $h(s) = (s^2+1)\exp(3s^2 - si)$

4. (a) Show that
   $$\frac{1}{1+z^2} = \sum_{n=0}^{\infty}(-1)^n z^{2n}$$
   for all z with $|z| < 1$.

   (b) How does (a) help explain why, for real values of x, the Taylor series
   $$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$
   converges only on the interval $(-1, 1)$?

5. (a) If $z = x + yi$, show that
   $$\Re(e^z) = e^x\cos(y)$$
   and
   $$\Im(e^z) = e^x\sin(y).$$

   (b) If $z = x + yi$, find $|e^z|$ and $\arg(e^z)$.

6. Show that $e^{i\pi} + 1 = 0$.

7. Verify the differentiation formulas for $\sin(z)$, $\sinh(z)$, and $\cosh(z)$.
8. (a) Show that
$$\int_{-2}^{-1} \frac{1}{x}\, dx = -\log(2).$$
   (b) Some computer algebra systems evaluate the integral in (a) as
$$\int_{-2}^{-1} \frac{1}{x}\, dx = \mathrm{Log}(-1) - \mathrm{Log}(-2).$$
   Reconcile this answer with the answer in (a).
9. Let z and w be complex numbers. Verify the following two properties of the complex logarithm.
   (a) $\mathrm{Log}(wz) = \mathrm{Log}(w) + \mathrm{Log}(z)$
   (b) $\mathrm{Log}\left(\frac{w}{z}\right) = \mathrm{Log}(w) - \mathrm{Log}(z)$
10. For a positive integer n, an *nth root of unity* is a complex number z with the property that $z^n = 1$. Show that for $m = 0, 1, \ldots, n-1$,
$$z_m = e^{i\frac{2m\pi}{n}}$$
   is an nth root of unity. Plot these points in the complex plane for $n = 10$.
11. (a) Use the fact that
$$\sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz})$$
   to find the complex power series representation for $\sin(z)$.
   (b) Use the fact that
$$\cos(z) = \frac{1}{2}(e^{iz} + e^{-iz})$$
   to find the complex power series representation for $\cos(z)$.
12. Define a complex version of the tangent function and show that
$$\tan(z) = \frac{1}{i}\left(\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}\right).$$
13. (a) Show that $\sin(ix) = i\sinh(x)$ for every real number x.
   (b) Show that $\cos(ix) = \cosh(x)$ for every real number x.
14. Let $z = x + yi$.
   (a) Show that
$$\Re(\sin(z)) = \sin(x)\cosh(y)$$

and

$$\Im(\sin(z)) = \cos(x)\sinh(y).$$

(b) Show that

$$\Re(\cos(z)) = \cos(x)\cosh(y)$$

and

$$\Im(\sin(z)) = -\sin(x)\sinh(y).$$

15. (a) Show that for any nonzero complex number z, $e^{\mathrm{Log}(z)} = z$.
    (b) If z is a nonzero complex number, does it necessarily follow that $\mathrm{Log}(e^z) = z$?

Difference Equations to **D**ifferential Equations

Section 7.3

Complex-Valued Functions: Motion in the Plane

In Section 7.2 we considered the problem of extending the elementary functions of calculus to complex-valued functions of a complex variable, while at the same time extending many of the concepts of the first six chapters to these new functions. In this section we will consider complex-valued functions of a real variable, that is, functions of the form $f : \mathbb{R} \to \mathbb{C}$. Such functions are often used to describe the motion of an object in the plane; if we think of the real variable t as measuring time, then we may interpret $f(t)$ as the location of an object in the complex plane at time t.

Since limits are at the foundation of most concepts in calculus, we begin with a definition of limit in this setting.

Definition Suppose $f : \mathbb{R} \to \mathbb{C}$ and f is defined for all t in an interval about the point a. We say that the *limit* of $f(t)$ as t approaches a is L, denoted

$$\lim_{t \to a} f(t) = L,$$

if whenever $\{t_n\}$ is a sequence of real numbers with $t_n \neq a$ for all n and

$$\lim_{n \to \infty} t_n = a,$$

then

$$\lim_{n \to \infty} f(t_n) = L.$$

Suppose $f : \mathbb{R} \to \mathbb{C}$. If we let $x(t) = \Re(f(t))$ and $y(t) = \Im(f(t))$, then

$$f(t) = x(t) + iy(t).$$

Hence, from our work in Section 7.2,

$$\lim_{t \to a} f(t) = \lim_{t \to a} x(t) + i \lim_{t \to a} y(t). \tag{7.3.1}$$

This result also holds if we modify our definition of limit to include one-sided limits and limits to ∞ or $-\infty$.

Example Suppose a particle moves in the plane so that its position at time t is given by

$$f(t) = \cos(2\pi t) + i \sin(2\pi t) = e^{2\pi i t}.$$

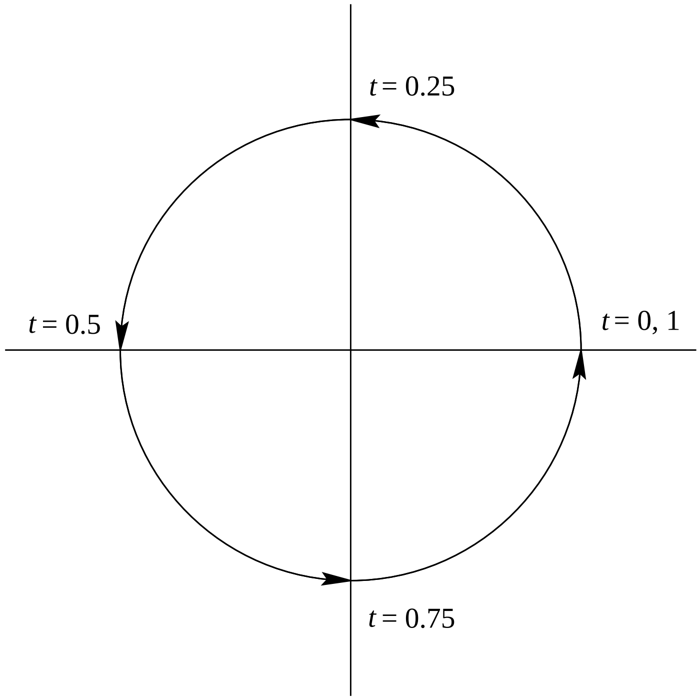


Figure 7.3.1 Motion of a particle on the unit circle centered at the origin

If we let C denote the unit circle centered at the origin, then $f(t)$ is a point in the complex plane on C, $2\pi t$ units from $(1,0)$ in the counterclockwise direction along the circumference of C. For example, at time $t=0$ the particle is at $f(0)=1$, at time $t=\frac{1}{4}$ the particle is at $f\left(\frac{1}{4}\right)=i$, at time $t=\frac{1}{2}$ the particle is at $f\left(\frac{1}{2}\right)=-1$, at $t=\frac{3}{4}$ the particle is at $f\left(\frac{3}{4}\right)=-i$, and at time $t=1$ the particle is at $f(1)=1$. Note that f has period 1, so the particle traverses C once in the counterclockwise direction as t goes from 0 to 1, after which the particle will repeat this motion over every interval of time of length 1. See Figure 7.3.1. As an example of a limit, we note that

$$\lim_{t\to-\frac{1}{4}} f(t) = \lim_{t\to-\frac{1}{4}} \cos(2t) + i \lim_{t\to-\frac{1}{4}} \sin(2t) = \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right) = -i.$$

Note that the path of the particle shown in Figure 7.3.1 is not the graph of the function f, but rather a plot of $f(t)$ for values of t from 0 to 1. In general, if $f(t)=x(t)+iy(t)$ represents the position of a particle moving in the complex plane at time t, we can obtain a good representation of the path of the particle over an interval of time $[a,b]$ by plotting the points $(x(t),y(t))$ for a large number of points in $[a,b]$ and connecting these points with straight lines, similar to the procedure we used for plotting the graph of a function in Section 2.1.

Example Suppose a particle moves in the plane so that its position at time t is given by

$$z(t) = \tanh(t) + i\operatorname{sech}(t).$$

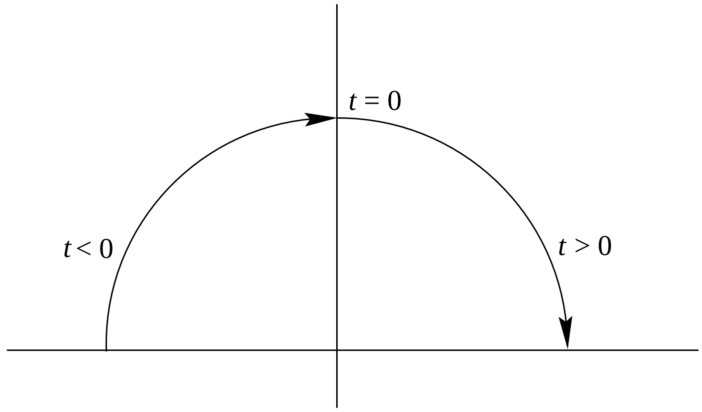


Figure 7.3.2 Motion on the upper half of the unit circle

Then $\Re(z(t)) = \tanh(t)$, $\Im(z(t)) = \operatorname{sech}(t)$, and

$$|z(t)| = \sqrt{\tanh^2(t) + \operatorname{sech}^2(t)} = 1.$$

Thus the particle is moving along the unit circle C as in the previous example. However, since $\operatorname{sech}(t) > 0$ for all t, the particle is always on the upper half of C. Moreover, $z(0) = i$,

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} \tanh(t) + i \lim_{t\to\infty} \operatorname{sech}(t) = 1$$

and

$$\lim_{t\to-\infty} z(t) = \lim_{t\to-\infty} \tanh(t) + i \lim_{t\to-\infty} \operatorname{sech}(t) = -1.$$

Combining these results with the fact that $\Re(z(t)) = \tanh(t)$ is an increasing function, we see that as time flows from $-\infty$ to ∞, the particle moves from left to right on the upper half of C, coming from the point $(-1, 0)$ as t increases from $-\infty$ and approaching the point $(1, 0)$ as t increases toward ∞. See Figure 7.3.2.

We may now define continuity and differentiability in analogy with our previous definitions.

Definition We say a function $f : \mathbb{R} \to \mathbb{C}$ is *continuous* at a if $\lim_{t\to a} f(t) = f(a)$.

If $x(t) = \Re(f(t))$ and $y(t) = \Im(f(t))$, then

$$\lim_{t\to a} f(t) = f(a)$$

if and only if

$$\lim_{t\to a} x(t) = x(a)$$

and

$$\lim_{t\to a} y(t) = y(a).$$

Hence f is continuous at a if and only if both x and y are continuous at a. For example, the functions in the previous examples are both continuous for every t in $(-\infty, \infty)$ since the functions $\cos(t)$, $\sin(t)$, $\tanh(t)$, and $\operatorname{sech}(t)$ are continuous for all t in $(-\infty, \infty)$.

Definition If $f : \mathbb{R} \to \mathbb{C}$, then the derivative of f at a, denoted $f'(a)$, is given by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}, \tag{7.3.2}$$

provided the limit exists.

Now if $f(t) = x(t) + iy(t)$, where x and y are both differentiable, then

$$\begin{aligned} f'(t) &= \lim_{h \to 0} \frac{f(t+h) - f(t)}{h} \\ &= \lim_{h \to 0} \frac{x(t+h) + iy(t+h) - (x(t) + iy(t))}{h} \\ &= \lim_{h \to 0} \frac{x(t+h) - x(t)}{h} + i \lim_{h \to 0} \frac{y(t+h) - y(t)}{h} \\ &= x'(t) + iy'(t). \end{aligned}$$

Hence differentiating a function $f : \mathbb{R} \to \mathbb{C}$ reduces to differentiating the real and complex parts of f.

Proposition If $x : \mathbb{R} \to \mathbb{R}$ and $y : \mathbb{R} \to \mathbb{R}$ are differentiable and $f(t) = x(t) + iy(t)$, then

$$f'(t) = x'(t) + iy'(t). \tag{7.3.3}$$

Example If $f(t) = \cos(t) + i\sin(t)$, then $f'(t) = -\sin(t) + i\cos(t)$.

Of course, it is also possible to express a function $f : \mathbb{R} \to C$ in polar form. If we let $r(t) = |f(t)|$ and $\theta(t) = \arg(f(t))$, then r and θ are real-valued functions and

$$f(t) = r(t)e^{i\theta(t)}. \tag{7.3.4}$$

If r and θ are differentiable, then

$$\begin{aligned} f'(t) &= \frac{d}{dt} r(t)e^{i\theta(t)} \\ &= \frac{d}{dt}(r(t)\cos(\theta(t)) + ir(t)\sin(\theta(t))) \\ &= -r(t)\sin(\theta(t))\theta'(t) + r'(t)\cos(\theta(t)) + ir(t)\cos(\theta(t))\theta'(t) + ir'(t)\sin(\theta(t)) \\ &= r(t)\theta'(t)(-\sin(\theta(t)) + i\cos(\theta(t))) + r'(t)(\cos(\theta(t)) + i\sin(\theta(t))) \\ &= ir(t)\theta'(t)\left(-\frac{\sin(\theta(t))}{i} + \cos(\theta(t))\right) + r'(t)e^{i\theta(t)} \\ &= ir(t)\theta'(t)(i\sin(\theta(t)) + \cos(\theta(t))) + r'(t)e^{i\theta(t)} \\ &= ir(t)\theta'(t)e^{i\theta(t)} + r'(t)e^{i\theta(t)}. \end{aligned}$$

Note that this result is exactly what we would obtain if we treated i as a real constant and differentiated (7.3.4) using the product and chain rules. Hence instead of remembering the formula, we need only remember that we should differentiate a function given in polar form using the product rule and treating i as we would any constant. In particular, taking $r(t) = 1$ for all t, we have

$$\frac{d}{dt}e^{i\theta(t)} = i\theta'(t)e^{i\theta(t)}. \tag{7.3.5}$$

Example If $f(t) = 4te^{it^2}$, then

$$f'(t) = 4t(2it)e^{it^2} + 4e^{it^2} = (4 + 8it^2)e^{it^2}.$$

To understand the derivative geometrically, consider the setting where $z(t) = x(t) + iy(t)$ represents the position at time t of a particle moving in the plane. Then $x'(t)$ represents the velocity of the particle in the x direction and $y'(t)$ represents the velocity of the particle in the y direction. In other words, if all forces acting on the particle were to cease at time t_0, then, according to Newton's first law, during the next unit of time the particle would move in a straight line, $x'(t_0)$ units in the x direction and $y'(t_0)$ units in the y direction. That is, in one unit of time the particle would travel along a straight line from $(x(t_0), y(t_0))$ to $(x(t_0) + x'(t_0), y(t_0) + y'(t_0))$. Hence the particle would move, in a straight line, from $z(t_0)$ to $z(t_0) + z'(t_0)$. Moreover, since the distance traveled in this unit of time is $|z'(t_0)|$, the speed of the particle at time t_0 is given by $|z'(t_0)|$. In short, $\arg(z'(t_0))$ tells us the direction in which the particle is moving at time t_0 and $|z'(t_0)|$ tells us the speed at which the particle is traveling at that instant. These considerations make the next definition reasonable.

Definition If $z(t)$ gives the position at time t of a particle moving in the plane, then we call $z'(t)$ the *velocity* of the particle and we call $|z'(t)|$ the *speed* of the particle.

Notice that this definition is directly analogous to our treatment of motion along a straight line in earlier chapters. In that case, if $f(t)$ represented the position of a particle moving along a straight line, then we called $f'(t)$ the velocity of the particle and $|f'(t)|$ the speed of the particle at time t.

Also notice that if for a given time t_0 we draw an arrow from $z(t_0)$ to $z(t_0) + z'(t_0)$, then this arrow points in the direction of motion of the particle at time t_0. Moreover, if $x'(t_0) \neq 0$, then this arrow has slope

$$\frac{y'(t_0)}{x'(t_0)}.$$

Now, from the chain rule, we have

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt},$$

from which we obtain

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(t)}{x'(t)}.$$

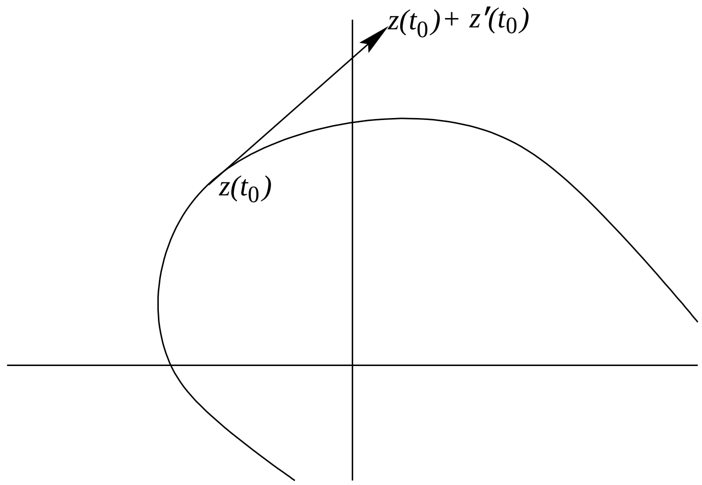


Figure 7.3.3 Motion in the complex plane

Hence the arrow from $z(t_0)$ to $z(t_0) + z'(t_0)$ points along the line tangent to the curve of motion at $z(t_0)$. Moreover, the length of this arrow is the speed of the particle at the instant t_0

Example If the position of a particle moving in the plane is given by $z(t) = 2e^{i\frac{t}{2}}$ at time t, then the particle is traveling counterclockwise on the circle C of radius 2 centered at the origin. The velocity of the particle is given by

$$z'(t) = ie^{i\frac{t}{2}} = e^{i\frac{\pi}{2}}e^{i\frac{t}{2}} = e^{i\left(\frac{t}{2}+\frac{\pi}{2}\right)}$$

and $|z'(t)| = 1$. Hence at any time t, the particle is moving at unit speed with velocity pointing in the direction of $z(t)$ rotated counterclockwise through an angle of $\frac{\pi}{2}$. See Figure 7.3.4.

Also in analogy with the one-dimensional case, if $z(t)$ is the position of a particle moving in the plane at time t, then the *acceleration* of the particle is given by $z''(t)$, the derivative of the velocity. Newton's second law of motion applies in this setting, telling us that if the particle has mass m, then the force acting on the particle at time t is

$$F(t) = mz''(t).$$

Hence the magnitude of the force acting on the particle is

$$|F(t)| = m|z''(t)|$$

and the force acts in the direction of an arrow pointing from $z(t)$ to $z(t) + z''(t)$.

Example In our previous example, position was given by

$$z(t) = 2e^{i\frac{t}{2}}$$

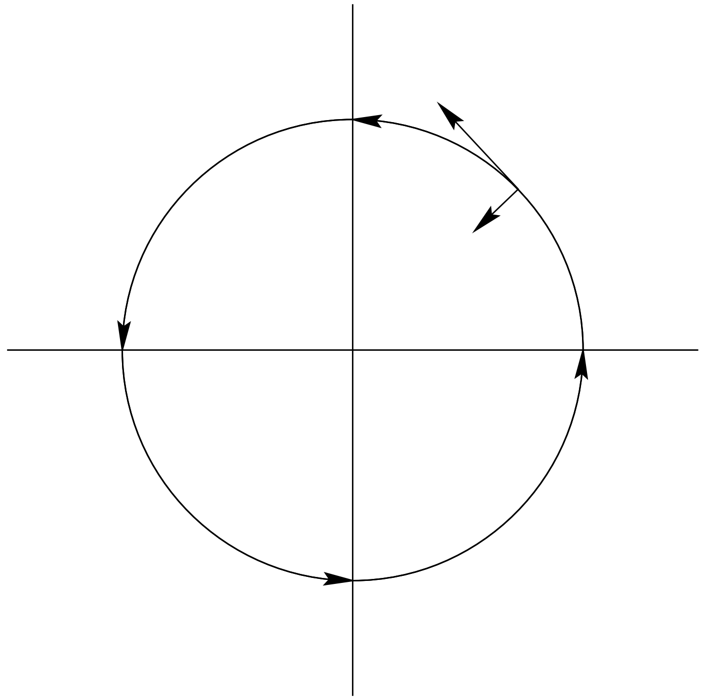


Figure 7.3.4 Arrows indicating velocity and acceleration at time $t = \frac{\pi}{2}$

and velocity by

$$z'(t) = ie^{i\frac{t}{2}}.$$

Thus the acceleration of the particle is

$$z''(t) = -\frac{1}{2}e^{i\frac{t}{2}}.$$

Note that

$$z''(t) = -\frac{1}{4}z(t),$$

showing that the force acting on the particle is directed toward the center of the circle C. See Figure 7.3.4.

Since we may compute the derivative of a complex-valued function by differentiating its complex and real parts separately, it is reasonable to define the definite integral of such a function in terms of the integrals of its real and complex parts.

Definition Suppose $f : \mathbb{R} \to \mathbb{C}$ with $f(t) = x(t) + iy(t)$. If x and y are both integrable on the interval $[a, b]$, then we define the *definite integral* of f over the interval $[a, b]$ by

$$\int_a^b f(t)dt = \int_a^b x(t)dt + i\int_a^b y(t)dt. \tag{7.3.6}$$

Example If

$$f(t) = \sin(t) + i\cos\left(\frac{t}{2}\right),$$

then

$$\begin{aligned}\int_0^\pi f(t)dt &= \int_0^\pi \sin(t)dt + i\int_0^\pi \cos\left(\frac{t}{2}\right)dt \\ &= -\cos(t)\Big|_0^\pi + 2i\sin\left(\frac{t}{2}\right)\Big|_0^\pi \\ &= (1+1) + i(2-0) \\ &= 2+2i.\end{aligned}$$

If $F:\mathbb{R}\to\mathbb{C}$ and $f:\mathbb{R}\to C$ are continuous on $[a,b]$, with $F(t) = X(t) + iY(t)$, $f(t) = x(t) + iy(t)$, and $f(t) = F'(t)$ for all t in (a,b), then

$$\begin{aligned}\int_a^b f(t)dt &= \int_a^b x(t)dt + i\int_a^b y(t)dt \\ &= X(t)\Big|_a^b + iY(t)\Big|_a^b \\ &= (X(b) - X(a)) + i(Y(a) - Y(b)) \\ &= (X(b) + iY(b)) - (X(a) + iY(a)) \\ &= F(b) - F(a).\end{aligned}$$

Hence we have a version of the Fundamental Theorem of Integral Calculus which may be applied directly to complex-valued functions of a real variable.

Proposition If $F:\mathbb{R}\to\mathbb{C}$ and $f:\mathbb{R}\to C$ are continuous on $[a,b]$ with $f(t) = F'(t)$ for all t in $[a,b]$, then

$$\int_a^b f(t)dt = F(b) - F(a).$$

Example If $f(t) = e^{3it}$, then

$$\int_0^\pi f(t)dt = \int_0^\pi e^{3it}dt = \frac{1}{3i}e^{3it}\Big|_0^\pi = \frac{1}{3i}(e^{3i\pi} - e^0) = \frac{1}{3i}(-1-1) = -\frac{2}{3i} = \frac{2}{3}i.$$

Example Integration by parts, which we derived in Section 4.5 from the product rule, is applicable in our current situation. For example, to evaluate

$$\int_0^\pi 3te^{it}dt,$$

we let

$$u = 3t \qquad dv = e^{it}dt$$

$$du = 3dt \quad v = \frac{1}{i}e^{it} = -ie^{it}.$$

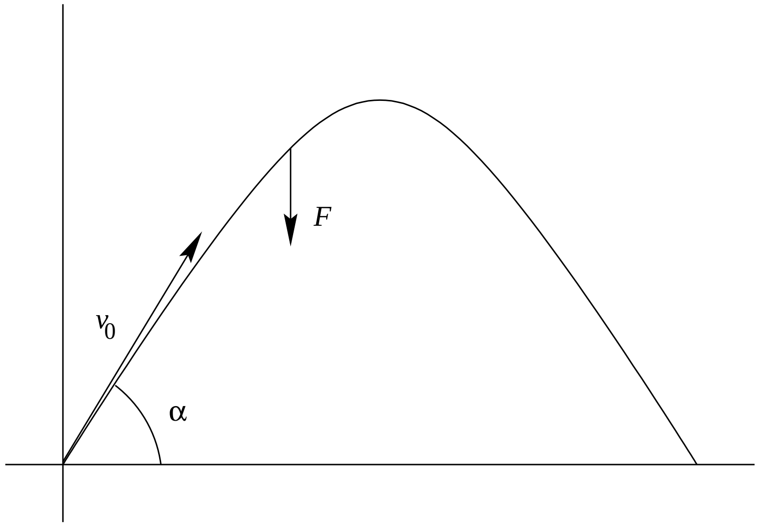


Figure 7.3.5 Motion of a projectile

Then

$$\begin{aligned}\int_0^\pi 3te^{it}dt &= -3tie^{it}\Big|_0^\pi + \int_0^\pi 3ie^{it}dt \\ &= -3\pi ie^{i\pi} + 0 + 3e^{it}\Big|_0^\pi \\ &= -3\pi i(-1) + 3e^{i\pi} - 3e^0 \\ &= 3\pi i - 3 - 3 \\ &= -6 + 3\pi i.\end{aligned}$$

In our final example for this section, we consider the problem of finding the motion of a projectile moving close to the surface of the earth. This problem will not only tie together many of the concepts of this section, but it will also provide a preview of Section 7.4 and our discussion of differential equations in Chapter 8.

Example Suppose a projectile of mass m is fired from the surface of the earth at an angle α, where $0 < \alpha < \frac{\pi}{2}$. We will consider the motion of the projectile as a path in the complex plane with its position at time t given by $z(t)$. Further, we assume that its initial position is $z(0) = 0$ and its initial velocity is $z'(0) = v_0$. Ignoring the effects of air resistance, the only force acting on the projectile during its flight is the force of gravity, acting vertically downward. Hence at any time t the force is given by $F = -mgi$, where g is the acceleration due to gravity (32 feet/second^2 or 9.8 meters/second^2). See Figure 7.3.5. Thus Newton's second law of motion gives us

$$-mgi = mz''(t),$$

that is,

$$-gi = z''(t),$$

at any time t. If we let $v(t)$ be the velocity of the projectile at time t, then

$$v'(t) = z''(t) = -gi.$$

Hence, by the previous proposition,

$$v(t) - v(0) = \int_0^t v'(s)ds = -\int_0^t gids = -gis\Big|_0^t = -gti.$$

Thus

$$v(t) = -gti + v_0.$$

Integrating again, we have, since $z'(t) = v(t)$,

$$z(t) - z(0) = \int_0^t (-gsi + v_0)ds = \left(-\frac{1}{2}gis^2 + v_0 s\right)\Big|_0^t = v_0 t - \frac{1}{2}gt^2 i.$$

Now if $s_0 = |v_0|$, that is, if s_0 is the initial speed of the projectile, then

$$v_0 = s_0 e^{i\alpha} = s_0 \cos(\alpha) + s_0 \sin(\alpha) i.$$

Hence

$$z(t) = (s_0 \cos(\alpha) + s_0 \sin(\alpha) i)t - \frac{1}{2}gt^2 i = s_0 \cos(\alpha)t + \left(s_0 \sin(\alpha)t - \frac{1}{2}gt^2\right) i.$$

Thus

$$\Re(z(t)) = s_0 \cos(\alpha)t$$

and

$$\Im(z(t)) = s_0 \sin(\alpha)t - \frac{1}{2}gt^2.$$

That is, if we write $z(t) = x(t) + iy(t)$, where $x : \mathbb{R} \to \mathbb{R}$ and $y : \mathbb{R} \to \mathbb{R}$, then

$$x(t) = s_0 \cos(\alpha)t \tag{7.3.7}$$

and

$$y(t) = s_0 \sin(\alpha)t - \frac{1}{2}gt^2. \tag{7.3.8}$$

Note that $x(t)$ gives the horizontal distance traveled at time t and $y(t)$ gives the height of the projectile above the ground at time t. For example, if the projectile is fired at an angle of $\alpha = \frac{\pi}{6}$ with an initial speed of $s_0 = 50$ feet per second, then its position at time t is specified by

$$x(t) = 25\sqrt{3}t \text{ feet}$$

and

$$y(t) = 25t - 16t^2 \text{ feet}.$$

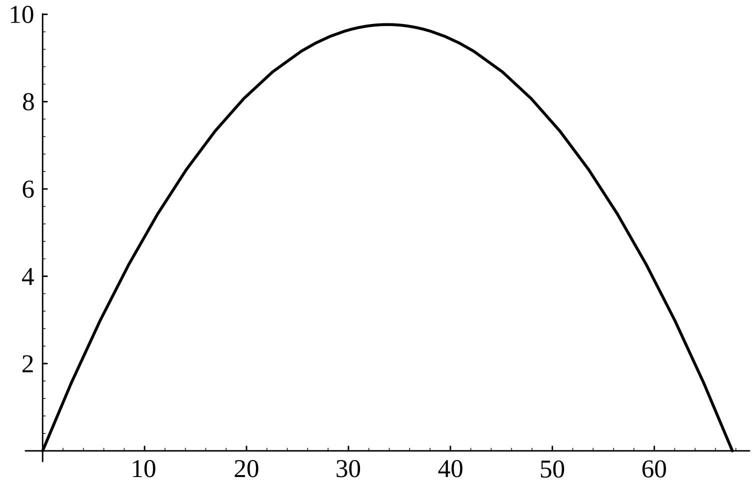


Figure 7.3.6 Motion of a projectile

A plot of this motion is shown in Figure 7.3.6.

In the next section we will consider a more complicated motion problem, namely, the two-body problem, the problem of determining the orbit of a planet about its sun.

Problems

1. For each of the following, suppose the given function specifies the position of a particle moving in the complex plane. Plot the path of the motion over the given time interval, indicating the direction of motion with arrows on the curve.
   (a) $f(t) = \cos(2t) + i\sin(2t)$, $0 \le t \le \pi$
   (b) $z(t) = 4\cos(t) + i\sin(t)$, $0 \le t \le 2\pi$
   (c) $g(t) = \sin\left(\frac{t}{2}\right) + i\cos\left(\frac{t}{2}\right)$, $0 \le t \le 3\pi$
   (d) $z(t) = \operatorname{sech}(2t) + i\tanh(2t)$, $-\infty < t < \infty$
   (e) $f(t) = 2t + it^2$, $-4 \le t \le 4$
   (f) $g(t) = t^2 + it^4$, $-2 \le t \le 2$
   (g) $z(t) = 3e^{it}$, $-\pi \le t \le \pi$
   (h) $h(t) = 3te^{it}$, $0 \le t \le 6\pi$
   (i) $z(t) = \frac{3}{t}e^{2it}$, $1 \le t \le 20$
2. Differentiate each of the functions in the previous problem.
3. For each of the following, suppose the given function specifies the position of a particle moving in the complex plane. Find the velocity, speed, and the acceleration for each at the specified time.
   (a) $z(t) = \cos(2t) + i\sin(2t)$, $t = \frac{\pi}{6}$

(b) $f(t) = 3\sin(t) + i\cos(2t)$, $t = \pi$

(c) $z(t) = \tanh(t) + i\text{sech}(t)$, $t = 3$

(d) $h(t) = 4t^2 + i(4t - 1)$, $t = 1$

(e) $z(t) = 5e^{it}$, $t = \dfrac{\pi}{2}$

(f) $f(t) = 4t^2 e^{2it}$, $t = \dfrac{5\pi}{3}$

4. Evaluate the following integrals.

(a) $\displaystyle\int_0^4 (2t + it)dt$

(b) $\displaystyle\int_0^\pi (\sin(t) + i\cos(3t))dt$

(c) $\displaystyle\int_0^{\frac{\pi}{2}} (-3\sin(2t) + it^3)dt$

(d) $\displaystyle\int_0^{\frac{\pi}{3}} 5e^{it}dt$

(e) $\displaystyle\int_0^\pi 2te^{3it}dt$

(f) $\displaystyle\int_0^\pi t^2 e^{it}dt$

5. Suppose $z(t)$ specifies the position at time t of a particle moving in the complex plane. If we know $z(0) = 1 + i$ and $z'(t) = \cos(t) + i\sin(t)$, find $z(t)$ and plot the path of the object for $0 \le t \le 2\pi$.

6. In the last example of this section we saw that if a projectile is fired from the surface of the earth at an angle α, $0 < \alpha < \frac{\pi}{2}$, with an initial speed of s_0 feet per second, then the x and y coordinates of its position after t seconds are given by

$$x = s_0\cos(\alpha)t$$

and

$$y = s_0\sin(\alpha)t - 16t^2.$$

(a) Find the time t at which the projectile strikes the ground.

(b) The *range* R of the projectile is the value of x when the projectile strikes the ground. Use your result from (a) to find R.

(c) Show that R is maximized when $\alpha = \frac{\pi}{4}$.

(d) Solve the first equation for t in terms of x and substitute this result into the second equation to show that path of the projectile is a parabola.

7. A projectile is fired from the surface of the earth at an angle α, $0 < \alpha < \frac{\pi}{2}$, with an initial speed of 150 feet per second.

(a) Using the results of the previous problem, find the maximum range for the projectile.

(b) What is the range of the projectile if $\alpha = \frac{\pi}{6}$? If $\alpha = \frac{\pi}{3}$? In each case, when does the projectile strike the ground?

(c) Plot the path of motion for $\alpha = \frac{\pi}{6}$, $\alpha = \frac{\pi}{4}$, and $\alpha = \frac{\pi}{3}$.

8. Suppose a particle moves in the complex plane so that its position at time t is given by $z(t) = x(t) + iy(t)$, where

$$x(t) = \int_0^t \cos\left(\frac{\pi s^2}{2}\right) ds$$

and

$$y(t) = \int_0^t \sin\left(\frac{\pi s^2}{2}\right) ds.$$

   (a) Plot the path of motion for $-5 \leq t \leq 5$, indicating the direction of motion with arrows on the curve.

   (b) Find the velocity and acceleration of the particle.

9. If $f : \mathbb{R} \to \mathbb{R}$ is such that

$$\int_{-\infty}^{\infty} |f(t)| dt < \infty,$$

then the function

$$\varphi(\lambda) = \int_{-\infty}^{\infty} f(t) e^{i\lambda t} dt$$

is called the *Fourier transform* of f.

   (a) Show that

$$\varphi(\lambda) = \sum_{n=0}^{\infty} \frac{(i\lambda)^n}{n!} \int_{-\infty}^{\infty} t^n f(t) dt.$$

   (b) Show that

$$\varphi'(\lambda) = \sum_{n=0}^{\infty} \frac{i^{n+1}\lambda^n}{n!} \int_{-\infty}^{\infty} t^{n+1} f(t) dt.$$

   (c) Show that

$$\frac{\varphi'(0)}{i} = \int_{-\infty}^{\infty} t f(t) dt.$$

   (d) Show that

$$\frac{\varphi^{(n)}(0)}{i^n} = \int_{-\infty}^{\infty} t^n f(t) dt$$

   for $n = 0, 1, 2, \ldots$.

10. Let

$$f(t) = \begin{cases} e^{-t}, & \text{for } t \geq 0, \\ 0, & \text{for } t < 0. \end{cases}$$

   (a) With reference to the previous problem, show that the Fourier transform of f is

$$\varphi(\lambda) = \frac{1}{1 - i\lambda}.$$

(b) Use the results from (a) and Problem 9 to evaluate

$$\int_0^\infty t^n e^{-t} dt$$

for $n = 0, 1, 2, 3, 4$.

(c) For $s > 0$, the function

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$$

is called the *gamma function*. Show that

$$\Gamma(n+1) = n!$$

for $n = 0, 1, 2, \ldots$.

11. With reference to Problem 9, find the Fourier transform of

$$f(t) = e^{-\frac{t^2}{2}}$$

and use it to evaluate

$$\int_{-\infty}^\infty t^n e^{-\frac{t^2}{2}} dt$$

for $n = 0, 1, 2, 3, 4$.

Difference Equations to Differential Equations

Section 7.4

The Two-Body Problem

In 1609 Johann Kepler (1571-1630) published the first two of his three laws of planetary motion. The first of these states that the orbit of a planet about the sun is an ellipse with the sun at one focus. He had reached this conclusion after painstaking analysis of the data Tycho Brahe (1546-1601) had collected from observing the motion of Mars over a period of more than 20 years. His work was a scientific triumph because it created a model for the solar system that was not only more accurate than the models of Copernicus and Ptolemy, but simpler as well. Yet, however brilliant, Kepler's result amounted to fitting a curve to a set of data without discovering any fundamental principles underlying the motion of planets that would cause their orbits to be as we observe them. In 1687 Newton provided the missing principles. In his great work, *Philosophiae naturalis principia mathematica*, Newton demonstrated that the elliptical orbit of a planet is a consequence of his three laws of motion and the inverse square law of gravitation. Hence the behavior of the planets could be explained by the same laws which govern the path of an apple as it falls from a tree to the ground; for the first time it became clear that the so-called heavenly bodies behaved no differently than the seemingly more substantial bodies of our everyday experience.

In this section we will see how the motion of the planets may be explained using only Newton's laws and tools from our study of calculus. The solution of this problem is one of the greatest triumphs of the human intellect in general and of calculus in particular.

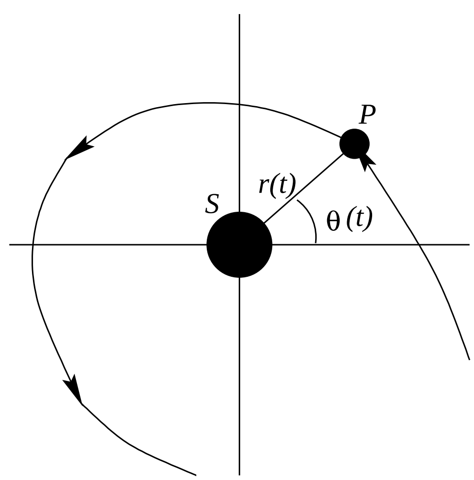


Figure 7.4.1 Possible orbit of a body P about a body S

To begin, suppose we have two bodies, one of mass m, which we denote by P, and the other of mass M, which we denote by S. We may think of S as representing the sun and

P as representing a planet. It is possible to show that Newton's laws of motion hold in a coordinate system with the origin located at the center of mass of the two bodies; for simplicity, we will assume that M is significantly larger than m (as it is if S is the sun and P is a planet, asteroid, or comet), allowing us to assume that the center of mass is located at S. Thus we choose a coordinate system for the complex plane so that S is at the origin and we let $z(t)$ represent the position of P with respect to S at time t. If we express $z(t)$ in polar coordinates, then

$$z(t) = r(t)e^{i\theta(t)}, \tag{7.4.1}$$

where r and θ are real-valued functions, as shown in Figure 7.4.1. For simplicity of notation, we will usually drop the explicit reference to t and simply write

$$z = re^{i\theta}. \tag{7.4.2}$$

By Newton's law of gravitation the magnitude of the gravitational force of attraction between the two bodies is

$$|F| = \frac{GMm}{r^2}, \tag{7.4.3}$$

where G is a constant, approximately

$$6.67 \times 10^{-11} \frac{\text{nt m}^2}{\text{kg}^2}$$

if we measure force in Newtons, distance in meters, and mass in kilograms. Since gravity is an attractive force and we are assuming S to be at rest at the origin, F is directed from P toward the origin. Hence we have

$$F = -\frac{GMm}{r^2}e^{i\theta}. \tag{7.4.4}$$

Moreover, we assume that this is the only force acting on the two bodies. Now if $v(t)$ and $a(t)$ represent the velocity and acceleration, respectively, of P at time t, then, by Newton's second law of motion, $F = ma$, we must have

$$ma = -\frac{GMm}{r^2}e^{i\theta}. \tag{7.4.5}$$

Letting $k = GM$, this simplifies to

$$a = -\frac{k}{r^2}e^{i\theta}. \tag{7.4.6}$$

From our work in Section 7.3 we know that

$$v = \frac{dz}{dt} = \frac{d}{dt}re^{i\theta} = ire^{i\theta}\frac{d\theta}{dt} + e^{i\theta}\frac{dr}{dt} \tag{7.4.7}$$

and

$$\begin{aligned}
a &= \frac{dv}{dt} \\
&= \frac{d}{dt}\left(ire^{i\theta}\frac{d\theta}{dt} + e^{i\theta}\frac{dr}{dt}\right) \\
&= ire^{i\theta}\frac{d}{dt}\left(\frac{d\theta}{dt}\right) + \frac{d\theta}{dt}\frac{d}{dt}(ire^{i\theta}) + e^{i\theta}\frac{d}{dt}\left(\frac{dr}{dt}\right) + \frac{dr}{dt}\frac{d}{dt}(e^{i\theta}) \\
&= ire^{i\theta}\frac{d^2\theta}{dt^2} + \frac{d\theta}{dt}\left(i^2re^{i\theta}\frac{d\theta}{dt} + ie^{i\theta}\frac{dr}{dt}\right) + e^{i\theta}\frac{d^2r}{dt^2} + \frac{dr}{dt}ie^{i\theta}\frac{d\theta}{dt} \\
&= ire^{i\theta}\frac{d^2\theta}{dt^2} - re^{i\theta}\left(\frac{d\theta}{dt}\right)^2 + ie^{i\theta}\frac{d\theta}{dt}\frac{dr}{dt} + e^{i\theta}\frac{d^2r}{dt^2} + ie^{i\theta}\frac{d\theta}{dt}\frac{dr}{dt} \\
&= -re^{i\theta}\left(\frac{d\theta}{dt}\right)^2 + e^{i\theta}\frac{d^2r}{dt^2} + i\left(re^{i\theta}\frac{d^2\theta}{dt^2} + 2e^{i\theta}\frac{d\theta}{dt}\frac{dr}{dt}\right). \qquad (7.4.8)
\end{aligned}$$

Putting (7.4.6) and (7.4.8) together gives us

$$-\frac{k}{r^2}e^{i\theta} = -re^{i\theta}\left(\frac{d\theta}{dt}\right)^2 + e^{i\theta}\frac{d^2r}{dt^2} + i\left(re^{i\theta}\frac{d^2\theta}{dt^2} + 2e^{i\theta}\frac{d\theta}{dt}\frac{dr}{dt}\right).$$

After dividing through by $e^{i\theta}$ we have

$$-\frac{k}{r^2} = -r\left(\frac{d\theta}{dt}\right)^2 + \frac{d^2r}{dt^2} + i\left(r\frac{d^2\theta}{dt^2} + 2\frac{d\theta}{dt}\frac{dr}{dt}\right). \qquad (7.4.9)$$

The equality in (7.4.9) implies that the the real part of the left-hand side of the equation is equal to the real part of the right-hand side of the equation and the imaginary part of the left-hand side of the equation is equal to the imaginary part of the right-hand side of the equation. That is,

$$-\frac{k}{r^2} = -r\left(\frac{d\theta}{dt}\right)^2 + \frac{d^2r}{dt^2} \qquad (7.4.10)$$

and

$$0 = r\frac{d^2\theta}{dt^2} + 2\frac{d\theta}{dt}\frac{dr}{dt}. \qquad (7.4.11)$$

Multiplying both sides of (7.4.11) by r gives us

$$0 = r^2\frac{d^2\theta}{dt^2} + 2r\frac{d\theta}{dt}\frac{dr}{dt}. \qquad (7.4.12)$$

However,

$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = r^2\frac{d^2\theta}{dt^2} + 2r\frac{d\theta}{dt}\frac{dr}{dt},$$

so (7.4.12) implies that

$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0. \tag{7.4.13}$$

Since a function with 0 for its derivative must be a constant function, it follows that

$$r^2\frac{d\theta}{dt} = c \tag{7.4.14}$$

for some constant c. In any interval of time of interest, we will have $r > 0$, that is, S and P are not a the same point in space, and so $r^2 > 0$. It follows that if $c = 0$, then $\frac{d\theta}{dt} = 0$ for all t, corresponding to the relatively uninteresting case when θ is a constant and P moves along a straight line passing through S. The more interesting cases are when $c < 0$ or $c > 0$. Since the former case implies that $\frac{d\theta}{dt} < 0$ for all t and the latter implies $\frac{d\theta}{dt} > 0$ for all t, the choice of sign for c ultimately depends on our choice of orientation in our coordinate system, that is, the direction in which we measure positive angles. Hence, without loss of generality, we may assume $c > 0$ or, equivalently, $\frac{d\theta}{dt} > 0$.

We will now use the substitution $s = \frac{1}{r}$ to put (7.4.10) into a simpler form. With this substitution, $r = \frac{1}{s}$, so

$$\frac{dr}{dt} = \frac{d}{dt}\left(\frac{1}{s}\right) = -\frac{1}{s^2}\frac{ds}{dt} = -\frac{1}{s^2}\frac{ds}{d\theta}\frac{d\theta}{dt}. \tag{7.4.15}$$

Since, from (7.4.14),

$$\frac{d\theta}{dt} = \frac{c}{r^2} = cs^2, \tag{7.4.16}$$

we have

$$\frac{dr}{dt} = -c\frac{ds}{d\theta}. \tag{7.4.17}$$

Differentiating again,

$$\frac{d^2r}{dt^2} = \frac{d}{dt}\left(-c\frac{ds}{d\theta}\right) = -c\frac{d}{dt}\left(\frac{ds}{d\theta}\right) = -c\frac{d}{d\theta}\left(\frac{ds}{d\theta}\right)\frac{d\theta}{dt} = -c\frac{d\theta}{dt}\frac{d^2s}{d\theta^2}. \tag{7.4.18}$$

Hence, using (7.4.16),

$$\frac{d^2r}{dt^2} = -c^2s^2\frac{d^2s}{d\theta^2}. \tag{7.4.19}$$

Finally, substituting (7.4.16), (7.4.19), and $s = \frac{1}{r}$ into (7.4.10) gives us

$$-ks^2 = -\frac{1}{s}(cs^2)^2 - c^2s^2\frac{d^2s}{d\theta^2} = -c^2s^3 - c^2s^2\frac{d^2s}{d\theta^2}. \tag{7.4.20}$$

Dividing both sides of this equation by $-c^2s^2$, we have

$$\frac{d^2s}{d\theta^2} + s = \frac{k}{c^2}. \tag{7.4.21}$$

This is the differential equation to which all our work has been leading. The solution of this equation will be an expression for s as a function of θ; since r is in turn a function of s, namely, $r = \frac{1}{s}$, this will give us r as a function of θ and allow us to determine the path of motion of P. Note, however, that we will not have found r as a function of t. In other words, we will be able to determine the path of motion of P, but we will not be able to determine where along that path P is at any specific time t.

To solve (7.4.21), we first note that if $y(\theta)$ is a solution of the equation

$$\frac{d^2y}{d\theta^2} + y = 0,$$

then the function

$$x(\theta) = y(\theta) + \frac{k}{c^2}$$

satisfies the equation

$$\frac{d^2x}{d\theta^2} + x = \frac{k}{c^2}$$

since

$$\frac{d^2}{d\theta^2}\left(y + \frac{k}{c^2}\right) + \left(y + \frac{k}{c^2}\right) = \frac{d^2y}{d\theta^2} + y + \frac{k}{c^2} = 0 + \frac{k}{c^2} = \frac{k}{c^2}.$$

Hence to solve (7.4.21), we need only solve the equation

$$\frac{d^2s}{d\theta^2} + s = 0. \tag{7.4.22}$$

That is, we need only find a function s of θ such that

$$\frac{d^2s}{d\theta^2} = -s. \tag{7.4.23}$$

Now (7.4.22) simply says that s is a function with the property that its second derivative is the negative of itself. But we already know two such functions, namely, $\sin(\theta)$ and $\cos(\theta)$; moreover, for any constants A and B, the function $A\sin(\theta) + B\cos(\theta)$ also has this property. Although the justification is beyond our resources at this point, it is in fact true that any solution of (7.4.23) must be of the form

$$A\sin(\theta) + B\cos(\theta) \tag{7.4.24}$$

for some constants A and B. From this it now follows that our sought after solution to (7.4.21) must have the form

$$s = A\sin(\theta) + B\cos(\theta) + \frac{k}{c^2} \tag{7.4.25}$$

for some constants A and B.

We will now find values for the constants A and B so that

$$\left.\frac{ds}{d\theta}\right|_{\theta=0} = 0 \tag{7.4.26}$$

and

$$\left.\frac{d^2s}{d\theta^2}\right|_{\theta=0} \leq 0. \tag{7.4.27}$$

Intuitively, this means we are looking for values which satisfy conditions for s to have a local maximum at $\theta = 0$. Equivalently, these conditions will hold if r has a local minimum at $\theta = 0$. We make think of this as choosing the constants A and B in such a way that P is closest to S when the path of P crosses the positive real axis. Now

$$\frac{ds}{d\theta} = A\cos(\theta) - B\sin(\theta),$$

so

$$\left.\frac{ds}{d\theta}\right|_{\theta=0} = A, \tag{7.4.28}$$

and

$$\frac{d^2s}{d\theta^2} = -A\sin(\theta) - B\cos(\theta),$$

so

$$\left.\frac{d^2s}{d\theta^2}\right|_{\theta=0} = -B. \qquad (7.4.29).$$

Hence the conditions (7.4.26) and (7.4.27) are satisfied if we set $A = 0$ and we require $B \geq 0$. In other words, the conditions (7.4.26) and (7.4.27) are satisfied by

$$s = B\cos(\theta) + \frac{k}{c^2}, \tag{7.4.30}$$

where $B \geq 0$.

In terms of r, (7.4.30) gives us

$$\frac{1}{r} = B\cos(\theta) + \frac{k}{c^2} = \frac{c^2B\cos(\theta) + k}{c^2},$$

so

$$r = \frac{c^2}{c^2B\cos(\theta) + k} = \frac{\dfrac{c^2}{k}}{1 + \dfrac{c^2B}{k}\cos(\theta)}. \tag{7.4.31}$$

If we let $\alpha = \frac{c^2}{k}$ and $\epsilon = \alpha$B, then our expression for r as a function of θ reduces to

$$r = \frac{\alpha}{1 + \epsilon\cos(\theta)}, \tag{7.4.32}$$

where $\epsilon \geq 0$ and $\alpha > 0$ are constants.

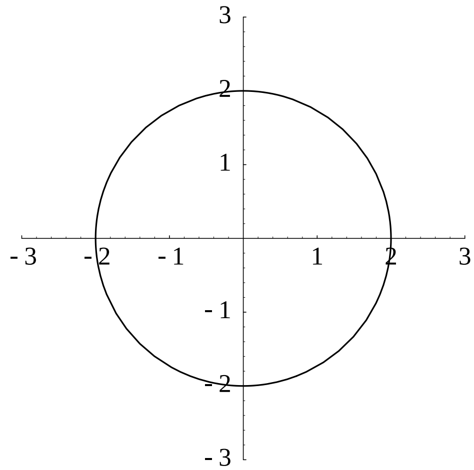


Figure 7.4.2 Circular orbit for P when $\epsilon = 0$ and $\alpha = 2$

Note that, as indicated above, our solution does not give us values of r and θ for specified values of t, but rather (7.4.32) gives us a value of r for any specified value of θ. In other words, our solution does not give us the position of P for a given time t, but it does tell us the location of P as a function of θ. Indeed, if we plot the points $z = re^{i\theta}$ for all values of θ in the interval $[-\pi, \pi]$, with r given by (7.4.32), then the resulting curve will be the path of the orbit of P about S. For example, if $\epsilon = 0$, then $r = \alpha$ for all t and the orbit of P is a circle of radius α with center at S, as shown in Figure 7.4.2 for $\alpha = 2$. Note that because of our assumption that $\frac{d\theta}{dt} > 0$, the motion along this curve, and all subsequent curves, will be in the counter-clockwise direction.

If $0 < \epsilon < 1$, then $\epsilon\cos(\theta)$ has a maximum value of ϵ when $\theta = 0$ and a minimum value of $-\epsilon$ when $\theta = -\pi$ or $\theta = \pi$. Thus the minimum value of r is

$$r(0) = \frac{\alpha}{1+\epsilon}$$

and the maximum value of r is

$$r(-\pi) = r(\pi) = \frac{\alpha}{1-\epsilon}.$$

Hence the orbit of P about S is a closed curve with

$$\frac{\alpha}{1+\epsilon} \le r \le \frac{\alpha}{1-\epsilon}$$

for all θ. An example for $\alpha = 2$ and $\epsilon = 0.5$, in which case $\frac{4}{3} \le r \le 4$ for all θ, is shown in Figure 7.4.3.

Note that

$$\lim_{\epsilon \to 1^-} r(0) = \lim_{\epsilon \to 1^-} \frac{\alpha}{1+\epsilon} = \frac{\alpha}{2},$$

whereas

$$\lim_{\epsilon \to 1^-} r(\pi) = \lim_{\epsilon \to 1^-} \frac{\alpha}{1-\epsilon} = \infty.$$

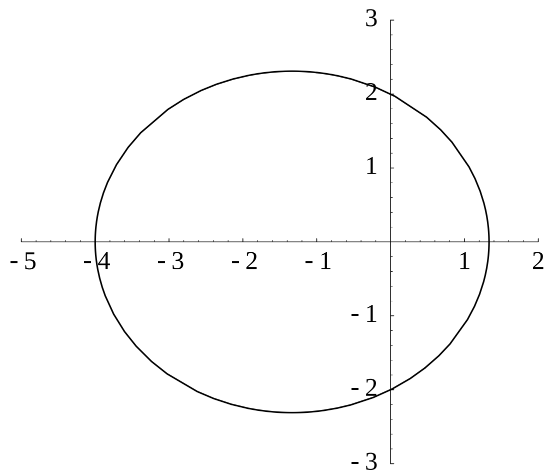


Figure 7.4.3 Orbit of P for $\epsilon = 0.5$ and $\alpha = 2$

Hence as ϵ approaches 1 from the left, the point of closest approach of P to S shrinks toward $\frac{\alpha}{2}$, but the point at which P is farthest from S increases without bound. Thus, as ϵ varies from 0 to 1, the orbit of P flattens out, changing from a circle to a long oblong shape. Figure 7.4.4 shows the orbit of P for α =2 and $\epsilon = 0.95$, in which case $1.026 \leq r \leq 40$ for all θ . Because of this behavior, ϵ is called the *eccentricity* of the orbit of P.

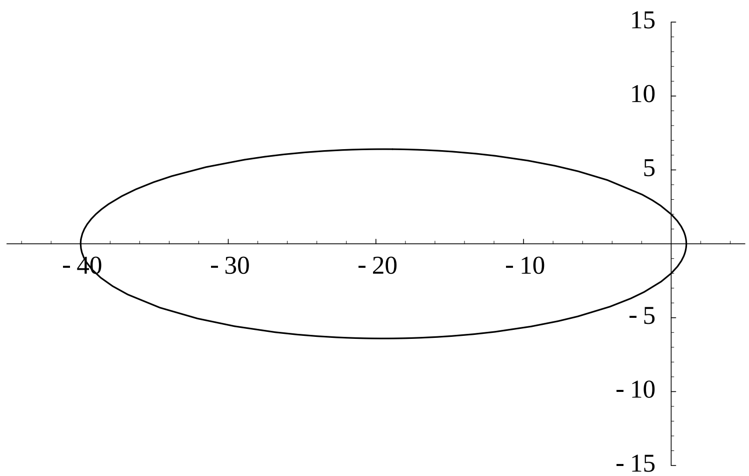


Figure 7.4.4 Orbit of P for $\epsilon = 0.95$ and $\alpha = 2$

When $\epsilon = 1$, r is not defined for $\theta = -\pi$ and $\theta = \pi$. In fact, in this case

$$\lim_{\theta \to \pi^-} r(\theta) = \lim_{\theta \to \pi^-} \frac{\alpha}{1 + \cos(\theta)} = \infty$$

and

$$\lim_{\theta \to -\pi^+} r(\theta) = \lim_{\theta \to -\pi^+} \frac{\alpha}{1 + \cos(\theta)} = \infty.$$

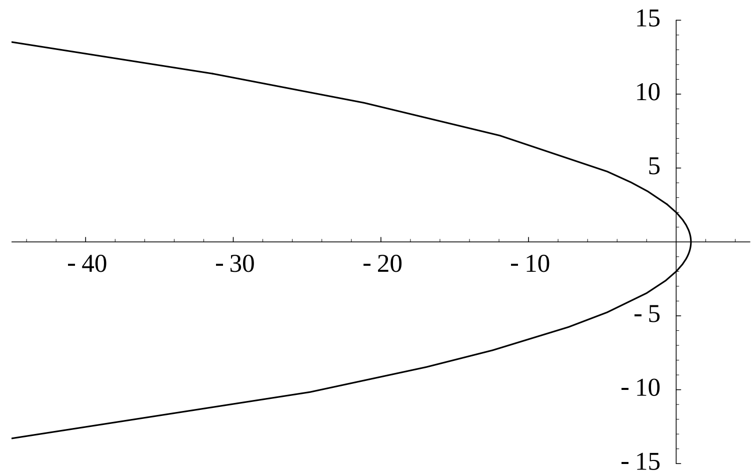


Figure 7.4.5 Orbit of P for $\epsilon = 1$ and $\alpha = 2$

Hence the orbit of P is not closed; P makes its closest approach to S when $\theta = 0$, at which point the distance from P to S is $\frac{\alpha}{2}$, and then follows a path which takes it ever farther away from S. The situation for $\alpha = 2$ and $\epsilon = 1$ is shown in Figure 7.4.5.

For $\epsilon > 1$, there are angles θ_1 and θ_2, with

$$-\pi < \theta_1 < -\frac{\pi}{2}$$

and

$$\frac{\pi}{2} < \theta_2 < \pi,$$

such that

$$\cos(\theta_1) = \cos(\theta_2) = -\frac{1}{\epsilon}. \tag{7.4.33}$$

Whenever $-\pi \leq \theta \leq \theta_1$ or $\theta_2 \leq \theta \leq \pi$ we have $1 + \epsilon\cos(\theta) \leq 0$. Since $\alpha > 0$ and $r \geq 0$ for all θ, the orbit of P in this case is defined by (7.4.32) only when $\theta_1 < \theta < \theta_2$. Moreover,

$$\lim_{\theta \to \theta_1^+} r(\theta) = \lim_{\theta \to \theta_1^+} \frac{\alpha}{1 + \epsilon\cos(\theta)} = \infty$$

and

$$\lim_{\theta \to \theta_2^-} r(\theta) = \lim_{\theta \to \theta_2^-} \frac{\alpha}{1 + \epsilon\cos(\theta)} = \infty$$

Thus again the orbit of P is not closed; P approaches S to within a distance of $\frac{\alpha}{1+\epsilon}$ at $\theta = 0$ and then follows a path away from S. See Figure 7.4.6 for the case $\alpha = 2$ and $\epsilon = 2$.

The curves in Figures 7.4.3 through 7.4.6 should look familiar. Indeed, the curves in Figures 7.4.3 and 7.4.4 are both ellipses, the curve in Figure 7.4.5 is a parabola, and the curve in Figure 7.4.6 is a hyperbola. This is not hard to see if we rewrite the equation

$$r = \frac{\alpha}{1 + \epsilon\cos(\theta)} \tag{7.4.34}$$

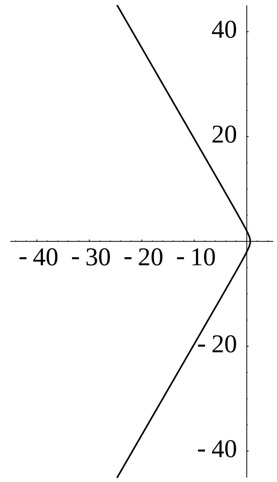


Figure 7.4.6 Orbit of P for $\epsilon = 2$ and $\alpha = 2$

in rectangular coordinates. Recall that if x and y are, respectively, the real and imaginary parts of $z = re^{i\theta}$, then

$$r = \sqrt{x^2 + y^2}$$

and

$$\cos(\theta) = \frac{x}{\sqrt{x^2 + y^2}}.$$

Hence if z is a point on the curve with equation (7.4.34), we have

$$\sqrt{x^2 + y^2} = \frac{\alpha}{1 + \dfrac{\epsilon x}{\sqrt{x^2 + y^2}}} = \frac{\alpha\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2} + \epsilon x}.$$

Dividing both sides by $\sqrt{x^2 + y^2}$ gives us

$$1 = \frac{\alpha}{\sqrt{x^2 + y^2} + \epsilon x},$$

and so

$$\sqrt{x^2 + y^2} = \alpha - \epsilon x.$$

Squaring, we have

$$x^2 + y^2 = \alpha^2 - 2\alpha\epsilon x + \epsilon^2 x^2,$$

from which we obtain

$$(1 - \epsilon^2)x^2 + y^2 + 2\alpha\epsilon x - \alpha^2 = 0. \tag{7.4.35}$$

Thus if the polar coordinates of z satisfy (7.4.34), then the rectangular coordinates of z must satisfy (7.4.35). Moreover, we know from analytic geometry that a curve in the plane with equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0, \tag{7.4.36}$$

where a, b, c, d, e, and f are all constants, is an ellipse if $b^2 - 4ac < 0$, a parabola if $b^2 - 4ac = 0$, and a hyperbola if $b^2 - 4ac > 0$. Because of this result, we call the number

$$D = b^2 - 4ac \tag{7.4.37}$$

the *discriminant* of (7.4.36). In the case of (7.4.35), we have

$$D = 0 - 4(1 - \epsilon^2) = -4(1 - \epsilon^2). \tag{7.4.38}$$

Thus $D < 0$ when $0 \le \epsilon < 1$, $D = 0$ when $\epsilon = 1$, and $D > 0$ when $\epsilon > 1$. Since we have already seen that the orbit of P is a circle when $\epsilon = 0$ (a circle being a particular case of an ellipse), we now have the following classification of the orbit of P about S in terms of the eccentricity ϵ:

Eccentricity	**Orbit of P**
$\epsilon = 0$	Circle
$0 < \epsilon < 1$	Ellipse
$\epsilon = 1$	Parabola
$\epsilon > 1$	Hyperbola

Recall that, collectively, these curves are known as the conic sections.

We have seen that starting with the assumptions of Newton's law of gravitation and his second law of motion, we may conclude that the orbit of a body P about another body S must be a conic section. As great as Newton's accomplishment was, scientifically, mathematically, and philosophically, it is not the end of the story. The work we have done only accounts for the interaction of two bodies, isolated without any forces acting on them other than their mutual gravitational attraction. In reality, to model our entire solar system we would have to consider, at the minimum, the effects of the gravitational fields of the sun plus at least nine planets, as well as numerous moons, asteroids, and comets. Because of these other considerations, the orbits of the planets are not true ellipses, although, since by far the most dominant force acting on any one planet is the gravitational attraction between it and the sun, the deviation from elliptical paths is small. The problem of the motion of three or more bodies interacting under the influence of their mutual gravitational attraction has challenged mathematicians since the time of Newton. However, we now know that this problem, known as the *n-body problem*, cannot, in general, be solved exactly. Since the work of Henri Poincaré (1854-1912), advances on this problem have been directed toward qualitative and numerical descriptions of the orbits, not toward exact analytic solutions. In fact it was Poincaré who first showed that even in the case of only three bodies, the orbits can be highly complex, revealing a sensitivity to initial conditions that would make predictions about the future path of a given body effectively impossible. The work on this problem continues to the present.

Problems

1. The *perihelion* of the orbit of a planet is the point of the orbit which is closest to the sun. The following table gives the eccentricity and the distance from the sun at perihelion for each of the known planets in our solar system. Note the distances are given in astronomical units, where one astronomical unit is approximately 92.9 million miles, the mean distance from the earth to the sun.

Planet	**Eccentricity**	**Distance at Perihelion**
Mercury	0.21	0.31
Venus	0.01	0.72
Earth	0.02	0.98
Mars	0.09	1.38
Jupiter	0.05	4.95
Saturn	0.06	9.02
Uranus	0.05	18.3
Neptune	0.01	29.8
Pluto	0.25	29.8

(a) Plot the orbits of each of the planets.

(b) The aphelion of the orbit of a planet is the point of the orbit which is farthest from the sun. Find the distance of each planet from the sun at aphelion.

(c) Which orbits are closest to being circular? Which ones deviate the most from being circular?

(d) Plot the orbits of Neptune and Pluto together. How do they differ?

2. The orbit of the Comet Kohoutek has an eccentricity of 0.9999 and its distance from the sun at perihelion is 0.14 astronomical units. Plot the orbit of Comet Kohoutek and compare it with the orbit of Pluto from Problem 1. How far away from the sun is Comet Kohoutek at aphelion?

3. The orbit of Halley's comet has an eccentricity of 0.967 and its distance from the sun at perihelion is 0.59 astronomical units. Plot the orbit of Halley's comet and compare it with the orbits of Pluto and Comet Kohoutek as found in Problems 1 and 2. How far away from the sun is Halley's comet at aphelion?

4. The orbit of Encke's comet has an eccentricity of 0.847 and its distance from the sun at perihelion is 0.34 astronomical units. Plot the orbit of Encke's comet and compare it with the orbits of Pluto, Comet Kohoutek, and Halley's comet as found in Problems 1, 2, and 3. How far away from the sun is Encke's comet at aphelion?

5. (a) Use the information in Problem 1 to find the equation for the orbit of the earth in rectangular coordinates (that is, an equation of the form (7.4.35)).

   (b) Use your result from (a) and the techniques of Section 4.8 to find the length of the earth's orbit. Convert your answer into miles.

   (c) What is the average speed of the earth in miles per hour?

6. (a) Use the information in Problem 1 to find the equation for the orbit of Pluto in rectangular coordinates (that is, an equation of the form (7.4.35)).

   (b) Use your result from (a) and the techniques of Section 4.8 to find the length of Pluto's orbit. Convert your answer into miles.

   (c) What is the average speed of Pluto in miles per hour? You will need to know that it takes Pluto 248 years to complete one orbit about the sun.

7. To solve the two-body problem we had to solve a differential equation of the form

$$\frac{d^2y}{dt^2} = -y.$$

In this problem we consider the equation

$$\frac{d^2y}{dt^2} = y. \tag{7.4.39}$$

(a) Find two functions, $y_1(t)$ and $y_2(t)$, which satisfy (7.4.39) and are such that $y_2(t)$ is not a constant multiple of $y_1(t)$.

(b) Show that

$$y(t) = Ay_1(t) + By_2(t)$$

satisfies (7.4.39) for any constants A and B.

(c) Find a solution $y(t)$ of (7.4.39) such that $y(0) = 2$ and

$$\left.\frac{dy}{dt}\right|_{t=0} = 4.$$

D*ifference Equations*
to
D*ifferential Equations*

Section 8.1

Numerical Solutions of Differential Equations

If x is a function of a real variable t and f is a function of both x and t, then the equation

$$\dot{x}(t) = f(x(t), t) \tag{8.1.1}$$

is called a *first order differential equation*. Solving such an equation involves more than algebraic manipulation; indeed, although the equation itself involves three quantities, x, $\dot{x}$, and t, to find a solution we must identify a function x, defined solely in terms of the independent variable t, which satisfies the relationship of (8.1.1) for all t in some open interval. For many equations, exact solution is not possible and we have to rely on approximations. In this chapter we will discuss techniques for finding both approximate and, where possible, exact solutions to differential equations.

We have already seen many examples of differential equations: in Section 4.8 when we discussed finding the position of an object moving in a straight line given its velocity function and its initial position, in Section 6.3 when we discussed models for growth and decay, in Section 7.3 when we discussed the motion of a projectile, and in Section 7.4 when we considered the two-body problem. Indeed, in many ways the study of differential equations is at the heart of calculus. To study the interaction of physical bodies in the world is to study the ramifications of physical laws such as the law of gravitation and Newton's second law of motion, laws which frequently lead, as we saw in Section 7.4, to questions involving the solution of differential equations. Newton was the first to realize the power of calculus for solving a vast array of physical problems. The mathematicians that followed him enlarged and refined his techniques until they began to believe that the entire future of the universe, as well as its past, could be discerned from a knowledge of the current positions and velocities of all physical bodies and the forces at work between them. In such a world view, nothing is undetermined in itself; what appears to us as undetermined is simply a reflection of our ignorance of the forces involved. As an example of this view, writing in 1795, Pierre Simon Laplace (1749-1827) said:

> Given for one instant an intelligence which could comprehend all the forces by which nature is animated and the respective situation of the beings who compose it — an intelligence sufficiently vast to submit these data to analysis — it would embrace in the same formula the movements of the greater bodies of the universe and those of the lightest atom; for it, nothing would be uncertain and the future, as the past, would be present to its eyes. *

* P. S. Laplace, *Essai philosphique sur les probabilités* (Paris, 1814), translated by F. W. Truscott and F. L. Emory, *A Philosophical Essay on Probabilities* (New York, 1951), page 3.

Today we know more about the limits to our knowledge, but, nevertheless, the study of differential equations remains a key component to our understanding of the universe.

To begin our study, consider the equation

$$\dot{x}(t) = f(x(t), t) \tag{8.1.2}$$

with the initial condition $x(t_0) = x_0$. To simplify notation, we will frequently omit the independent variable when referring to $x(t)$ and write simply

$$\dot{x} = f(x, t). \tag{8.1.3}$$

Now if $f(x, t)$ depends only on the value of t, that is, if $f(x, t) = g(t)$ for all values of x, where g is a function of t alone, then we may solve (8.1.3) by integration. That is, integrating both sides (8.1.3) gives us

$$\int_{t_0}^{t} \dot{x}(s)ds = \int_{t_0}^{t} g(s)ds. \tag{8.1.4}$$

Substituting

$$\int_{t_0}^{t} \dot{x}(s)ds = x(t) - x(t_0),$$

into (8.1.4), we have

$$x(t) = x(t_0) + \int_{t_0}^{t} g(s)ds. \tag{8.1.5}$$

Assuming we can compute the definite integral, which, provided g is continuous, can at least always be done numerically, we have solved the differential equation. This is the type of differential equation we considered in Section 4.8.

Example Consider the equation

$$\dot{x} = 4\sin(3t)$$

with initial condition $x(0) = 5$. Then

$$\begin{aligned}
x(t) &= 5 + \int_0^t 4\sin(3s)ds \\
&= 5 - \frac{4}{3}\cos(3s)\Big|_0^t \\
&= 5 - \frac{4}{3}\cos(3t) + \frac{4}{3} \\
&= \frac{1}{3}(19 - 4\cos(3t)).
\end{aligned}$$

More generally, suppose f in (8.1.3) depends on both x and t. In that case, since the right-hand side of the equation involves the unknown function x, we cannot simply integrate both sides of the equation. We have solved some equations like this in earlier sections, such as the inhibited population growth model

$$\dot{x} = \frac{\alpha}{M}x(M - x)$$

in Section 6.3, and we will discuss general techniques for finding exact solutions for several types of equations in the coming sections. However, in this section we will concentrate on direct numerical techniques for approximating solutions. Indeed, knowing the difficulty of evaluating ordinary definite integrals, it is not hard to believe that many, if not most, differential equations may be solved only through numerical approximation.

Although the function x in equation (8.1.3) is unknown, we do have enough information to find its best affine approximation at t_0. Namely, the best affine approximation to x at t_0 is

$$T(t) = x_0 + \dot{x}(t_0)(t - t_0) = x_0 + f(x_0, t_0)(t - t_0). \tag{8.1.6}$$

Hence, from our work with Taylor's theorem in Section 5.2 and assuming $f(x(t), t)$ is a continuous function, we have

$$x(t) = x_0 + f(x_0, t_0)(t - t_0) + O((t - t_0)^2). \tag{8.1.7}$$

Equivalently,

$$x(t_0 + h) = x_0 + hf(x_0, t_0) + O(h^2). \tag{8.1.8}$$

Thus for a small value of h,

$$x_1 = x_0 + hf(x_0, t_0)$$

will provide a good approximation to $x(t_0 + h)$. However, we want to do more than this; since x is a function, we want to be able to approximate its values over an entire interval, say $[t_0, t_1]$. To do so, we choose a small value for h and iterate the process that gave us x_1. Specifically, we let $s_k = t_0 + kh$, $k = 0, 1, \ldots, n$, where n is chosen large enough that $s_n \geq t_1$, and compute

$$x_{k+1} = x_k + hf(x_k, s_k) \tag{8.1.9}$$

for $k = 0, 1, \ldots, n - 1$. That is, we compute an approximation for $x(s_{k+1})$ by applying the best affine approximation to our approximation for $x(s_k)$, repeating the process enough times until we have approximated x over the entire interval $[t_0, t_1]$. This method of approximation is known as *Euler's method*.

Euler's method To approximate a solution to the equation

$$\dot{x} = f(x, t)$$

with initial condition $x(t_0) = x_0$ on an interval $[t_0, t_1]$, choose a small value for $h > 0$ and an integer n such that $t_0 + nh \geq t_1$. Letting $s_k = t_0 + kh$, $k = 0, 1, \ldots, n$, compute $x_1, x_2, \ldots, x_n$ using the difference equation

$$x_{k+1} = x_k + hf(x_k, s_k). \tag{8.1.10}$$

Then x_k is an approximation for $x(t + kh)$.

Note that (8.1.10) makes use of a difference equation, a discrete time equation that we first met in Section 1.4, in order to approximate the solution of a differential equation.

Example Consider the differential equation $\dot{x} = 0.04x$ with initial condition $x(0) = 50$. From our work in Chapter 6 we know that the solution to this equation is

$$x(t) = 50e^{0.04t}.$$

In particular, $x(50) = 50e^2 = 369.45$, rounding our answer to the second decimal place. To approximate x on the interval $[0, 50]$ using Euler's method, we will first take $h = 1$ and, starting with $x_0 = 50$, compute $x_1, x_2, \ldots, x_{50}$, where in this case x_k will approximate $x(k)$. Using (8.1.10) with $f(x, t) = 0.04x$ and rounding to two decimal places, we have

$$\begin{aligned}
x_1 &= x_0 + h(0.04x_0) = 50 + (1)(0.04)(50) = 50 + 2 + 52.00, \\
x_2 &= x_1 + h(0.04x_1) = 52 + (1)(0.04)(52) = 52 + 2.08 = 54.08, \\
x_3 &= x_2 + h(0.04x_2) = 54.08 + (1)(0.04)(54.08) = 54.08 + 2.16 = 56.24,
\end{aligned}$$

and so on, ending with $x_{50} = 355.33$. The following table gives the values of x_t and $x(t)$ for $t = 0, 5, 10, \ldots, 50$:

t	x_t	$x(t)$
0	50.00	50.00
5	60.83	61.07
10	74.01	74.59
15	90.05	91.11
20	109.56	111.28
25	133.29	135.91
30	162.17	166.01
35	197.30	202.76
40	240.05	247.65
45	292.06	302.48
50	355.33	369.45

Notice that the error in our approximation, that is, the difference between x_k and $x(k)$, increases as k increases. For example,

$$x(5) - x_5 = 0.24,$$

whereas

$$x(50) - x_{50} = 14.12.$$

This should be expected since the error of the approximation on each step is compounded by the errors made in each of the preceding steps. The only way we can control the amount of error in our approximations is to decrease the step size. For example, if we reduce the step size to $h = 0.1$, we obtain

$$\begin{aligned}
x_1 &= x_0 + h(0.04x_0) = 50 + (0.1)(0.04)(50) = 50 + 0.2 = 50.2000, \\
x_2 &= x_1 + h(0.04x_1) = 50.2 + (0.1)(0.04)(50.2) = 50.2 + 0.2008 = 50.4008, \\
x_3 &= x_2 + h(0.04x_2) = 50.4008 + (0.1)(0.04)(50.4008) = 50.4008 + 0.2016 = 50.6024,
\end{aligned}$$

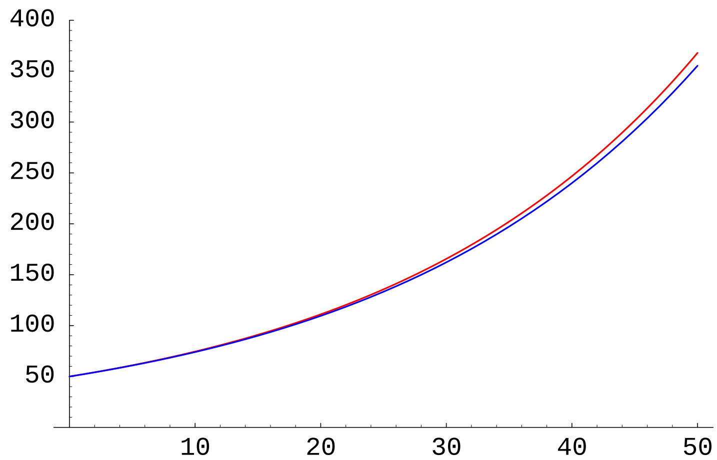


Figure 8.1.1 Approximate solutions of $\dot{x} = 0.04x$

and so on, ending with $x_{500} = 367.98$ as our approximation for $x(50)$. In general, with $h = 0.1$, x_k is an approximation for $x(0+kh) = x(0.1k)$. Equivalently, $x(t)$ is approximated by x_k where $k = 10t$, provided $10t$ is an integer. The following table gives the values of x_{10t} and $x(t)$ for $t = 0, 5, 10, \ldots, 50$:

t	x_{10t}	$x(t)$
0	50.00	50.00
5	61.05	61.07
10	74.53	74.59
15	91.00	91.11
20	111.10	111.28
25	135.64	135.91
30	165.61	166.01
35	202.19	202.76
40	246.86	247.65
45	301.40	302.48
50	367.98	369.45

As expected, reducing the step size from $h = 1$ to $h = 0.1$ greatly reduces the error in the approximations. Figure 8.1.1 shows the graphs of both our approximations to x (the graph of x is not shown since, on this scale, it is essentially the same as our second approximation).

Example The differential equation in the previous example could be used to model a population which is growing at a rate of approximately 4% per unit of time, starting with an initial population of 50. As a modification of this model, consider a case where the rate of growth of the population is decreasing over time. For example, the rate of growth might start out at 4% but decrease over time toward 2%. If $x(t)$ is the population after t units of time, then the differential equation

$$\dot{x} = 0.02(1 + e^{-\frac{t}{10}})x$$

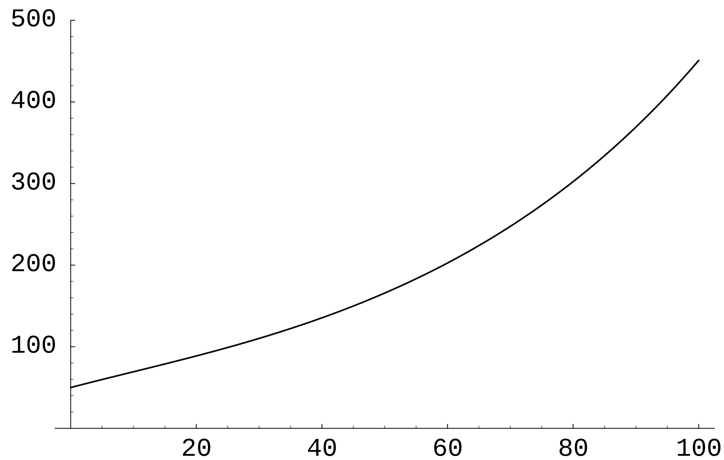


Figure 8.1.2 An approximate solution of $\dot{x} = 0.02(1 + e^{-\frac{t}{10}})x$

would describe one such situation. To approximate the solution to this equation over the interval $[0, 100]$, again assuming an initial population of $x_0 = 50$, we will use Euler's method with $h = 0.05$ and compute $x_1, x_2, \ldots, x_{2000}$. Then, with

$$f(x,t) = 0.02(1 + e^{-\frac{t}{10}})x,$$

we have

$$\begin{aligned} x_1 &= x_0 + hf(50,0) = 50 + (0.05)(2) = 50 + 0.1 = 50.10000, \\ x_2 &= x_1 + hf(50.1, 0.05) = 50.1 + (0.05)(1.99900) = 50.1 + 0.09995 = 50.19995, \\ x_3 &= x_2 + hf(50.19995, 0.10) = 50.19995 + (0.05)(1.99801) \\ &= 50.19995 + 0.09990 = 50.29985, \end{aligned}$$

and so on, ending with $x_{2000} = 450.91$ as our approximation to $x(20)$, where we have rounded x_1, x_2, and x_3 to five decimal places and x_{2000} to two decimal places. In general, x_k is an approximation for $x(0 + 0.05k) = x(0.05k)$; that is, $x(t)$ is approximated by x_{20t}. The following table lists the values of x_{20t}, rounded to two decimal places, for $t = 0, 10, 20, \ldots, 100$:

t	x_{20t} ($\approx x(t)$)
0	50.00
10	69.30
20	88.67
30	110.17
40	135.40
50	165.74
60	202.59
70	247.50
80	302.30
90	369.20
100	450.91

The graph of our approximate solution is shown in Figure 8.1.2.

As we have seen, the accuracy of Euler's method depends on the step size h. In theory, we can obtain any level of accuracy we desire by choosing h small enough; however, in practice there are limitations on how small we can make h. These limitations include the level of precision with which a given computer represents numbers, the time it takes to perform the necessary computations (smaller values of h require more iterations of (8.1.10)), and the accumulation of round-off error resulting from requiring large numbers of iterations. Fortunately, there are many ways to improve upon Euler's method, one of which we will consider now.

For the equation

$$\dot{x} = f(x,t)$$

with $x(t_0) = x_0$, Euler's method is based on using $hf(x_0, t_0)$ as an approximation to the difference $x(t_0 + h) - x_0$. The accuracy of this approximation will depend upon how much $\dot{x}(t)$ differs from $f(x_0, t_0)$ over the interval $[t_0, t_0 + h]$. In general, the accuracy of our approximation will improve if we use

$$\dot{x}\left(t_0 + \frac{h}{2}\right),$$

the slope of x at the midpoint of the interval $[t_0, t_0 + h]$, in place of $\dot{x}(t_0) = f(x_0, t_0)$, the slope of x at the left-hand endpoint of this interval. Now

$$\dot{x}\left(t_0 + \frac{h}{2}\right) = f\left(x\left(t_0 + \frac{h}{2}\right), t_0 + \frac{h}{2}\right). \tag{8.1.11}$$

However, we do not know $x\left(t_0 + \frac{h}{2}\right)$. To get around this difficulty, we will first use Euler's method to approximate $x\left(t_0 + \frac{h}{2}\right)$ by

$$x_0 + \frac{h}{2} f(x_0, t_0),$$

and then approximate $\dot{x}\left(t_0 + \frac{h}{2}\right)$ by

$$f\left(x_0 + \frac{h}{2} f(x_0, t_0), t_0 + \frac{h}{2}\right). \tag{8.1.12}$$

Replacing $f(x_0, t_0)$ by (8.1.12) in Euler's method, we have

$$x_1 = x_0 + hf\left(x_0 + \frac{h}{2} f(x_0, t_0), t_0 + \frac{h}{2}\right) \tag{8.1.13}$$

as our approximation for $x(t_0 + h)$. It can be shown that in this case

$$x(t_0 + h) - x_1 = O(h^3),$$

whereas we have seen that the error in one step of Euler's method is $O(h^2)$. This method of approximation is known as the *Runge-Kutta method of order 2*, after the mathematicians

Carl Runge (1856-1927) and M. W. Kutta (1867-1944). In general, an approximation method is said to be of order n if the error in one step is $O(h^{n+1})$. There are Runge-Kutta formulas for approximations of higher order, but we will consider only this order 2 formula.

Second order Runge-Kutta To approximate the solution of the equation

$$\dot{x} = f(x,t)$$

with initial condition $x(t_0) = x_0$ on an interval $[t_0, t_1]$, choose a small value for $h > 0$ and an integer n such that $t_0 + nh \geq t_1$. Letting $s_k = t_0 + kh$, $k = 0, 1, 2, \ldots n$, compute

$$m = \frac{h}{2} f(x_k, s_k)$$

and

$$x_{k+1} = x_k + hf\left(x_k + m, s_k + \frac{h}{2}\right) \tag{8.1.14}$$

for $k = 0, 1, \ldots n-1$. Then x_k is an approximation for $x(t_0 + kh)$.

Example We will approximate the solution of $\dot{x} = 0.04x$ with $x(0) = 50$ on the interval $[0, 50]$ using the second order Runge-Kutta method with $h = 0.1$. Using $f(x,t) = 0.04x$ and rounding to four decimal places, we have

$$\begin{aligned}
x_1 &= 50 + 0.1f(50 + 0.05f(50, 0), 0.05) \\
&= 50 + 0.1f(50.1, 0.05) \\
&= 50 + 0.2004 \\
&= 50.2004, \\
x_2 &= 50.2004 + 0.1f(50.2004 + 0.05f(50.2004, 0.1), 0.15) \\
&= 50.2004 + 0.1f(50.3008, 0.15) \\
&= 50.2004 + 0.2012 \\
&= 50.4016,
\end{aligned}$$

and so on, up to $x_{500} = 369.4508$. Since $h = 0.1$, x_{10t} gives us an approximation for $x(t)$ whenever $10t$ is an integer. The following table gives the values of x_{10t} and $x(t)$ for $t = 0, 5, 10, \ldots, 50$:

t	x_{10t}	$x(t)$
0	50.0000	50.0000
5	61.0701	61.0701
10	74.5912	74.5912
15	91.1058	91.1059
20	111.2768	111.2770
25	135.9137	135.9141
30	166.0053	166.0058
35	202.7592	202.7600
40	247.6506	247.6516
45	302.4809	302.4824
50	369.4508	369.4528

Comparing these values with the values we obtained with Euler's method for the same equation, we see that we have decreased our error significantly without decreasing the step size. Hence we have gained accuracy without greatly increasing the number of computations required.

Example In Problem 7 in Section 4.8 (which is repeated in Problem 2 at the end of this section) we discussed the problem of a body in free fall near the surface of the earth, neglecting the effects of air resistance. In that case, if the body has mass m and F is the force acting on the body, then, since we are neglecting all forces except for that of gravity, $F = -mg$, where $g = 32$ feet/second^2. However, if we consider the effects of air resistance, then we have to modify F to account for this additional force acting in a direction opposite to that of gravity. One common assumption is that air resistance is proportional to velocity; in that case, we have

$$F = -32m - kv, \tag{8.1.15}$$

where $k > 0$ is a constant which depends on the particular body and v is the velocity of the body. Note that $kv < 0$ since $v < 0$, so the additional force $-kv$ is acting in the opposite direction of gravity. Now if a is the acceleration of the body, $F = ma$ implies

$$ma = -32m - kv, \tag{8.1.16}$$

and so

$$a = -32 - \frac{k}{m}v. \tag{8.1.17}$$

Since $a = \dot{v}$, (8.1.17) gives us the differential equation

$$\dot{v} = -32 - cv, \tag{8.1.18}$$

where

$$c = \frac{k}{m}$$

is a constant which depends both on the mass of the body and its air resistance. For example, suppose we have a situation where $c = 0.1$ and the body is released from rest high above the ground. Then we are interested in the solution of the equation

$$\dot{v} = -32 - 0.1v$$

with the initial condition $v(0) = 0$. The following table gives the results from applying the second order Runge-Kutta method with step size $h = 0.1$ over the interval $[0, 70]$:

t	v_{10t} ($\approx v(t)$)
0	0.0
5	-125.9
10	-202.3
15	-248.6
20	-276.7
25	-293.7
30	-304.1
35	-310.3
40	-314.1
45	-316.4
50	-317.8
55	-318.7
60	-319.2
65	-319.5
70	-319.7

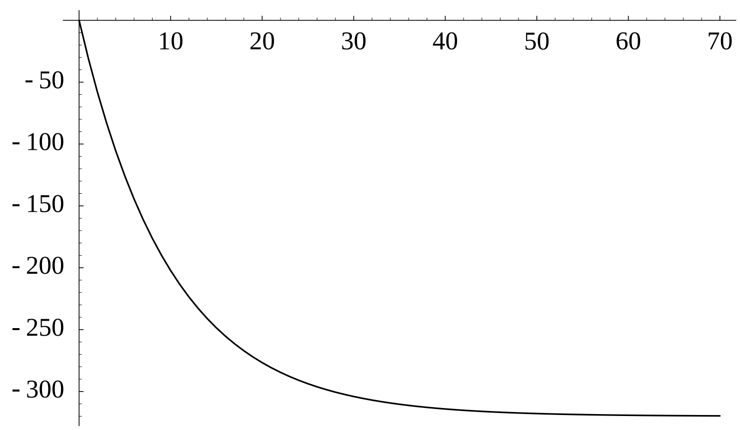


Figure 8.1.3 Velocity of a body in free fall

From the table of values and from the graph of the approximate solution in Figure 8.1.3, it appears that the velocity of the body approaches a limiting value. We shall see in the next section that this is indeed the case. For this example, the velocity will approach -320 feet per second. We call this velocity the *terminal velocity* of the body.

Problems

1. Solve each of the following differential equations using the given initial condition.

   (a) $\dot{x} = t^2 - 2$, $x(0) = 3$ (b) $\dot{x} = -\sin(t)$, $x(0) = 2$

   (c) $\dot{y} = \sqrt{t}$, $y(1) = -3$ (d) $\dot{w} = te^{-t}$, $w(0) = 2$

2. Let $x(t)$, $v(t)$, and $a(t)$ be the height, velocity, and acceleration, respectively, of an object of mass m in free fall near the surface of the earth. Let x_0 and v_0 be the

height and velocity, respectively, of the object at time t_0. If we ignore the effects of air resistance, the force acting on the body is $-mg$, where g is a constant ($g =$ 9.8 meters/second^2 or $g = 32$ feet/second^2). Thus by Newton's second law of motion,

$$-mg = ma(t),$$

and so

$$a(t) = -g.$$

Show that

$$x(t) = -\frac{1}{2}gt^2 + v_0 t + x_0.$$

3. Suppose an object is projected vertically upward from a height of 100 feet with an initial velocity of 20 feet per second. Use Problem 2 to answer the following questions.
   (a) Find $x(t)$, the height of the object at time t.
   (b) At what time does the object reach its maximum height?
   (c) What is the maximum height reached by the object?
   (d) At what time will the object strike the ground?
4. For each of the following, use Euler's method to approximate the solution of the equation over the given interval I using the step size h specified. Plot your result.
   (a) $\dot{x} = e^{-2t^2}$, $x(0) = 1$, $I = [0, 10]$, $h = 0.1$
   (b) $\dot{x} = -0.96x$, $x(0) = 100$, $I = [0, 100]$, $h = 0.5$
   (c) $\dot{y} = 0.05y - 5$, $y(0) = 200$, $I = [0, 50]$, $h = 0.05$
   (d) $\dot{x} = 0.002x(100 - x)$, $x(0) = 10$, $I = [0, 200]$, $h = 0.1$
   (e) $\dot{y} = 0.02(1 + 0.5e^{-\frac{t}{15}})y$, $y(0) = 10$, $I = [0, 150]$, $h = 0.5$
   (f) $\dot{x} = \cos(x^2)$, $x(0) = 0$, $I = [0, 10]$, $h = 0.01$
   (g) $\dot{x} = 0.045x + 0.4t$, $x(0) = 30$, $I = [0, 40]$, $h = 0.1$
   (h) $\dot{x} = 0.1x + \cos(t)$, $x(0) = 0$, $I = [0, 10]$, $h = 0.05$
5. Use the second order Runge-Kutta method to approximate solutions to the equations in Problem 4.
6. In 1990 the population of India was 853.4 million. If $P(t)$ is the population of India t years after 1990, suppose P satisfies the differential equation

$$\dot{P} = k(t)P,$$

   where $k(t)$ is the rate of growth of the population at time t. For example, at the start of 1990 the population of India was growing at a rate of 2.1% per year, so $k(0) = 0.021$.
   (a) Suppose the rate of growth remains constant; that is, suppose $k(t) = 0.021$ for all $t \geq 0$. Find $P(t)$. In what year will the population reach 1 billion? In what year will it reach 2 billion? In what year will it reach 3 billion?

(b) Now suppose $k(t)$ is decreasing toward 1% in such a way that

$$k(t) = 0.01(1 + 1.1e^{-\frac{t}{30}}).$$

Use the second order Runge-Kutta method to approximate P over the interval $[0, 100]$ using a step size of $h = 0.1$. In what year will the population reach 1 billion? In what year will it reach 2 billion? In what year will it reach 3 billion?

(c) Plot your results from (a) and (b) together.

7. In the final example in this section we considered the problem of an object in free fall when the air resistance is proportional to the velocity of the object. Now consider the case where the air resistance is proportional to the square root of the speed.

   (a) If s is the speed of the object, in feet per second, t seconds after it is released, explain why s satisfies the differential equation

   $$\dot{s} = 32 - c\sqrt{s},$$

   $s(0) = 0$, for some constant $c > 0$.

   (b) Using $c = 1$, use the second order Runge-Kutta method to solve the equation in (a) over the interval $[0, 500]$ using a step size of $h = 0.5$. Plot your results.

   (c) Does the object appear to approach a limiting speed? If so, what is the terminal speed?

   (d) Solve the equation

   $$32 - c\sqrt{s} = 0$$

   for s. Explain the connection between this answer and your answer in (c).

8. In the final example in this section we considered the problem of an object in free fall when the air resistance is proportional to the velocity of the object. Now consider the case where the air resistance is proportional to the square of the velocity.

   (a) If v is the velocity of the object, in feet per second, t seconds after it is released, explain why v satisfies the differential equation

   $$\dot{v} = -32 + cv^2,$$

   $v(0) = 0$, for some constant $c > 0$.

   (b) Using $c = 0.01$, use the second order Runge-Kutta method to approximate the solution to the equation in (a) over the interval $[0, 20]$ using a step size of $h = 0.02$. Plot your results.

   (c) Does the object appear to approach a limiting velocity? If so, what is the terminal velocity?

   (d) Solve the equation

   $$-32 + cv^2 = 0$$

   for v. Explain the connection between this answer and your answer in (c).

9. Suppose the population of a certain country was 56 million in 2000 and the natural rate of the growth of the population was 2% per year. Moreover, suppose $k(t)$ represents the net rate of growth of the population due to immigration and emigration t years after 2000.

   (a) Let $P(t)$ be the population of the country t years after 2000. Explain why P should satisfy the differential equation

   $$\dot{P} = 0.02P + k(t),$$

   with $P(0) = 56$.

   (b) If $k(t) = 0.04t$, use the second order Runge-Kutta method to approximate the solution to the equation in (a) over the interval $[0, 25]$ using a step size of $h = 0.05$. Plot your results.

   (c) What does this model predict for the population of the country in the year 2010?

   (d) When will the population of the country reach 100 million?

Difference Equations to Differential Equations

Section 8.2

Separation of Variables

In the previous section we discussed two methods for approximating the solution of a differential equation

$$\dot{x} = f(x, t)$$

with initial condition $x(t_0) = x_0$. We will now consider, in this section as well as in Sections 8.3 and 8.4, techniques for finding *closed form* solutions for such equations, that is, solutions expressible in terms of the elementary functions of calculus. To do so will require considering different classes of equations depending on the form of the function f. As in ordinary integration, finding a closed form expression for the solution of a differential equation is frequently a difficult, if not impossible, problem which requires us to exploit whatever information we can gain from the form of the function. In this section we will consider a class of equations known as *separable equations* and in Sections 8.3 and 8.4 we will consider linear equations.

We call a differential equation

$$\dot{x} = f(x, t) \tag{8.2.1}$$

with initial condition $x(t_0) = x_0$ *separable*, or say it has *separable variables*, if $f(x, t) = g(x)h(t)$ for some functions g and h, where g depends only on x and h depends only on t. We will assume that g and h are both continuous and hence, in particular, integrable. In that case, (8.2.1) becomes

$$\dot{x} = g(x)h(t) \tag{8.2.2}$$

which implies that

$$\frac{\dot{x}}{g(x)} = h(t) \tag{8.2.3}$$

at all points for which $g(x) \neq 0$. Integrating (8.2.3) from t_0 to t (assuming $g(x(s)) \neq 0$ for all s between t_0 and t), we have

$$\int_{t_0}^{t} \frac{1}{g(x(s))}\ \dot{x}(s)ds = \int_{t_0}^{t} h(s)ds, \tag{8.2.4}$$

where we have used s as the variable of integration so that our answer will be in terms of t. Now the substitution

$$\begin{aligned} u &= x(s) \\ du &= \dot{x}(s) \end{aligned}$$

gives us

$$\int_{t_0}^{t} \frac{1}{g(s)}\ \dot{x}(s)ds = \int_{x(t_0)}^{x(t)} \frac{1}{g(u)}\ du = \int_{x_0}^{x} \frac{1}{g(u)}\ du \tag{8.2.5}$$

for the integral on the right-hand side. Hence, putting (8.2.4) and (8.2.5) together,

$$\int_{x_0}^{x} \frac{1}{g(u)}\, du = \int_{t_0}^{t} h(s) ds. \tag{8.2.6}$$

Thus we can solve an equation with separable variables provided we are able to evaluate both of the integrals in (8.2.6) and then solve the resulting equation for x. The process may break down at either of these final two steps, in which case we must fall back on numerical approximations even though the equation is separable.

Separation of variables If g and h are continuous functions of x and t, respectively, and x satisfies the differential equation

$$\dot{x} = g(x)h(t) \tag{8.2.7}$$

with $x(t_0) = x_0$, then

$$\int_{x_0}^{x} \frac{1}{g(u)}\, du = \int_{t_0}^{t} h(s) ds, \tag{8.2.8}$$

provided $g(u) \neq 0$ for all u between x_0 and x.

Note that this is the same method we used to solve the inhibited growth model equation in Section 6.3.

Example Consider the equation

$$\dot{x} = 0.4x$$

with $x(0) = 100$. This is a separable equation with, in the notation used above, $g(x) = x$ and $h(t) = 0.4$. (Note that the choices for g and h are not unique.) Using (8.2.8), we have

$$\int_{100}^{x} \frac{1}{u}\, du = \int_{0}^{t} 0.4 ds,$$

Now, assuming $x > 0$,

$$\int_{100}^{x} \frac{1}{u}\, du = \log(u)\Big|_{100}^{x} = \log(x) - \log(1000) = \log\left(\frac{x}{100}\right),$$

and

$$\int_{0}^{t} 0.4 ds = 0.4s\Big|_{0}^{t} = 0.4t.$$

Hence we have

$$\log\left(\frac{x}{100}\right) = 0.4t,$$

from which we obtain

$$\frac{x}{100} = e^{0.4t}$$

and, finally,

$$x = 100e^{0.4t}.$$

Note that this is the solution we should expect from our study of equations of this form in Sections 6.1 and 6.3.

Example Consider the equation

$$\dot{y} = -2yt \tag{8.2.9}$$

with $y(0) = y_0 \neq 0$. This is a separable equation with, in the notation used above, $g(y) = y$ and $h(t) = -2t$. Using (8.2.8), we have

$$\int_{y_0}^{y} \frac{1}{u}\, du = -\int_0^t 2sds.$$

Now

$$\int_{y_0}^{y} \frac{1}{u}\, du = \log|u|\Big|_{y_0}^{y} = \log|y| - \log|y_0| = \log\left|\frac{y}{y_0}\right|$$

and

$$-\int_0^t 2sds = -s^2\Big|_0^t = -t^2.$$

Hence we have

$$\log\left|\frac{y}{y_0}\right| = -t^2,$$

from which it follows that

$$\left|\frac{y}{y_0}\right| = e^{-t^2}.$$

Now $e^{-t^2} > 0$ for all t, so $y(t)$ is never 0. Since y is continuous (which follows from our assumption that it is differentiable), this means that either $y(t) > 0$ for all t or $y(t) < 0$ for all t. Since $y(0) = y_0$, $y(t) > 0$ for all t if $y_0 > 0$ and $y(t) < 0$ for all t if $y_0 < 0$. In either case,

$$\frac{y(t)}{y_0} > 0$$

for all t, so

$$\left|\frac{y}{y_0}\right| = \frac{y}{y_0}.$$

Hence we have

$$\frac{y}{y_0} = e^{-t^2},$$

or

$$y = y_0 e^{-t^2}. \tag{8.2.10}$$

Note that (8.2.10) also specifies a solution of (8.2.9) when $y_0 = 0$, namely, the solution $y(t) = 0$ for all t. By leaving the value of y_0 unspecified, we have found the general form of all possible solutions for the equation. We call the family of all possible solutions given by (8.2.10) the *general solution* of the equation (8.2.9). Any solution obtained by specifying

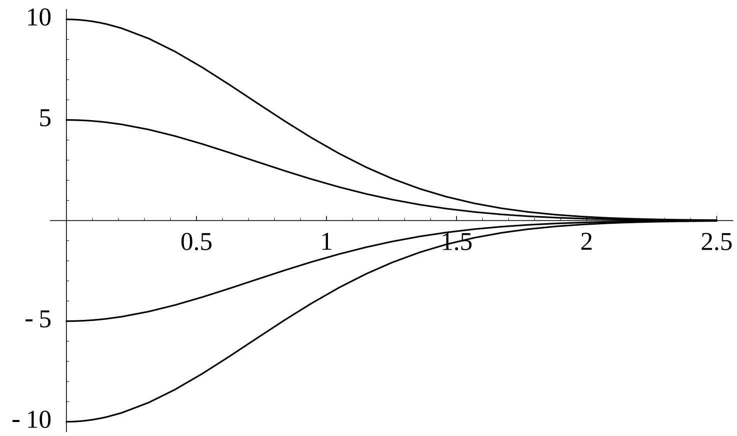


Figure 8.2.1 Four particular solutions of $\dot{y} = -2yt$

a value of y_0, say, for example, $y_0 = 10$, is called a *particular solution* of the equation. Figure 8.2.1 shows the graphs of four particular solutions for this equation.

As noted in the first example, the choices for g and h are not unique. For example, in the second example we could just as well have taken $g(y) = 2y$ and $h(t) = t$. However, one should attempt to choose g and h in such a way that the subsequent steps in the solution are as simple as possible.

Example Consider the equation

$$\dot{x} = -\frac{t}{x}$$

with $x(0) = x_0 \neq 0$. Separating the variables, we have

$$\int_{x_0}^{x} u du = -\int_0^t s ds.$$

Now

$$\int_{x_0}^{x} u du = u^2\Big|_{x_0}^{x} = x^2 - x_0^2$$

and

$$-\int_0^t s ds = -s^2\Big|_0^t = -t^2,$$

and so

$$x^2 - x_0^2 = -t^2,$$

or

$$x^2 + t^2 = x_0^2.$$

This equation implicitly defines x as a function of t. Indeed, from this equation we can see that the graph of x is part of circle of radius x_0 centered at the origin. Solving explicitly for x, we have

$$x = \sqrt{x_0^2 - t^2}$$

if $x_0 > 0$ and

$$x = -\sqrt{x_0^2 - t^2}$$

if $x_0 < 0$. Note that x is only defined for $-x_0 < t < x_0$.

Example In Section 8.1 we considered the equation

$$\dot{v} = -g - \frac{k}{m}v,$$

with $v(0) = 0$, as a model for the velocity of an object in free fall near the surface of the earth when the force due to air resistance is proportional to velocity. Here v is the velocity of the object, g, as usual, is 32 feet per second per second or 9.8 meters per second per second, m is the mass of the object, and $k > 0$ is a constant which depends on the air resistance of the particular object. If we write this equation in the form

$$\dot{v} = -g\left(1 + \frac{k}{gm}v\right) \tag{8.2.11}$$

and separate variables, using

$$f(v) = 1 + \frac{k}{gm}v$$

and

$$h(t) = -g,$$

then we have

$$\int_0^v \frac{1}{1 + \frac{k}{gm}v}\, du = -\int_0^t g ds.$$

Now

$$\int_0^v \frac{1}{1 + \frac{k}{gm}v}\, du = \frac{gm}{k} \log\left|1 + \frac{k}{gm}u\right|\Big|_0^v = \frac{gm}{k} \log\left|1 + \frac{k}{gm}v\right|$$

and

$$-\int_0^t g ds = -gs\Big|_0^t = -gt,$$

so

$$\frac{gm}{k} \log\left|1 + \frac{k}{gm}v\right| = -gt.$$

Hence

$$\log\left|1 + \frac{k}{gm}v\right| = -\frac{kt}{m},$$

from which it follows that

$$\left|1 + \frac{k}{gm}v\right| = e^{-\frac{kt}{m}}.$$

Thus either

$$1 + \frac{k}{gm}v = e^{-\frac{kt}{m}}$$

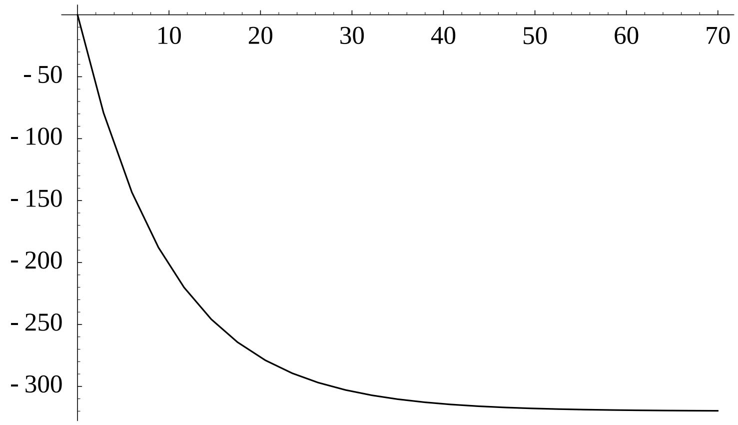


Figure 8.2.2 Graph of $v(t) = -320(1 - e^{-0.1t})$

or

$$1 + \frac{k}{gm}v = -e^{-\frac{kt}{m}}.$$

That is, either

$$v = -\frac{gm}{k}(1 - e^{-\frac{kt}{m}}).$$

or

$$v = \frac{gm}{k}(1 - e^{-\frac{kt}{m}}).$$

Since the object is following toward the earth, $v < 0$ and we must have

$$v = -\frac{gm}{k}(1 - e^{-\frac{kt}{m}}).$$

Hence we now have a closed form solution for this model of free fall, whereas in the previous section we could only compute a numerical approximation. Notice that one advantage of the closed form solution is that we did not have to specify values for the parameters k and m before finding the solution; as a result, we may now easily compute v for any specified values of k and m. For example, using $\frac{k}{m} = 0.1$ and $g = 32$ as in our example in Section 8.1, we obtain a plot of v as shown in Figure 8.2.2. You should compare this with the graph of our numerical solution in Figure 8.1.3. Also, the closed form solution allows us to compute

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} -\frac{gm}{k}(1 - e^{-\frac{kt}{m}}) = -\frac{gm}{k}, \tag{8.2.12}$$

showing that an object falling according to this model has a terminal velocity, as we suspected from our numerical work in Section 8.1. Moreover, (8.2.12) gives us a general expression for this velocity. For our example, $\frac{k}{m} = 0.1$ and $g = 32$ give us a terminal velocity of

$$-\frac{gm}{k} = -320 \text{ feet per second.}$$

Problems

1. Solve each of the following differential equations using the given initial condition.

   (a) $\dot{x} = -0.9x$, $x(0) = 75$

   (b) $\dot{x} = x^2$, $x(0) = 10$

   (c) $\dot{y} = \dfrac{t}{y}$, $y(0) = 5$

   (d) $\dot{w} = \dfrac{w}{t}$, $w(1) = 1$

   (e) $\dot{x} = \dfrac{t}{x + tx}$, $x(0) = 4$

   (f) $\dot{y} = 1 + y^2$, $y(0) = 0$

   (g) $\dot{x} = x(1 - x)$, $x(0) = 0.2$

2. (a) Solve the differential equation $\dot{x} = -x^2\text{t}$, $x(0) = x_0 \neq 0$.

   (b) Graph x on the interval $[5, 5]$ for $x_0 = 2$, $x_0 = 5$, and $x_0 = 10$. Are the graphs similar?

   (c) What is the domain of x if $x_0 > 0$? What is the domain of x if $x_0 < 0$?

   (d) Graph x for $x_0 = -1$ and $x_0 = 1$. Are the graphs similar?

3. (a) A curve is defined so that whenever (x_0, y_0), with $y_0 \neq 0$, is a point on the curve,

   $$\left.\frac{dy}{dx}\right|_{(x,y)=(x_0,y_0)} = -\frac{ax_0}{by_0},$$

   where $a > 0$ and $b > 0$ are constants. Show that the curve must be an ellipse. Under what conditions is the curve a circle?

   (b) A curve is defined so that whenever (x_0, y_0), with $y_0 \neq 0$, is a point on the curve,

   $$\left.\frac{dy}{dx}\right|_{(x,y)=(x_0,y_0)} = \frac{ax_0}{by_0},$$

   where $a > 0$ and $b > 0$ are constants. Show that the curve must be a hyperbola.

4. In Chapter 6 we considered the consequences of the population growth model

   $$\dot{x} = kx,$$

   with $x(0) = x_0$, where $x(t)$ represents the size of some population at time t and $k > 0$ is a constant which depends on the rate at which the population is growing. In this problem we will see what happens if $\dot{x}$ is proportional, not to x, but to some power of x. That is, consider the model

   $$\dot{x} = kx^b, \tag{8.2.13}$$

   with $x(0) = x_0$ and $b > 0$ a constant.

   (a) Solve (8.2.13) when $b = 2$ and show that

   $$\lim_{t \to \frac{1}{kx_0}^-} x(t) = \infty.$$

   Plot x for $x_0 = 50$ and $k = 0.001$, $k = 0.01$, and $k = 0.02$.

(b) Solve (8.2.13) when $b > 1$. Find c such that

$$\lim_{t\to c^-} x(t) = \infty.$$

Plot x for $x_0 = 50$, $k = 0.01$, and $b = 1.5$, $b = 1.2$, and $b = 1.01$.

(c) Solve (8.2.13) when $b = 0.5$. Show that x is a quadratic polynomial and

$$\lim_{t\to\infty} x(t) = \infty.$$

Plot x for $x_0 = 50$ and $k = 0.01$, $k = 0.02$, and $k = 0.05$.

(d) Solve (8.2.13) when $0 < b < 1$ and show that

$$\lim_{t\to\infty} x(t) = \infty.$$

Plot x for $x_0 = 50$, $k = 0.01$, and $b = 0.2$, $b = 0.4$, and $b = 0.9$.

(e) Compare the rates of growth for $0 < b < 1$, $b = 1$, and $b > 1$. Which model leads to the slowest population growth? Which model leads to the most rapid population growth? Why is the case $b > 1$ sometimes referred to as the *doomsday model*?

5. Suppose the force due to air resistance acting on a falling body of mass m is proportional to the square of the velocity v.

(a) Explain why v satisfies the differential equation

$$\dot{v} = -g + \frac{k}{m}v^2,$$

where $k > 0$ is a constant.

(b) Assuming $v(0) = 0$, solve the equation in (a) for v.

(c) Show that the terminal velocity of the object is $-\sqrt{\dfrac{mg}{k}}$.

(d) Plot v over the interval $[0, 20]$ using $g = 32$ and $\frac{k}{m} = 0.01$. Compare this plot with the plot of the numerical solution found in Problem 8 of Section 8.1.

6. In Section 1.4 we discussed the discrete time version of Newton's law of cooling. Briefly, this law says if an object with an initial temperature of T_0 is placed in an environment which is held at a constant temperature S, then the rate of change of the temperature T of the object is proportional to the difference between T and S. In terms of differential equations, this says that T must satisfy the equation

$$\dot{T} = k(T - S)$$

for some constant k.

(a) Show that

$$T = S + (T_0 - S)e^{kt}$$

and verify that

$$\lim_{t\to\infty} T(t) = S.$$

(b) A cup of coffee, initially at a temperature of 115°F, is placed on a table in a room held at a constant temperature of 72°F. If after five minutes the coffee has cooled to 105°F, what is the temperature of the coffee after 20 minutes? How long will it take the coffee to cool to 80°F? Graph T.

(c) A glass of lemonade, initially at a temperature of 40°F, is placed on a table in a room held at a constant temperature of 75°F. If after 10 minutes the lemonade has warmed to 48°F, what is the temperature of the lemonade after 30 minutes? How long will it take the lemonade to warm to 65°F? Graph T.

(d) A cup of coffee, initially at a temperature of 110°F, is placed on a table in a room. After five minutes the coffee has cooled to 100°F and after ten minutes the coffee has cooled to 92°F. What is the temperature of the room?

Difference Equations to Differential Equations

Section 8.3

First Order Linear Differential Equations

We will now consider closed form solutions for another important class of differential equations. A differential equation

$$\dot{x} = f(x, t)$$

with $x(t_0) = x_0$ is called a *linear equation* if

$$f(x, t) = p(t)x + q(t) \tag{8.3.1}$$

for some functions p and q which depend only on t. We will assume that both p and q are continuous functions. Note that under certain circumstances, such as $q(t) = 0$ for all t, a linear equation is also separable. The solution of such equations is based on the following observation: If we let

$$P(t) = \int_{t_0}^{t} p(s)ds, \tag{8.3.2}$$

then

$$\frac{d}{dt}\left(xe^{-P(t)}\right) = -xp(t)e^{-P(t)} + \dot{x}e^{-P(t)} = e^{-P(t)}(\dot{x} - p(t)x). \tag{8.3.3}$$

Now we want $\dot{x} = p(t)x + q(t)$, that is, $\dot{x} - p(t)x = q(t)$, so we are looking for a function x such that

$$\frac{d}{dt}\left(xe^{-P(t)}\right) = q(t)e^{-P(t)}. \tag{8.3.4}$$

Integrating (8.3.4) from t_0 to t (using u for our variable of integration), we have

$$\int_{t_0}^{t} \frac{d}{du}\left(x(u)e^{-P(u)}\right)du = \int_{t_0}^{t} q(u)e^{-P(u)}du. \tag{8.3.5}$$

Now

$$\begin{aligned}\int_{t_0}^{t} \frac{d}{du}\left(x(u)e^{-P(u)}\right)du &= x(u)e^{-P(u)}\Big|_{t_0}^{t} \\ &= x(t)e^{-P(t)} - x(t_0)e^{-P(t_0)} \\ &= x(t)e^{-P(t)} - x_0\end{aligned} \tag{8.3.6}$$

since $P(t_0) = 0$ and $x(t_0) = x_0$. Hence we want

$$x(t)e^{-P(t)} - x_0 = \int_{t_0}^{t} q(u)e^{-P(u)}du. \tag{8.3.7}$$

Solving (8.3.7) fot $x(t)$, we have

$$x(t) = e^{P(t)} \left(\int_{t_0}^{t} q(u) e^{-P(u)} du + x_0 \right). \tag{8.3.8}$$

Similar to our situation with separable equations, (8.3.8) provides a closed form solution to our equation only if the requisite integrals may be computed in closed form. If not, numerical techniques will be necessary.

Linear equations If p and q are continuous, x satisfies the differential equation

$$\dot{x} = p(t)x + q(t) \tag{8.3.9}$$

with $x(t_0) = x_0$, and

$$P(t) = \int_{t_0}^{t} p(s) ds, \tag{8.3.10}$$

then

$$x(t) = e^{P(t)} \left(\int_{t_0}^{t} q(u) e^{-P(u)} du + x_0 \right). \tag{8.3.11}$$

Example Consider the equation

$$\dot{x} = \frac{x}{t} + 4t$$

with $x(1) = 5$. This is a linear equation with, in the notation used above,

$$p(t) = \frac{1}{t}$$

and

$$q(t) = 4t.$$

Then

$$P(t) = \int_{1}^{t} \frac{1}{s}\, ds = \log(s)\Big|_{1}^{t} = \log(t),$$

where, in order for the integral to exist, we have restricted t to be positive. Thus, using (8.3.11),

$$\begin{aligned} x &= e^{\log(t)} \left(\int_{1}^{t} 4u e^{-\log(u)} du + 5 \right) \\ &= t \left(\int_{1}^{t} \frac{4u}{u}\, du + 5 \right) \\ &= t \left(\int_{1}^{t} 4 du + 5 \right) \\ &= t \left(4t\Big|_{1}^{t} + 5 \right) \\ &= t(4t - 4) + 5t \\ &= 4t^2 + t. \end{aligned}$$

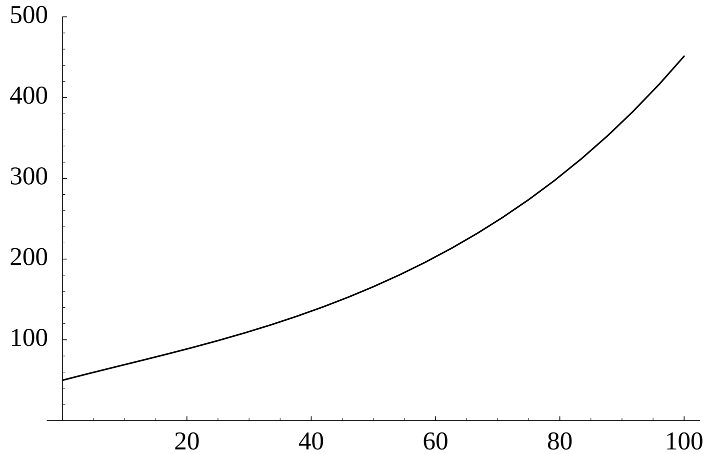


Figure 8.3.1 A solution of $\dot{x} = 0.02(1 + e^{-0.1t})x$

Example The equation
$$\dot{x} = p(t)x$$
with $x(0) = x_0$ may be used as a model for growth or decay where the rate of growth or decay is not necessarily a constant. Such an equation may be solved by separating variables, but the solution also follows from (8.3.11): Since $q(t) = 0$ for all t, we have
$$x = x_0 e^{\int_0^t p(s)ds}.$$
For example, if $p(t) = k$ for all t, where k is a constant, then we obtain the familiar solution
$$x = x_0 e^{kt}.$$
If $p(t) = 0.02(1 + e^{-0.1t})$, as in an example in Section 8.1, then
$$\begin{aligned}\int_0^t p(s)ds &= \int_0^t 00.02(1 + e^{-0.1s})ds \\ &= 0.02(s - 10e^{-0.1s})\Big|_0^t \\ &= 0.02(t - 10e^{-0.1t}) - 0.02(-10) \\ &= 0.02t - 0.02e^{-0.1t} + 0.2.\end{aligned}$$
Hence
$$x = x_0 e^{0.02t - 0.02e^{-0.1t} + 0.2.}.$$
The graph of x when $x_0 = 50$ is shown in Figure 8.3.1. You should compare this with the plot of an approximate solution in Figure 8.1.2.

Example A small reservoir holds 10,000 cubic feet of water. Water flows in at a rate of 100 cubic feet per hour and out at the same rate. Suppose initially the water in the reservoir contains 5 grams of salt per cubic foot, but the water flowing in contains 10 grams

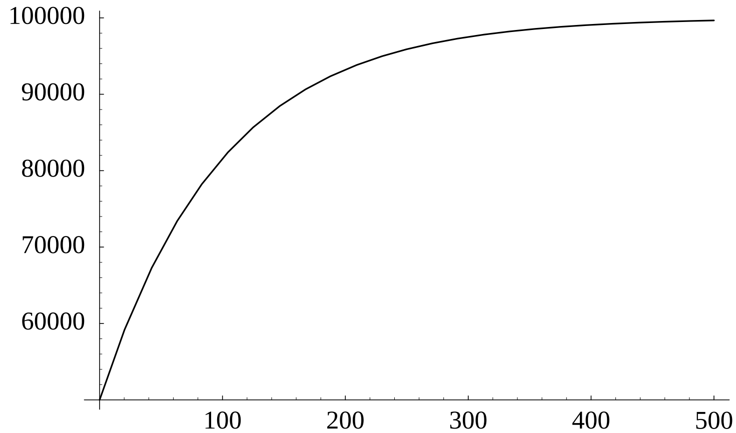


Figure 8.3.2 Graph of $x = 100,000 - 50,000e^{-0.01t}$

of salt per cubic foot. Let $x(t)$ be the amount of salt in the reservoir after t hours. Note that salt is entering the reservoir at a rate of 1000 grams per hour. Assuming the water in the reservoir is well-mixed at all times, the concentration of salt in the reservoir at time t is

$$\frac{x(t)}{10,000}$$

grams per cubic foot, from which it follows that salt is leaving the reservoir at a rate of

$$100\frac{x(t)}{10,000} = \frac{x(t)}{100}$$

grams per hour. Thus the rate of change of salt in the reservoir is given by

$$\dot{x} = 1000 - 0.01x.$$

That is, x satisfies a linear differential equation with $p(t) = -0.01$ and $q(t) = 1000$. Then

$$P(t) = -\int_0^t 0.01ds = -0.01t,$$

and so, using $x(0) = (5)(10,000) = 50,000$ grams, we have

$$\begin{aligned} x &= e^{-0.01t}\left(\int_0^t 1000e^{0.01u}du + 50,000\right) \\ &= e^{-0.01t}\left(100,000e^{0.01u}\Big|_0^t + 50,000\right) \\ &= e^{-0.01t}(100,000e^{0.01t} - 100,000 + 50,000) \\ &= 100,000 - 50,000e^{-0.01t}. \end{aligned}$$

In particular, note that

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} 100,000 - 50,000e^{-0.01t} = 100,000,$$

and we see that over time the concentration of salt in the reservoir will approach the concentration of salt in its intake water. The graph of x is shown in Figure 8.3.2.

Problems

1. Solve each of the following linear differential equations.

   (a) $\dot{x} = 3x + 2t$, $x(0) = 2$
   (b) $\dot{x} = 2x - t^2$, $x(0) = 1$
   (c) $\dot{y} = 0.4y + 3$, $y(0) = 5$
   (d) $\dot{w} = -w + e^{-2t}$, $w(0) = 3$
   (e) $\dot{x} = \frac{2x}{t} + t^2$, $x(1) = 4$
   (f) $\dot{y} = -y + 2e^{-t} + t^2$, $y(0) = 1$

2. In 1990 the population of Botswana was 1.2 million. If $x(t)$ is the population of Botswana t years after 1990, suppose x satisfies the differential equation

$$\dot{x} = k(t)x,$$

where $k(t)$ represents the rate of growth of the population at time t. At the start of 1990 the population of Botswana was growing at the rate of 2.9%, so $k(0) = 0.029$.

   (a) Suppose the rate of growth of the population is decreasing toward 1.5% in such a way that
$$k(t) = 0.015(1 + 0.93e^{0.04t}).$$
   Solve for x.

   (b) Compare your result in (a) with the result for a constant rate of growth of $k(t) = 0.029$ by plotting both solutions together.

3. Suppose the population of a certain country was 56 million in 2000 and the natural rate of the growth of the population was 2% per year. Moreover, suppose $k(t)$ represents the net rate of growth of the population due to immigration and emigration t years after 2000.

   (a) Let $y(t)$ be the population of the country t years after 2000. Explain why y should satisfy the differential equation

$$\frac{dy}{dt} = 0.02y + k(t),$$

   with $y(0) = 56$.

   (b) Solve the equation if $k(t) = 0.04t$. Plot your results.

   (c) What does this model predict for the population of the country in the year 2010?

   (d) When will the population of the country reach 100 million?

   (e) Compare your results with the numerical results obtained for the same problem in Problem 9 of Section 8.1.

4. A 500 gallon tank is initially filled with water with a concentration of 4 grams of salt per gallon. Water flows into the tank at the rate of 10 gallons per minute and is drawn off at the same rate. The concentration of salt in the intake water is 2 grams per gallon. Assume that the water in the tank is well-mixed at all times.

(a) Let $x(t)$ be the amount of salt in the tank at time t. Find a linear differential equation for x which models this situation.

(b) Solve the equation from (a) and graph the solution. What happens to x as t increases?

5. Suppose a tank holds V liters of a liquid which contains a certain chemical at a concentration of k_0 grams per liter. Liquid flows into the tank at rate of q liters per second and is drawn off at the same rate. The concentration of the chemical in the intake liquid is k grams per liter.

(a) If $x(t)$ is the amount of the chemical in the tank at time t, show that x satisfies the linear differential equation

$$\dot{x} = qk - \frac{q}{V}x$$

with $x(0) = k_0 V$.

(b) Solve the equation in (a). What happens to x as $t \to \infty$?

6. An equation of the form

$$\dot{x} = p(t)x + q(t)x^n \tag{8.3.12}$$

is called a *Bernoulli equation*. Note that the equation is linear if either $n = 0$ or $n = 1$.

(a) Assume $n \neq 0$ and $n \neq 1$. Show that the substitution $w = x^{1-n}$ in (8.3.12) results in the linear differential equation

$$\dot{w} = (1-n)p(t)w + (1-n)q(t).$$

(b) Use the result of (a) to solve the equation

$$\dot{x} = \frac{x}{t} - x^2$$

with $x(1) = 1$.

(c) Use the result of (a) to solve the equation

$$\dot{=}x(1-x)$$

with $x(0) = 0.5$.

7. Note that (c) of Problem 6 is a particular example of the logistic differential equation that we studied in Section 6.3 in our discussion of the inhibited population growth model. In general, we considered the logistic equation

$$\dot{x} = \frac{\alpha}{M}x(M-x)$$

with $x(0) = x_0$, where $x(t)$ is the size of the population at time t, α is the natural growth rate of the population, and M is the maximum size of the population that is sustainable in the given environment. Write this equation in the form of a Bernoulli equation and use the result from (a) of Problem 6 to show that

$$x = \frac{M}{1 + \beta e^{-\alpha t}}$$

where

$$\beta = \frac{M - x_0}{x_0}.$$

Difference Equations to **D**ifferential Equations

Section 8.4

Second Order Linear Differential Equations

To this point we have we have considered only first order differential equations. However, many of the most interesting differential equations involve second derivatives. Indeed, since acceleration is the second derivative of position, Newton's second law of motion, $F = ma$, is a second order differential equation. In general, if f is a known function of three variables, then the equation

$$\ddot{x} = f(\dot{x}, x, t) \tag{8.4.1}$$

is called a *second order differential equation.* If we let $y = \dot{x}$, then (8.4.1) may be written as a pair of first order differential equations

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= f(y, x, t). \end{aligned} \tag{8.4.2}$$

Hence moving from the study of first order differential equations to the study of second order differential equations is analogous to moving from the study of one algebraic equation in one unknown to the study of two algebraic equations in two unknowns. We will make use of this fact when we consider numerical approximations to solutions of second order equations in Section 8.6.

As was the case with first order equations, the existence of a closed form solution to a second order differential equation and our ability to find one when it exists depends very much on the form of the function f in (8.4.1). We shall consider closed form solutions for only one class of such equations, leaving other equations for either the numerical approximations of Section 8.6 or the infinite series techniques of Section 8.7. Here we are concerned with equations of the form

$$\ddot{x} + b\dot{x} + cx = 0, \tag{8.4.3}$$

which we call a *second order homogeneous linear differential equation with constant coefficients*, corresponding to

$$f(\dot{x}, x, t) = -b\dot{x} - cx$$

in (8.4.1). The term *homogeneous* refers to the fact that the function $x(0) = 0$ for all t is a solution of the equation and the phrase *constant coefficients* refers to the fact that b and c are assumed to be constants.

To begin our study of these equations, suppose $x_1(t)$ and $x_2(t)$ are both solutions of (8.4.3) and let $x(t) = c_1x_1(t) + c_2x_2(t)$ for constants c_1 and c_2. Then

$$\begin{aligned}\ddot{x} + b\dot{x} + cx &= (c_1\ddot{x}_1 + c_2\ddot{x}_2) + b(c_1\dot{x} + c_2\dot{x}_2) + c(c_1x_1 + c_2x_2)\\ &= c_1(\ddot{x}_1 + b\dot{x}_1 + cx_1) + c_2(\ddot{x}_2 + b\dot{x}_2 + cx_2)\\ &= (c_1)(0) + (c_2)(0) = 0.\end{aligned}$$

That is, x is also a solution of (8.4.3).

Proposition If x_1 and x_2 are both solutions of the equation

$$\ddot{x} + b\dot{x} + cx = 0,$$

then $x = c_1x_1 + c_2x_2$ is also a solution of this equation for any constants c_1 and c_2.

The next proposition is key to our method of solving equations of the form (8.4.3), although we will leave its justification to a more advanced course. First we introduce a definition which will make the proposition, as well as our later results, easier to state.

Definition If f and g are functions for which neither one is a constant multiple of the other, then we say f and g are *linearly independent*.

Proposition Suppose x_1 and x_2 are linearly independent solutions of the equation

$$\ddot{x} + b\dot{x} + cx = 0.$$

Then for any solution x, there exist constants c_1 and c_2 such that

$$x = c_1x_1 + c_2x_2. \tag{8.4.4}$$

The equation

$$\ddot{x} + b\dot{x} + cx = 0 \tag{8.4.5}$$

will have a unique solution only when we place some restrictions on x. For example, if we specify initial conditions for both x and $\dot{x}$, say, $x(t_0) = x_0$ and $\dot{x}(t_0) = y_0$, then (8.4.4) will have a unique solution which satisfies these conditions. This statement is far from obvious, but should appear reasonable in light of the observation, made above, that we could write this equation as a pair of first order equations

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -by - cx.\end{aligned} \tag{8.4.6}$$

Hence our method of attack in solving (8.4.4) will be to first find two linearly independent solutions, say, x_1 and x_2, and then find values for constants c_1 and c_2 such that $x = c_1x_1 + c_2x_2$ satisfies the given initial conditions.

To find two linearly independent solutions of (8.4.4), we begin with the observation that if x satisfies this equation, then $\ddot{x}$ is equal to a sum of constant multiples of x and $\dot{x}$. Hence it would be reasonable to begin with $x = e^{kt}$, for some constant k, as an initial guess. In that case,

$$\dot{x} = ke^{kt}$$

and

$$\ddot{x} = k^2 e^{kt},$$

so x will be a solution of (8.4.4) if and only if

$$k^2 e^{kt} + bke^{kt} + ce^{kt} = e^{kt}(k^2 + bk + c) = 0 \tag{8.4.7}$$

for all t. Since $e^{kt} \neq 0$ for all t, this will happen if and only if

$$k^2 + bk + c = 0. \tag{8.4.8}$$

Hence $x = e^{kt}$ is a solution to (8.4.4) if and only if k is a root of (8.4.8).

Definition The equation

$$k^2 + bk + c = 0 \tag{8.4.9}$$

is called the *characteristic equation* of the differential equation

$$\ddot{x} + b\dot{x} + cx = 0.$$

Since the characteristic equation is quadratic in k, its roots are given by the quadratic formula, namely,

$$k_1 = \frac{-b - \sqrt{b^2 - 4c}}{2} \tag{8.4.10}$$

and

$$k_2 = \frac{-b + \sqrt{b^2 - 4c}}{2}. \tag{8.4.11}$$

At this point, our search for solutions breaks into three cases, depending on whether k_1 and k_2 are (1) distinct real numbers (that is, $b^2 - 4c > 0$), (2) distinct complex numbers (that is, $b^2 - 4c < 0$), or (3) real, but equal (that is, $b^2 - 4c = 0$).

Case 1: Distinct real roots

Suppose k_1 and k_2 are distinct real roots of the characteristic equation. In that case, $x_1 = e^{k_1 t}$ and $x_2 = e^{k_2 t}$ are linearly independent solutions of

$$\ddot{x} + b\dot{x} + cx = 0$$

and all that remains is to find constants c_1 and c_2 such that

$$x = c_1 e^{k_1 t} + c_2 e^{k_2 t} \tag{8.4.12}$$

satisfies the given initial conditions.

Example Consider the equation

$$\ddot{x} + \dot{x} - 6x = 0$$

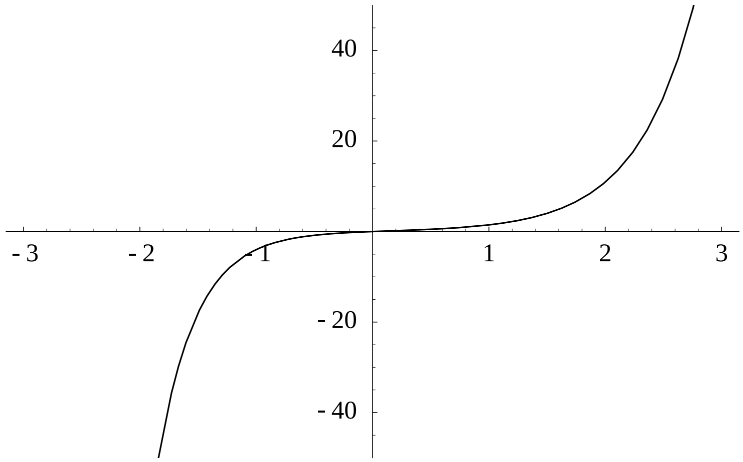


Figure 8.4.1 Solution of $\ddot{x} + \dot{x} - 6x = 0$ with $x(0) = 0$ and $\dot{x}(0) = 1$

with initial conditions $x(0) = 0$ and $\dot{x}(0) = 1$. For the characteristic equation we have

$$0 = k^2 + k - 6 = (k + 3)(k - 2).$$

Hence the roots of the characteristic equation are $k_1 = -3$ and $k_2 = 2$. Thus we must have

$$x = c_1 e^{-3t} + c_2 e^{2t}$$

for some constants c_1 and c_2. Now

$$\dot{x} = -3c_1 e^{-3t} + 2c_2 e^{2t},$$

so

$$x(0) = c_1 + c_2$$

and

$$\dot{x}(0) = -3c_1 + 2c_2.$$

Hence the initial conditions imply that

$$\begin{aligned} c_1 + c_2 &= 0 \\ -3c_1 + 2c_2 &= 1. \end{aligned}$$

The first equation implies $c_1 = -c_2$. Substituting into the second equation, we have

$$1 = 3c_2 + 2c_2 = 5c_2.$$

Hence

$$c_2 = \frac{1}{5}$$

and

$$c_1 = -\frac{1}{5}.$$

Thus

$$x = -\frac{1}{5} e^{-3t} + \frac{1}{5} e^{2t}.$$

The graph of x is shown in Figure 8.4.1

Case 2: Complex roots

Suppose k_1 and k_2 are distinct complex roots of the characteristic equation. As before, $e^{k_1 t}$ and $e^{k_2 t}$ are linearly independent solutions of

$$\ddot{x} + b\dot{x} + cx = 0.$$

However, these are complex-valued functions and for most applications we are looking for real-valued solutions. Now if we let

$$p = -\frac{b}{2} \tag{8.4.13}$$

and

$$q = \frac{\sqrt{4c - b^2}}{2}, \tag{8.4.14}$$

then $k_1 = p - qi$ and $k_2 = p + qi$. Hence

$$e^{k_1 t} = e^{(p-qi)t} = e^{pt}e^{-iqt} = e^{pt}(\cos(qt) - i\sin(qt)) \tag{8.4.15}$$

and

$$e^{k_2 t} = e^{(p+qi)t} = e^{pt}e^{iqt} = e^{pt}(\cos(qt) + i\sin(qt)). \tag{8.4.16}$$

Since these are both solutions, we know that

$$x_1 = \frac{1}{2}e^{k_1 t} + \frac{1}{2}e^{k_2 t} = e^{pt}\cos(qt) \tag{8.4.17}$$

and

$$x_2 = \frac{1}{2i}e^{k_2 t} - \frac{1}{2i}e^{k_1 t} = e^{pt}\sin(qt) \tag{8.4.18}$$

are also solutions. Then x_1 and x_2 are linearly independent real-valued solutions, so any real-valued solution must be of the form

$$x = c_1 x_1 + c_2 x_2 = e^{pt}(c_1\cos(qt) + c_2\sin(qt)) \tag{8.4.19}$$

for some constants c_1 and c_2 .

Example Consider the equation

$$\ddot{x} + 2\dot{x} + 5x = 0$$

with initial conditions $x(0) = 2$ and $\dot{x}(0) = 0$. The characteristic equation is

$$k^2 + 2k + 5 = 0,$$

which has roots

$$\frac{-2 \pm \sqrt{4 - 20}}{2} = -1 \pm 2i.$$

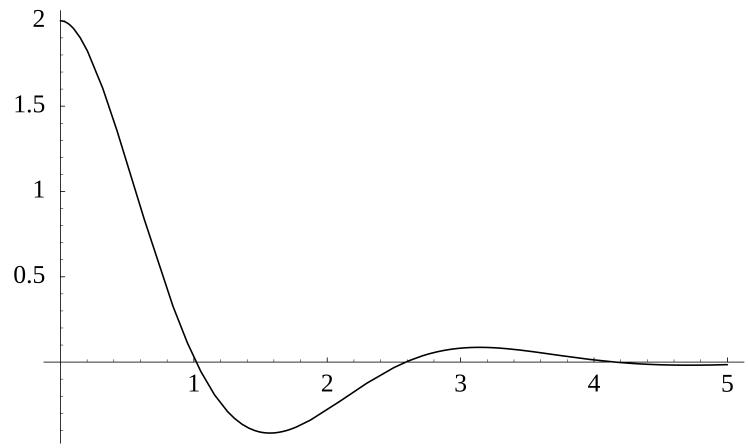


Figure 8.4.2 Solution of $\ddot{x} + 2\dot{x} + 5x = 0$ with $x(0) = 2$ and $\dot{x}(0) = 0$

Hence, by (8.4.19), we must have

$$x = e^{-t}(c_1 \cos(2t) + c_2 \sin(2t))$$

for some constants c_1 and c_2. Now

$$\dot{x} = e^{-t}(-2c_1 \sin(2t) + 2c_2 \cos(2t)) - e^{-t}(c_1 \cos(2t) + c_2 \sin(2t)),$$

so

$$x(0) = c_1$$

and

$$\dot{x}(0) = 2c_2 - c_1.$$

Hence the initial conditions $x(0) = 2$ and $\dot{x}(0) = 0$ imply that $c_1 = 2$ and

$$0 = 2c_2 - 2.$$

Thus $c_2 = 1$ and we have

$$x = e^{-t}(2\cos(2t) + \sin(2t)).$$

The graph of x is shown in Figure 8.4.2.

Case 3: Single real root

Suppose the characteristic equation has a single real root. In this case,

$$k_1 = k_2 = -\frac{b}{2}. \tag{8.4.20}$$

For simplicity, let us call this common value k. Then $x_1 = e^{kt}$ is a solution of the equation

$$\ddot{x} + b\dot{x} + cx = 0,$$

but in order to specify all possible solutions we need to find another solution which is linearly independent of x_1. We will show that, in this case, $x_2 = te^{kt}$ is such a solution. Now

$$\dot{x}_2 = kte^{kt} + e^{kt} = (1 + kt)e^{kt} \tag{8.4.21}$$

and

$$\ddot{x}_2 = (1 + kt)ke^{kt} + ke^{kt} = (2k + k^2t)e^{kt}. \tag{8.4.22}$$

Hence, remembering that k is a root of the characteristic equation (that is, $k^2 + bk + c = 0$) and $k = -\frac{b}{2}$, we have

$$\begin{aligned}
\ddot{x}_2 + b\dot{x}_2 + cx_2 &= (2k + k^2t)e^{kt} + b(1 + kt)e^{kt} + cte^{kt} \\
&= e^{kt}(2k + k^2t + b + bkt + ct) \\
&= e^{kt}((k^2 + bk + c)t + 2k + b) \\
&= e^{kt}(2k + b) \\
&= e^{kt}\Big(-\frac{2b}{2} + b\Big) \\
&= 0
\end{aligned}$$

for all t. Hence x_2 is another solution, clearly linearly independent of x_1. Thus for any solution x, there exist constants c_1 and c_2 such that

$$x = c_1x_1 + c_2x_2 = c_1e^{kt} + c_2te^{kt}. \tag{8.4.23}$$

Example Consider the equation

$$\ddot{x} + 2\dot{x} + x = 0$$

with initial conditions $x(0) = 10$ and $\dot{x}(0) = -20$. The characteristic equation is

$$0 = k^2 + 2k + 1 = (k + 1)^2$$

which has the single root $k = -1$. Hence

$$x = c_1e^{-t} + c_2te^{-t}$$

for some constants c_1 and c_2. Now

$$\dot{x} = -c_1e^{-t} - c_2te^{-t} + c_2e^{-t},$$

so $x(0) = c_1$ and $\dot{x}(0) = -c_1 + c_2$. Hence the initial conditions $x(0) = 10$ and $\dot{x}(0) = -20$ imply that $c_1 = 10$ and

$$-20 = -10 + c_2.$$

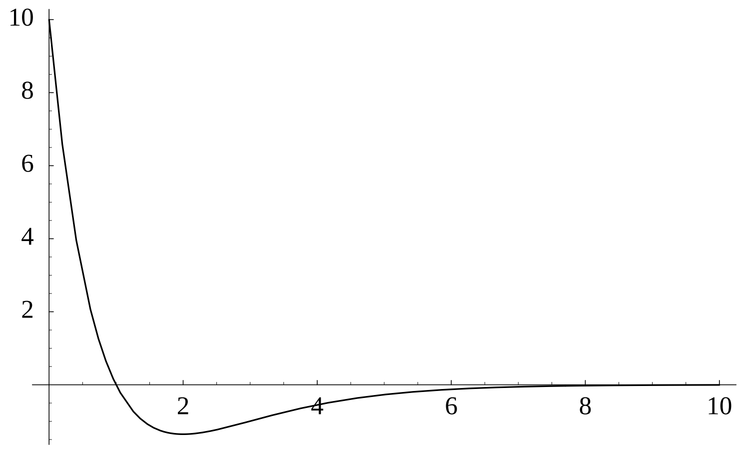


Figure 8.4.3 Solution of $\ddot{x} + 2\dot{x} + x = 0$ with $x(0) = 10$ and $\dot{x}(0) = -20$

Thus $c_2 = -10$ and we have

$$x = 10e^{-t} - 10te^{-t} = 10(1-t)e^{-t}.$$

The graph of x is shown in Figure 8.4.3.

Summary

If x_1 and x_2 are linearly independent solutions of the equation

$$\ddot{x} + b\dot{x} + cx = 0, \tag{8.4.24}$$

then any solution of (8.4.24) is of the form $x = c_1x_1 + c_2x_2$ for some constants c_1 and c_2. The family of all solutions $x = c_1x_1 + c_2x_2$ is called the *general solution* of (8.4.24). A solution with specified values for c_1 and c_2 is called a *particular solution*.

Let k_1 and k_2 be the roots of the characteristic equation

$$k^2 + bk + c = 0. \tag{8.4.25}$$

If k_1 and k_2 are real numbers with $k_1 \neq k_2$, then the general solution of (8.4.24) is

$$x = c_1e^{k_1t} + c_2e^{k_2t}. \tag{8.4.26}$$

If k_1 and k_2 are complex numbers with $k_1 = p - qi$ and $k_2 = p + qi$, then the general solution of (8.4.24) is

$$x = e^{pt}(c_1\cos(qt) + c_2\sin(qt)). \tag{8.4.27}$$

Finally, if $k = k_1 = k_2$, then the general solution of (8.4.24) is

$$x = c_1e^{kt} + c_2te^{kt}. \tag{8.4.28}$$

In the next section we will discuss the motion of a pendulum and the motion of a mass vibrating at the end of a spring as applications of the equations considered in this section.

Problems

1. Solve each of the following differential equations and plot the solution.
   (a) $\ddot{x} + \dot{x} - 2x = 0$, $x(0) = 0$, $\dot{x}(0) = 2$
   (b) $\ddot{x} = -x$, $x(0) = 10$, $\dot{x}(0) = 5$
   (c) $\ddot{x} + 3\dot{x} + 2x = 0$, $x(0) = 1$, $\dot{x}(0) = 0$
   (d) $\ddot{x} - \dot{4}x + 4x = 0$, $x(0) = 5$, $\dot{x}(0) = 0$
   (e) $\ddot{x} - 2\dot{x} + 2x = 0$, $x(0) = 10$, $\dot{x}(0) = 4$
   (f) $-\ddot{x} + 2\dot{x} - 4x = 0$, $x(0) = 1$, $\dot{x}(0) = 0$
   (g) $\ddot{x} + 4\dot{x} + 20x = 0$, $x(0) = 0$, $\dot{x}(0) = 3$
   (h) $2\ddot{x} + 3\dot{x} - 2x = 0$, $x(0) = 0$, $\dot{x}(0) = -2$
   (i) $\ddot{x} + 6\dot{x} + 9x = 0$, $x(0) = -6$, $\dot{x}(0) = 4$

2. Consider the equation $\ddot{x} + 2\dot{x} - 3x = 0$.
   (a) If $\dot{x}(0) = 1$, plot the solutions for $x(0) = 0$, $x(0) = -5$, and $x(0) = 5$. How do these solutions compare?
   (b) If $x(0) = 0$, plot the solutions for $\dot{x}(0) = 0$, $\dot{x}(0) = -2$, and $\dot{x}(0) = 2$. How do these solutions compare?

3. Consider the equation $\ddot{x} + 2\dot{x} + 10x = 0$.
   (a) If $\dot{x}(0) = 1$, plot the solutions for $x(0) = 0$, $x(0) = -10$, and $x(0) = 10$. How do these solutions compare?
   (b) If $x(0) = 10$, plot the solutions for $\dot{x}(0) = 0$, $\dot{x}(0) = -5$, and $\dot{x}(0) = 5$. How do these solutions compare?

4. Consider the equation $\ddot{x} + 4\dot{x} + 4x = 0$.
   (a) If $\dot{x}(0) = -15$, plot the solutions for $x(0) = 0$, $x(0) = -5$, and $x(0) = 5$. How do these solutions compare?
   (b) If $x(0) = 10$, plot the solutions for $\dot{x}(0) = 0$, $\dot{x}(0) = -20$, and $\dot{x}(0) = 20$. How do these solutions compare?

5. The techniques developed in this section may be used to solve higher order homogeneous linear differential equations with constant coefficients. Generalize the methods of this section to find the general solution for each of the following equations.
   (a) $\dfrac{d^3x}{dt^3} + 2\dfrac{d^2x}{dt^2} - \dfrac{dx}{dt} - 2x = 0$
   (b) $\dfrac{d^3x}{dt^3} + 3\dfrac{d^2x}{dt^2} + 3\dfrac{dx}{dt} + x = 0$

6. Show that if b and c are both positive and x is a solution of
$$\ddot{x} + b\dot{x} + cx = 0,$$
then $\lim_{t \to \infty} x(t) = 0$.

Difference Equations
to
Differential Equations

Section 8.5

Applications: Pendulums and Mass-Spring Systems

In this section we will investigate two applications of our work in Section 8.4. First, we will consider the motion of a pendulum, a problem originally mentioned in Section 2.2 in connection with the trigonometric functions. Second, we will discuss the motion of an object vibrating at the end of a spring.

The motion of a pendulum

Consider a pendulum consisting of a bob of mass m at the end of a rigid rod of length b. We will assume that the mass of the rod is negligible in comparison with the mass of the bob. Let $x(t)$ be the angle between the rod and the vertical at time t, with $x(t) > 0$ for angles measured in the counterclockwise direction and $x(t) < 0$ for angles measured in the clockwise direction. See Figure 8.5.1. Suppose the bob is pulled through an angle α and then released. That is, suppose our initial conditions are $x(0) = \alpha$ and $\dot{x}(0) = 0$. If we view the motion of the pendulum in the complex plane, with the real axis vertical, positive direction downward, and the imaginary axis horizontal, positive direction to the right, then the position of the bob at time t is given by

$$z(t) = be^{ix(t)}. \tag{8.5.1}$$

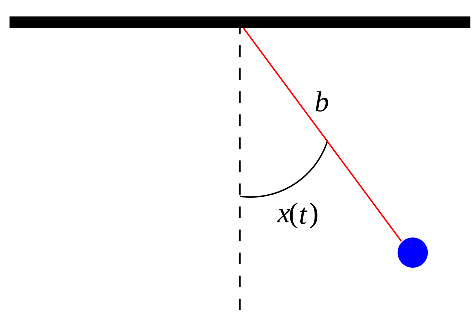


Figure 8.5.1 A pendulum

Then we have

$$\dot{z} = ib\dot{x}e^{ix} \tag{8.5.2}$$

and

$$\begin{aligned}\ddot{z} &= -b\dot{x}^2 e^{ix} + ib\ddot{x}e^{ix} \\ &= -b\dot{x}^2(\cos(x) + i\sin(x)) + ib\ddot{x}(\cos(x) + i\sin(x)) \\ &= (-b\dot{x}^2\cos(x) - b\ddot{x}\sin(x)) + i(-b\dot{x}^2\sin(x) + b\ddot{x}\cos(x)).\end{aligned} \tag{8.5.3}$$

Now $\ddot{z}$ is the acceleration of the pendulum, and so $m\ddot{z}$ must be equal to the force of gravity acting on the bob, namely, a force of magnitude mg acting in the downward direction, the direction of the positive real axis. Hence we must have $g = \ddot{z}$, that is,

$$g = (-b\dot{x}^2 \cos(x) - b\ddot{x}\sin(x)) + i(-b\dot{x}^2 \sin(x) + b\ddot{x}\cos(x)). \tag{8.5.4}$$

Equating the real and imaginary parts of the two sides of (8.5.4) gives us

$$g = -b\dot{x}^2 \cos(x) - b\ddot{x}\sin(x) \tag{8.5.5}$$

and

$$0 = -b\dot{x}^2 \sin(x) + b\ddot{x}\cos(x). \tag{8.5.6}$$

Multiplying (8.5.5) by $-\sin(x)$ and (8.5.6) by $\cos(x)$ gives us

$$-g\sin(x) = b\dot{x}^2 \cos(x)\sin(x) + b\ddot{x}\sin^2(x) \tag{8.5.7}$$

and

$$0 = -b\dot{x}^2 \sin(x)\cos(x) + b\ddot{x}\cos^2(x). \tag{8.5.8}$$

Adding (8.5.7) and (8.5.8) together yields

$$-g\sin(x) = b\ddot{x}(\sin^2(x) + \cos^2(x)) = b\ddot{x}. \tag{8.5.9}$$

Thus

$$\ddot{x} = -\frac{g}{b}\sin(x). \tag{8.5.10}$$

So we have reduced the problem of describing the motion of the pendulum to the problem of solving the second order differential equation (8.5.10) subject to the initial conditions $x(0) = \alpha$ and $\dot{x}(0) = 0$. Unfortunately, this equation is not linear. In fact, it is not possible to find a closed form solution for this equation. In Section 8.6 we will discuss how to study this equation using numerical approximations, but for now we will take a different approach to finding an approximate solution. Since we know

$$\sin(x) = x + o(x) \tag{8.5.11}$$

from our work on best affine approximations in Chapter 2, it is reasonable to replace $\sin(x)$ by x for small values of x. Hence, if we restrict to the case where α is small, we may replace (8.5.10) by the linear equation

$$\ddot{x} = -\frac{g}{b}x. \tag{8.5.12}$$

Since this equation is homogeneous with constant coefficients, we may solve it using the techniques of Section 8.4. Specifically, the characteristic equation for this equation is

$$k^2 + \frac{g}{b} = 0, \tag{8.5.13}$$

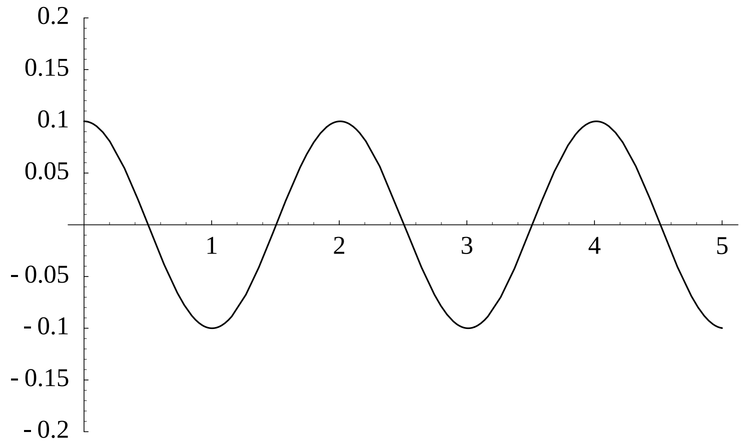


Figure 8.5.2 Motion of a pendulum

which has roots

$$k_1 = -i\sqrt{\frac{g}{b}} \qquad (8.5.14)$$

and

$$k_2 = i\sqrt{\frac{g}{g}} \qquad (8.5.15).$$

Hence the general solution is

$$x = c_1 \cos\left(\sqrt{\frac{g}{b}}t\right) + c_2 \sin\left(\sqrt{\frac{g}{b}}t\right). \qquad (8.5.16)$$

Then

$$\dot{x} = -c_1\sqrt{\frac{g}{b}} \sin\left(\sqrt{\frac{g}{b}}t\right) + c_2\sqrt{\frac{g}{b}} \cos\left(\sqrt{\frac{g}{b}}t\right), \qquad (8.5.17)$$

and so $x(0) = c_1$ and $\dot{x}(0) = c_2\sqrt{\frac{g}{b}}$. Hence the initial conditions $x(0) = \alpha$ and $\dot{x}(0) = 0$ imply $c_1 = \alpha$ and $c_2 = 0$. Thus

$$x = \alpha \cos\left(\sqrt{\frac{g}{b}}t\right) \qquad (8.5.18).$$

The graph of x for the case $b = 1$ meter and $\alpha = 0.1$ radians, in which case we use $g = 9.8$ meters per second per second, is shown in Figure 8.5.2.

One consequence of (8.5.18) is that the period of the motion, that is, the time it takes the bob to make one complete oscillation, is

$$\frac{2\pi}{\sqrt{\frac{g}{b}}} = 2\pi\sqrt{\frac{b}{g}}, \qquad (8.5.19)$$

independent of the value of α. Of course, we are working under the approximation $\sin(x) \approx x$, so (8.5.19) is actually only an approximation of the period. Nevertheless, the approximation is very good for small oscillations and is the reason pendulums were used to measure time in early clocks.

Vibrations in mechanical systems: mass-spring systems

In this example we consider the motion of an object of mass m suspended on a spring, as shown in Figure 8.5.3. We will measure the position of the object along a vertical axis, with the equilibrium position at 0 and the positive direction downward. Let $x(t)$ denote the position of the object at time t and suppose the object is released from rest at position x_0. That is, we suppose that $x(0) = x_0$ and $\dot{x}(0) = 0$. If we ignore any damping forces, such as resistance to the motion due to the surrounding medium, such as air or oil, then the only forces acting on the object are the force of gravity, contributing a term of mg, and the restorative force of the spring, given, according to Hooke's law, by $k\ell$ for some constant $k > 0$, where ℓ is the amount the spring is stretched or compressed from its natural length. If we let $\Delta\ell$ be the amount the spring is stretched when the object is at the equilibrium position, that is, when $x = 0$, then at any time the spring is stretched or compressed by $x + \Delta\ell$. Thus at any time t the force acting on the object is

$$F = mg - k(x + \Delta\ell). \tag{8.5.20}$$

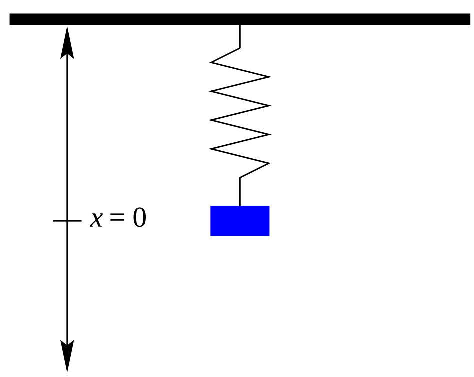


Figure 8.5.3 Mass on a spring at equilibrium

In particular, if the object is at rest at its equilibrium position, then both $x = 0$ and $F = 0$. Hence

$$0 = mg - k\Delta\ell, \tag{8.5.21}$$

and so

$$mg = k\Delta\ell. \tag{8.5.22}$$

Thus (8.5.20) simplifies to $F = -kx$. Applying Newton's second law of motion, we have

$$m\ddot{x} = -kx, \tag{8.5.23}$$

from which we obtain

$$\ddot{x} = -\frac{k}{m}x. \tag{8.5.24}$$

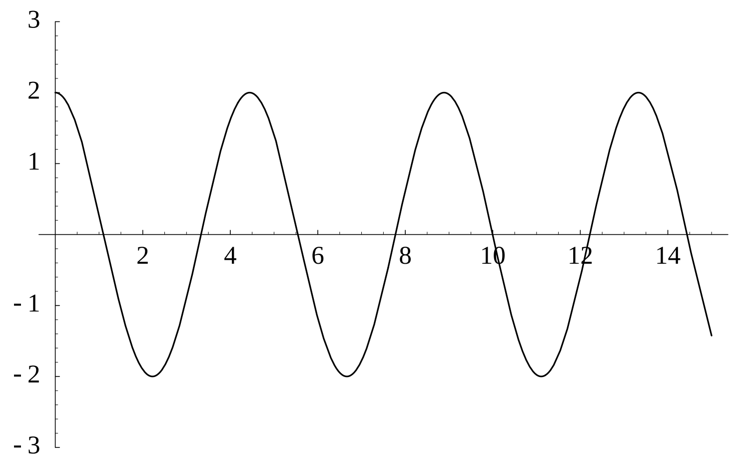


Figure 8.5.4 Motion of a mass-spring system without damping

This equation is of the same form as the equation derived above for approximating the motion of a pendulum. Hence, using the same reasoning, the solution is

$$x = x_0 \cos\left(\sqrt{\frac{k}{m}}t\right). \tag{8.5.25}$$

The graph of x for $k = 10$, $m = 5$, and $x_0 = 2$ is shown in Figure 8.5.4.

Notice that the period of the motion is

$$T = \frac{2\pi}{\sqrt{\frac{k}{m}}} = 2\pi\sqrt{\frac{m}{k}}. \tag{8.5.26}$$

The frequency of the motion, that is, the number of complete oscillations in one unit of time, is

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}. \tag{8.5.27}$$

Hence for a fixed mass, increasing the spring constant, that is, increasing the stiffness of the spring, decreases the period and increases the frequency; for a fixed spring constant, increasing the mass increases the period and decreases the frequency.

Now suppose there is a damping force, a force resisting the motion of the object, which is proportional to the velocity. This adds an additional term of $-c\dot{x}$, where c is a positive constant, to the force acting on the object, giving us $F = -kx - c\dot{x}$. Thus

$$m\ddot{x} = -kx - c\dot{x}, \tag{8.5.28}$$

and so

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0 \tag{8.5.29}$$

replaces (8.5.24) as the equation describing the motion of the object. To simplify the notation, we will let

$$b = \frac{c}{2m}$$

and

$$a = \sqrt{\frac{k}{m}}.$$

Then our differential equation becomes

$$\ddot{x} + 2b\dot{x} + a^2 x = 0, \tag{8.5.30}$$

with characteristic equation (using s for the variable)

$$s^2 + 2bs + a^2 = 0. \tag{8.5.31}$$

Hence the roots of the characteristic equation are

$$s_1 = \frac{-2b - \sqrt{4b^2 - 4a^2}}{2} = -b - \sqrt{b^2 - a^2} \tag{8.5.32}$$

and

$$s_2 = \frac{-2b + \sqrt{4b^2 - 4a^2}}{2} = -b + \sqrt{b^2 - a^2}. \tag{8.5.33}$$

Thus the behavior of the system depends on whether $b^2 - a^2 > 0$, $b^2 - a^2 = 0$, or $b^2 - a^2 < 0$. Equivalently, since

$$b^2 - a^2 = \frac{c^2}{4m^2} - \frac{k}{m},$$

the behavior of the system depends on whether $c^2 > 4mk$, $c^2 = 4mk$, or $c^2 < 4mk$. In the first case the system is said to be *overdamped*, in the second it is *critically damped*, and in the third it is *underdamped*.

First consider the overdamped case $b^2 - a^2 > 0$. In this case the characteristic equation has distinct real roots, so the general solution is

$$x = c_1 e^{s_1 t} + c_2 e^{s_2 t}. \tag{8.5.34}$$

Now

$$\dot{x} = c_1 s_1 e^{s_1 t} + c_2 s_2 e^{s_2 t}, \tag{8.5.35}$$

so $x(0) = c_1 + c_2$ and $\dot{x}(0) = c_1 s_1 + c_2 s_2$. Hence the initial conditions, $x(0) = x_0$ and $\dot{x}(0) = 0$, give us

$$x_0 = c_1 + c_2$$

and

$$0 = c_1 s_1 + c_2 s_2.$$

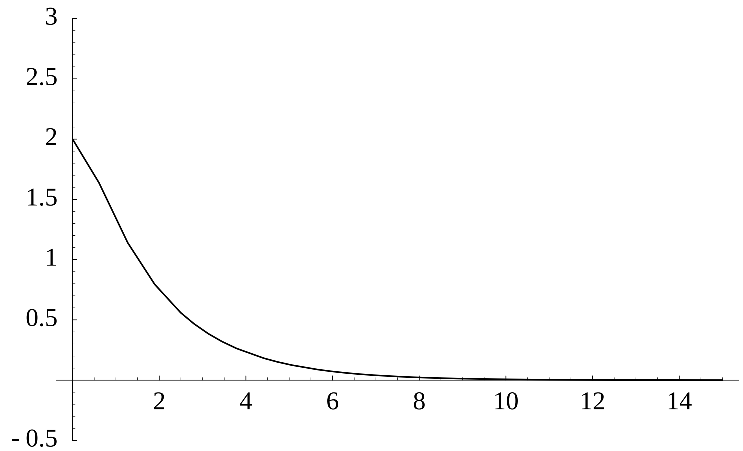


Figure 8.5.5 Motion of an overdamped mass-spring system

Multiplying the first equation by s_1 and subtracting from the second gives us

$$-x_0 s_1 = c_2(s_2 - s_1).$$

Hence

$$c_2 = -\frac{x_0 s_1}{s_2 - s_1}$$

and

$$c_1 = x_0 - c_2 = \frac{x_0(s_2 - s_1)}{s_2 - s_1} + \frac{x_0 s_1}{s_2 - s_1} = \frac{x_0 s_2}{s_2 - s_1}.$$

Thus

$$x = \frac{x_0}{s_2 - s_1}(s_2 e^{s_1 t} - s_1 e^{s_2 t}). \tag{8.5.36}$$

Now $b > 0$ and $b > \sqrt{b^2 - a^2}$, so

$$s_2 = -b + \sqrt{b^2 - a^2} < 0.$$

Hence

$$s_1 < s_2 < 0. \tag{8.5.37}$$

It follows that $e^{s_2 t} > e^{s_1 t}$, $s_2 - s_1 > 0$, and

$$s_2 e^{s_1 t} - s_1 e^{s_2 t} > s_2 e^{s_2 t} - s_1 e^{s_2 t} = e^{s_2 t}(s_2 - s_1) > 0$$

for all $t \geq 0$. Hence if $x_0 < 0$, then $x(t) < 0$ for all $t \geq 0$, and if $x_0 > 0$, then $x(t) > 0$ for all $t > 0$. Combining this with

$$\lim_{t \to \infty} x(t) = 0, \tag{8.5.38}$$

we see that in this case the system does not oscillate at all. After release, the object simply returns to the equilibrium position. Figure 8.5.5 shows this behavior for $k = 10$, $m = 5$, $c = 20$, and $x_0 = 2$.

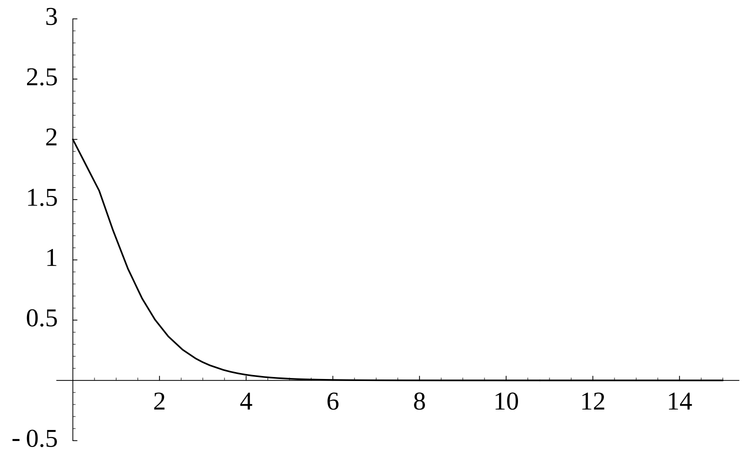


Figure 8.5.6 Motion of a critically damped mass-spring system

Next consider the case when $b^2 - a^2 = 0$. In this case the characteristic equation has only one real root, $s_1 = s_2 = -b$, so the general solution is

$$x = c_1 e^{-bt} + c_2 t e^{-bt}. \tag{8.5.39}$$

Then

$$\dot{x} = -bc_1 e^{-bt} - bc_2 t e^{-bt} + c_2 e^{-bt}, \tag{8.5.40}$$

so $x(0) = c_1$ and $\dot{x}(0) = -bc_1 + c_2$. Hence the initial conditions, $x(0) = x_0$ and $\dot{x}(0) = 0$, give us $c_1 = x0$ and $c_2 = bx_0$. Thus

$$x = x_0 e^{-bt} + bx_0 e^{-bt} = x_0 e^{-bt}(1 + bt). \tag{8.5.41}$$

Equivalently, since $b = \frac{c}{2m}$,

$$x = x_0 e^{-\frac{c}{2m}t}\left(1 + \frac{c}{2m}t\right). \tag{8.5.42}$$

Now for any $t \geq 0$,

$$1 + \frac{c}{2m}t > 0.$$

Hence, as in the overdamped case, the system does not oscillate. Once released, the object moves back to the equilibrium position without ever crossing it. Figure 8.5.6 shows this behavior for $k = 10$, $m = 5$, $c = 10\sqrt{2}$, and $x_0 = 2$. This motion is said to be critically damped because any increase in c results in overdamped motion, while any decrease in c results in underdamped motion, which we consider next.

Finally, consider the case when $b^2 - a^2 < 0$. The roots of the characteristic equation are now

$$s_1 = -b - \sqrt{b^2 - a^2} = -b - i\sqrt{a^2 - b^2} \tag{8.5.43}$$

and

$$s_2 = -b + \sqrt{b^2 - a^2} = -b + i\sqrt{a^2 - b^2} \tag{8.5.44}$$

If we let $\alpha = \sqrt{a^2 - b^2}$, then the general solution is

$$x = e^{-bt}(c_1 \cos(\alpha t) + c_2 \sin(\alpha t)). \tag{8.5.45}$$

Then

$$\dot{x} = e^{-bt}(-\alpha c_1 \sin(\alpha t) + \alpha c_2 \cos(\alpha t)) - be^{-bt}(c_1 \cos(\alpha t) + c_2 \sin(\alpha t)), \tag{8.5.46}$$

so $x(0) = c_1$ and $\dot{x}(0) = \alpha c_2 - bc_1$. Hence the initial conditions, $x(0) = x_0$ and $\dot{x}(0) = 0$, imply that $c_1 = x_0$ and

$$c_2 = \frac{bx_0}{\alpha}.$$

Thus

$$x = e^{-bt}\left(x_0 \cos(\alpha t) + \frac{bx_0}{\alpha}\sin(\alpha t)\right) = \frac{x_0}{\alpha}e^{-bt}(\alpha \cos(\alpha t) + b \sin(\alpha t)). \tag{8.5.47}$$

This expression simplifies somewhat if we introduce the angle

$$\theta = \tan^{-1}\left(\frac{b}{\alpha}\right). \tag{8.5.48}$$

Then

$$\cos(\theta) = \frac{\alpha}{\sqrt{\alpha^2 + b^2}}$$

and

$$\sin(\theta) = \frac{b}{\sqrt{\alpha^2 + b^2}}.$$

Moreover, since $\alpha = \sqrt{a^2 - b^2}$,

$$\sqrt{\alpha^2 + b^2} = \sqrt{(a^2 - b^2) + b^2} = a = \sqrt{\frac{k}{m}}.$$

Hence

$$\begin{aligned} x &= \frac{x_0\sqrt{\alpha^2 + b^2}}{\alpha}e^{-bt}\left(\frac{\alpha}{\sqrt{\alpha^2 + b^2}}\cos(\alpha t) + \frac{b}{\sqrt{\alpha^2 + b^2}}\sin(\alpha t)\right) \\ &= \frac{x_0}{\alpha}\sqrt{\frac{k}{m}}(\cos(\theta)\cos(\alpha t) + \sin(\theta)\sin(\alpha t)). \end{aligned}$$

Using the angle subtraction formula for cosine, this becomes

$$x = \frac{x_0}{\alpha}\sqrt{\frac{k}{m}}e^{-bt}\cos(\alpha t - \theta). \tag{8.5.49}$$

The presence of the cosine factor in this expression shows us that, even though we still have

$$\lim_{t \to \infty} x(t) = 0,$$

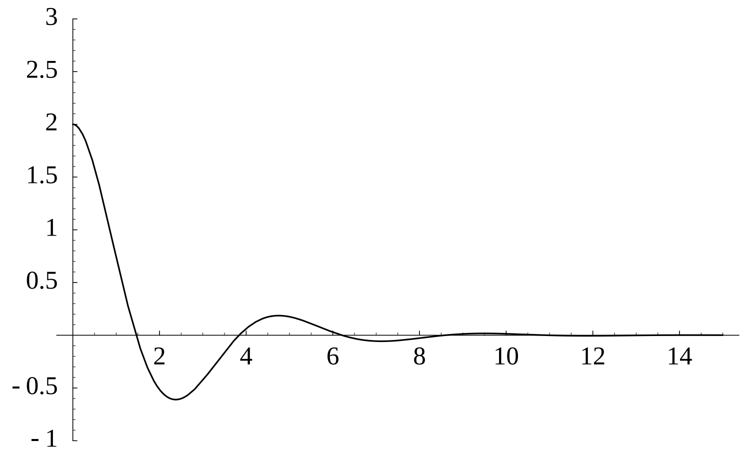


Figure 8.5.7 Motion of an underdamped mass-spring system

the underdamped mass-spring system will oscillate about the equilibrium position with a decreasing amplitude of

$$\frac{x_0}{\alpha}\sqrt{\frac{k}{m}}e^{-bt}. \tag{8.5.50}$$

Figure 8.5.7 shows this behavior for $k = 10$, $m = 5$, $c = 5$, and $x_0 = 2$.

Problems

1. In an experiment to determine g, a pendulum of length 50 centimeters is observed to have a period of oscillation of 1.42 seconds. Approximate g based on this observation.
2. The period of oscillation of a pendulum of length b given in (8.5.19) is, as mentioned, only an approximation of the true period. It can be shown that the true period of a pendulum released from an angle α is given by

$$T = 4\sqrt{\frac{b}{g}}\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2\sin^2(\phi)}}\, d\phi,$$

where $0 < \alpha < \pi$ and $k = \sin\left(\frac{\alpha}{2}\right)$.

(a) Find the period of oscillation for a pendulum of length 50 centimeters for $\alpha = \frac{\pi}{4}$, $\alpha = \frac{\pi}{6}$, $\alpha = \frac{\pi}{50}$, and $\alpha = \frac{\pi}{100}$. Compare these results with the approximation given in (8.5.19).

(b) Graph T as a function of α for $-\frac{\pi}{4} \leq \alpha \leq \frac{\pi}{4}$. For comparison, also plot the horizontal line

$$T = 2\pi\sqrt{\frac{b}{g}}.$$

3. Consider a mass-spring system with $x_0 = 10$, $\dot{x}(0) = 0$, $k = 10$, and $m = 10$. Plot $x(t)$ for $c = 0$, $c = 5$, $c = 10$, $c = 20$, $c = 25$, and $c = 30$. Identify each motion as overdamped, critically damped, underdamped, or undamped.

4. Consider a mass-spring system with $x_0 = 10$, $\dot{x}(0) = 0$, $m = 10$, and $c = 20$. Plot $x(t)$ for $k = 2$, $k = 5$, $k = 10$, and $k = 15$. Identify each motion as overdamped, critically damped, underdamped, or undamped.

5. Consider the underdamped motion of a mass-spring system expressed in (8.5.46).

   (a) Show that the maximum values of $x(t)$ occur at $t = 0, T, 2T, \ldots$, where

   $$T = \frac{2\pi}{\sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}}.$$

   Note that when $c = 0$, T reduces to the period of the motion for the mass-spring system without damping.

   (b) Show that if x_1 and x_2 are two successive maximum values of $x(t)$, then

   $$\frac{x_1}{x_2} = e^{\frac{cT}{2m}}.$$

6. Inside the earth, the force of gravity acting on an object is proportional to the distance between the object and the center of the earth.

   (a) Suppose a hole is drilled through the earth from pole to pole and a rock is dropped into the hole. If $x(t)$ is the distance from the object to the center of the earth at time t, show that, ignoring any resistive forces,

   $$x = R\cos\left(\sqrt{\frac{g}{R}}t\right),$$

   where R is the radius of the earth.

   (b) How long, in minutes, does it take for the rock to make one complete trip from pole to pole and back? Use $R = 3950$ miles.

   (c) What is the velocity of the rock, in miles per hour, when it reaches the center of the earth?

Difference Equations to **D**ifferential Equations

Section 8.6

The Geometry of Solutions: The Phase Plane

As mentioned in Section 8.4, we may represent a second order differential equation

$$\ddot{x} = f(x, \dot{x}, t) \tag{8.6.1}$$

as a system of two first order equations

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= f(x, y, t). \end{aligned} \tag{8.6.2}$$

More generally, if g and f are functions of x, y, and t, we may consider a system of equations

$$\begin{aligned} \dot{x} &= g(x, y, t) \\ \dot{y} &= f(x, y, t), \end{aligned} \tag{8.6.3}$$

of which (8.6.2) is a particular case when $g(x, y, t) = y$. In this section we shall consider the behavior of solutions to such systems of equations, paying particular attention to those arising in the manner of (8.6.2).

Definition Suppose $x(t)$ and $y(t)$ are solutions of the system

$$\begin{aligned} \dot{x} &= g(x, y, t) \\ \dot{y} &= f(x, y, t) \end{aligned}$$

for t in an interval $[a, b]$. The curve in the plane with coordinates $(x(t), y(t))$, $a \leq t \leq b$, is called a *phase curve* of the system. The plane in which the phase curve is plotted is called the *phase plane* of the system.

Note that if the system of equations arises from a second order differential equation, then a phase curve is a plot of $\dot{x}(t)$ versus $x(t)$. In many common cases, this is a plot of velocity versus position.

Definition Suppose the constant functions $x(t) = x_0$ and $y(t) = y_0$ is a solution of the system

$$\begin{aligned} \dot{x} &= g(x, y, t) \\ \dot{y} &= f(x, y, t) \end{aligned}.$$

Then the point (x_0, y_0) is called a *stationary point* of the system.

If (x_0, y_0) is a stationary point, then the phase curve of the solution

$$\begin{aligned} x(t) &= x_0 \\ y(t) &= y_0 \end{aligned}$$

consists of only the single point (x_0, y_0). That is, if (x_0, y_0) is a stationary point and the system has initial conditions $x(t_0) = x_0$ and $y(t_0) = y_0$, then the system will remain at the point (x_0, y_0) for all time. Moreover, note that for this solution

$$\dot{x}(t) = 0$$

and

$$\dot{y}(t) = 0$$

for all t. Since we must have

$$\begin{aligned} \dot{x} &= g(x, y, t) \\ \dot{y} &= f(x, y, t), \end{aligned}$$

it follows that stationary points are precisely the points (x_0, y_0) for which

$$g(x_0, y_0, t) = 0$$

and

$$f(x_0, y_0, t) = 0$$

for all t.

Example Consider the second order linear equation

$$\ddot{x} + \frac{k}{m}x = 0,$$

where k and m are positive constants. In Section 8.5 we saw how this equation models an undamped mass-spring system consisting of an object of mass m oscillating at the end of a spring with spring constant k. This equation is equivalent to the system of equations

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\frac{k}{m}x. \end{aligned} \tag{8.6.4}$$

Clearly, the only stationary point of this system is $(0, 0)$, corresponding to the object being at rest at the equilibrium position of the system. With initial conditions $x(0) = x_0$ and $y(0) = 0$, this system has solution

$$\begin{aligned} x &= x_0 \cos\left(\sqrt{\frac{k}{m}}t\right) \\ y &= -x_0\sqrt{\frac{k}{m}} \sin\left(\sqrt{\frac{k}{m}}\right). \end{aligned}$$

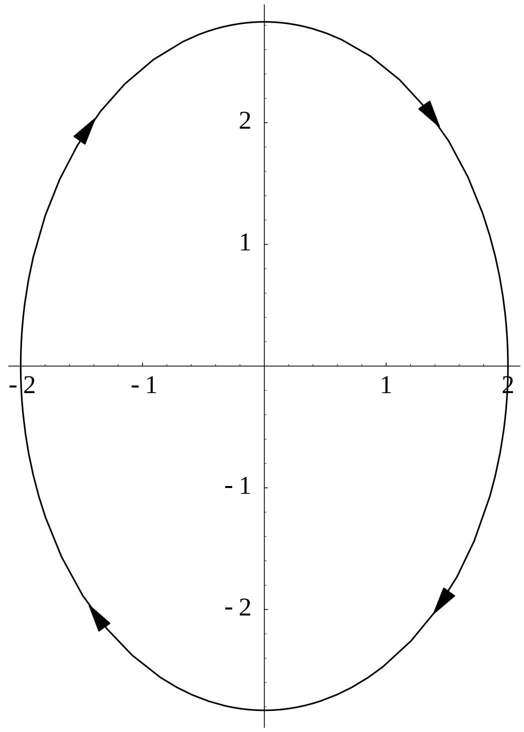


Figure 8.6.1 A phase curve for the system $\dot{x} = y$, $\dot{y} = -2x$

A plot of the phase curve for this solution is shown in Figure 8.6.1 for $k = 10$, $m = 5$, and $x_0 = 2$ for $0 \leq t \leq \sqrt{2}\pi$ (that is, for one full period of the motion). You should compare this plot with the graph of x in Figure 8.5.4. The arrows on the curve point in the direction of increasing t. At $t = 0$, the mass is released from a point 2 units below the equilibrium position, hence $x = 2$ and $y = 0$; as t increases from 0 to $\frac{\sqrt{2}\pi}{4}$, x decreases from 2 to 0 as the mass moves upward to the equilibrium position while y, the velocity, decreases from 0 to $-2\sqrt{2}$; as t increases from $\frac{\sqrt{2}\pi}{4}$ to $\frac{\sqrt{2}\pi}{2}$, x continues to decrease from 0 to -2 as the mass moves to its highest point, at which point its velocity is $y = 0$; at this time, the velocity becomes positive and the mass moves from -2, through the equilibrium position, back to 2, at which point the velocity is again 0 and the motion begins all over again. Notice that the phase curve is a closed curve because the motion is periodic: after a period of $\sqrt{2}\pi$ units of time, both the position and the velocity of the object have returned to their original values. Moreover, the stationary point $(0, 0)$ is at the center of this phase curve. In fact, all the phase curves for this equation are closed curves about the stationary point. Such a stationary point is called a *center*.

Note that in this example the phase curves are all ellipses. The curve in Figure 8.6.1 satisfies

$$\frac{x^2}{4} + \frac{y^2}{8} = 1. \tag{8.6.5}$$

Example Consider the second order linear equation

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0,$$

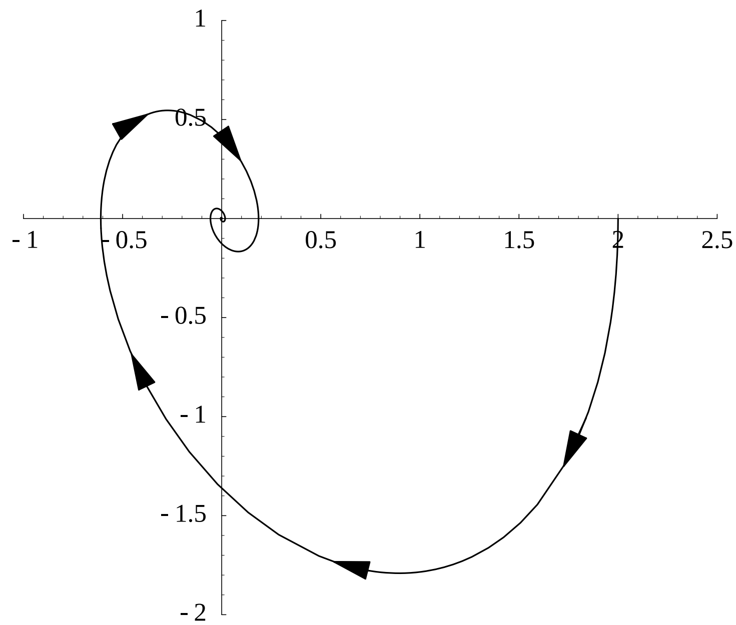


Figure 8.6.2 A phase curve for the system $\dot{x} = y$, $\dot{y} = -y - 2x$

where c, k, and m are positive constants. We saw in Section 8.5 that this equation models the motion of a damped mass-spring system consisting of an object of mass m attached to a spring with spring constant k and moving through a medium offering a resistive force proportional to $\dot{x}$. This second order equation is equivalent to the system

$$\begin{aligned} \dot{x} &= y \\ y &= -\frac{c}{m}y - \frac{k}{m}x. \end{aligned} \tag{8.6.6}$$

As in the previous example, the only stationary point of this system is $(0, 0)$. We will consider an example of the underdamped case, namely, $k = 10$, $m = 5$, and $c = 5$. In that case, with initial conditions $x(0) = x_0$ and $y(0) = 0$, the solution of (8.6.6) is

$$\begin{aligned} x &= 2\sqrt{\frac{2}{7}}x_0 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{7}}{2}t - \theta\right) \\ y &= -\sqrt{2}x_0 e^{-\frac{t}{2}} \left(\sin\left(\frac{\sqrt{7}}{2}t - \theta\right) + \frac{1}{\sqrt{7}}\cos\left(\frac{\sqrt{7}}{2}t - \theta\right)\right), \end{aligned}$$

where

$$\theta = \tan^{-1}\left(\frac{1}{\sqrt{7}}\right).$$

A plot of the phase curve for this solution is shown in Figure 8.6.2 for $x_0 = 2$ with $0 \leq t \leq 20$. You should compare this plot with the graph of x in Figure 8.5.7. Here we see that the phase curve is not closed and the motion is not periodic; as t increases, the curve spirals in toward the stationary point $(0, 0)$. This is in fact the general behavior of phase curves for this system: No matter what the initial condition, as t increases, both the position and velocity functions decay toward 0 as the mass performs smaller and smaller oscillations about the equilibrium. In this case we call the stationary point a *stable equilibrium.* In general, a stationary point (x_0, y_0) is a stable equilibrium if for any initial conditions sufficiently close to (x_0, y_0), the resulting phase curve approaches (x_0, y_0) in the limit as $t \to \infty$. In this example, every phase curve approaches $(0, 0)$ as $t \to \infty$.

A stationary point (x_0, y_0) is called an *unstable equilibrium* if there is a fixed distance d such that it is possible to find initial conditions arbitrarily close to (x_0, y_0) for which the resulting phase curve will eventually be farther than d away from (x_0, y_0).

Example Taking $k = 10$, $m = 5$, and $c = -5$ in the system (8.6.6) would lead to the solution

$$\begin{aligned} x &= 2\sqrt{\frac{2}{7}}x_0 e^{\frac{t}{2}} \cos\left(\frac{\sqrt{7}}{2}t - \theta\right) \\ y &= \sqrt{2}x_0 e^{\frac{t}{2}} \left(\frac{1}{\sqrt{7}} \cos\left(\frac{\sqrt{7}}{2}t - \theta\right) - \sin\left(\frac{\sqrt{7}}{2}t - \theta\right)\right), \end{aligned}$$

where

$$\theta = \tan^{-1}\left(-\frac{1}{\sqrt{7}}\right).$$

In this case, the stationary point $(0, 0)$ is an unstable equilibrium because, since $e^{\frac{t}{2}}$ increases with t, the phase curves spiral away from the stationary point $(0, 0)$. A plot of the phase curve for this solution is shown in Figure 8.6.3 for $x_0 = 0.01$ with $0 \leq t \leq 10$.

We will see another example of an unstable equilibrium when we return to the pendulum example below.

Numerical approximations

The ideas developed above are most helpful when exact solutions are not available and we must rely upon numerical approximations to understand the behavior of our solutions. However, before we can consider such examples, we must first modify our numerical techniques from Section 8.1 to the current situation. Since the second order Runge-Kutta method is more accurate than Euler's method, we will discuss only the modification of the former.

Suppose we wish to approximate the solution of the system

$$\begin{aligned} \dot{x} &= g(x, y, t) \\ \dot{y} &= f(x, y, t), \end{aligned} \tag{8.6.7}$$

with initial conditions $x(t_0) = x_0$ and $y(t_0) = y_0$, at time $t_0 + h$. First we approximate x at $t_0 + \frac{h}{2}$ by $x_0 + m_1$, where

$$m_1 = \frac{h}{2}\dot{x}(t_0) = \frac{h}{2}g(x_0, y_0, t_0), \tag{8.6.8}$$

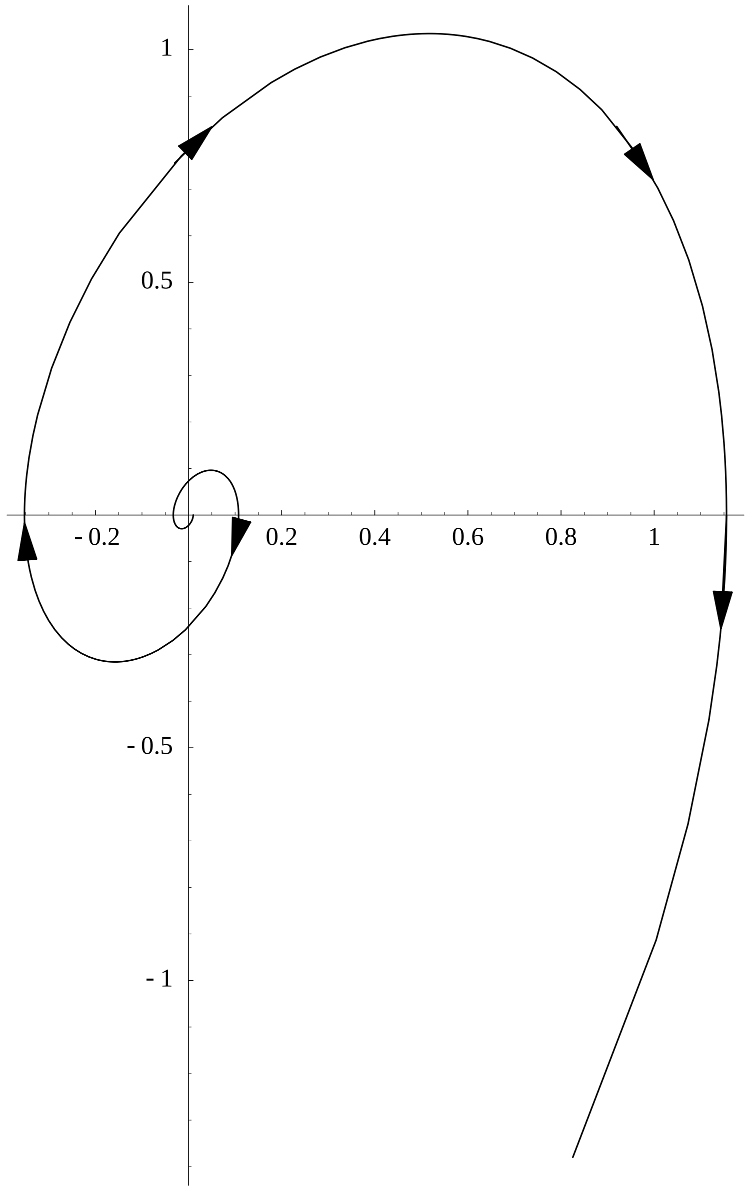


Figure 8.6.3 A phase curve for the system $\dot{x} = y$, $\dot{y} = y - 2x$

and y at $t_0 + \frac{h}{2}$ by $y_0 + m_2$, where

$$m_2 = \frac{h}{2}\dot{y}(t_0) = \frac{h}{2}f(x_0, y_0, t_0). \tag{8.6.9}$$

Then the second order Runge-Kutta approximations to $x(t_0 + h)$ and $y(t_0 + h)$ are given by

$$x_1 = x_0 + hg\Big(x_0 + m_1, y_0 + m_2, t_0 + \frac{h}{2}\Big) \tag{8.6.10}$$

and

$$y_1 = y_0 + hf\Big(x_0 + m_1, y_0 + m_2, t_0 + \frac{h}{2}\Big), \tag{8.6.11}$$

respectively. As before, to approximate the solution over an interval $[t_0, t_1]$, we iterate the above process as many times as necessary.

Second order Runge-Kutta To approximate the solution of the system of equations

$$\begin{aligned} \dot{x} &= g(x, y, t) \\ \dot{y} &= f(x, y, t) \end{aligned}$$

with initial conditions $x(t_0) = x_0$ and $y(t_0) = y_0$ on an interval $[t_0, t_1]$, choose a small value for $h > 0$ and an integer n such that $t_0 + nh \geq t_1$. Letting $s_k = t_0 + kh$, compute

$$\begin{aligned} m_1 &= \frac{h}{2} g(x_k, y_k, s_k) \\ m_2 &= \frac{h}{2} f(x_k, y_k, s_k) \end{aligned} \tag{8.6.12}$$

and

$$\begin{aligned} x_{k+1} &= x_k + hg\left(x_k + m_1, y_k + m_2, s_k + \frac{h}{2}\right) \\ y_{k+1} &= y_k + hf\left(x_k + m_1, y_k + m_2, s_k + \frac{h}{2}\right) \end{aligned} \tag{8.6.13}$$

for $k = 0, 1, 2, \ldots, n-1$. Then x_k is an approximation for $x(t_0 + kh)$ and y_k is an approximation for $y(t_0 + kh)$.

Example In this example we consider a simple case for modeling a predator-prey environment. Suppose animals of species A prey on animals of species B. For our example, species A will be foxes and species B will be rabbits, although they could be any two species that have the predator-prey relationship we are about to describe. We assume that the food supply of the rabbits is essentially unlimited and the foxes are their only natural enemy in the given environment, while, on the other hand, we assume the foxes are dependent upon the rabbits for the bulk of their food supply. We also assume that the foxes have no natural enemies. Let $y(t)$ be the size of the fox population and let $x(t)$ be the size of the rabbit population at time t. If there were no foxes, the rabbits would enjoy uninhibited growth and we would have

$$\dot{x} = \alpha x$$

for some constant $\alpha > 0$ representing the natural growth rate of rabbits in the given environment. However, if we assume that the number of encounters between rabbits and foxes is proportional to the product of the two populations and, furthermore, that a certain proportion of these encounters results in a rabbit becoming a meal for a fox, then $\dot{x}$ will be decreased by an amount βxy for some constant $\beta > 0$. Hence we have

$$\dot{x} = \alpha x - \beta xy = x(\alpha - \beta y).$$

At the same time, if there were no rabbits, the fox population would decline for want of food, that is, we would expect

$$\dot{y} = -\gamma y$$

for some constant $\gamma > 0$, while, if there are rabbits, encounters between rabbits and foxes contributes positively to the growth of the fox population. Thus y, the size of the fox population, should make a negative contribution to $\dot{y}$ and xy should make a positive contribution to $\dot{y}$. This leads us to suppose

$$\dot{y} = -\gamma y + \delta xy = -y(\gamma - \delta x)$$

for some constants $\gamma > 0$ and $\delta > 0$. Hence we now have a system of first order equations

$$\begin{aligned} \dot{x} &= x(\alpha - \beta y) \\ \dot{y} &= -y(\gamma - \delta x), \end{aligned} \tag{8.6.14}$$

where α, β, γ, and δ are all positive constants.

The stationary points for this system are solutions of

$$\begin{aligned} 0 &= x(\alpha - \beta y) \\ 0 &= -y(\gamma - \delta x). \end{aligned}$$

Clearly, $x = 0$, $y = 0$ is one solution. If $x \neq 0$, then, from the first equation, we must have

$$0 = \alpha - \beta y,$$

and so

$$y = \frac{\alpha}{\beta}.$$

Thus $y \neq 0$, so, from the second equation, we must have

$$0 = \gamma - \delta x,$$

from which we find

$$x = \frac{\gamma}{\delta}.$$

Hence the system (8.6.14) has two stationary points: $(0,0)$ and $(\frac{\gamma}{\delta}, \frac{\alpha}{\beta})$. The first corresponds to the uninteresting situation when there are no foxes and no rabbits; the second to an equilibrium condition in which the populations are in balance.

For example, consider the case with parameters $\alpha = 0.06$, $\beta = 0.0008$, $\gamma = 0.2$, and $\delta = 0.0008$, corresponding, in part, to a natural growth rate of 6% per year for the rabbits and a decay rate, in the absence of any rabbits, of 20% per year for the foxes. The system (8.6.14) then becomes

$$\begin{aligned} \dot{x} &= x(0.06 - 0.0008y) \\ \dot{y} &= -y(0.2 - 0.0008), \end{aligned} \tag{8.6.15}$$

with nonzero stationary point

$$\left(\frac{0.2}{0.0008}, \frac{0.06}{0.0008}\right) = (250, 75).$$

Hence a population of 250 rabbits and 75 foxes would be in equilibrium and would not change over time; the natural yearly increase in the rabbit population is accounted for exactly by the appetite of the foxes. To see what happens in other cases, suppose we start with an initial population of $x_0 = 400$ rabbits and $y_0 = 50$ foxes. We will approximate the solution to (8.6.15) over the interval $[0, 150]$ using the second order Runge-Kutta method with a step size of $h = 0.05$. To start, using

$$\begin{aligned} g(x, y, t) &= x(0.06 - 0.0008y) \\ f(x, y, t) &= -y(0.2 - 0.0008x), \end{aligned}$$

we compute

$$\begin{aligned} m_1 &= \frac{h}{2} g(x_0, y_0, t_0) = (0.025)(400)(0.06 - (0.0008)(50)) = 0.2 \\ m_2 &= \frac{h}{2} f(x_0, y_0, t_0) = (0.025)(50)(0.2 - (0.0008)(400)) = 0.15 \end{aligned}$$

and

$$\begin{aligned} x_1 &= x_0 + hg\Big(x_0 + m_1, y_0 + m_2, t_0 + \frac{h}{2}\Big) \\ &= 400 + (0.05)g(400.2, 50.15, 0.025) \\ &= 400 + (0.05)(400.2)(0.06 - (0.0008)(50.15)) \\ &= 400.3978 \\ y_1 &= y_0 + hf\Big(x_0 + m_1, y_0 + m_2, t_0 + \frac{h}{2}\Big) \\ &= 50 + (0.05)f(400.2, 50.15, 0.025) \\ &= 50 - (0.05)(50.15)(0.2 - (0.0008)(400.2)) \\ &= 50.3013, \end{aligned}$$

where we have rounded x_1 and y_1 to four decimal places. Then x_1 is an approximation for $x(0.05)$ and y_1 is an approximation for $y(0.05)$. In general, x_{20t} and y_{20t} are approximations for $x(t)$ and $y(t)$ when $20t$ is an integer. Table 8.6.1 gives our results for $t = 0, 5, 10, \ldots, 150$, where we have rounded the values to the nearest integer.

Notice the cyclic nature of both x and y. In the early years, the population of foxes increases due to the plentiful supply of rabbits for food. However, eventually (sometime between 15 and 20 years) the increasing fox population causes a decrease in the rabbit population to the point where the population of foxes begins to decline. As the fox population declines, there comes a point (between 30 and 35 years) when the rabbit population begins to increase, which in turn eventually leads to an increase in the fox population, starting sometime between 55 and 60 years. At this point, the cycle begins again. This behavior is most evident in Figure 8.6.4, where the numerical solutions for x and y have been plotted over the interval $[0, 150]$. Notice how the periods of the two curves are the same, but their phases are different. This phase difference occurs because, for example, a decrease in the rabbit population does not lead to an immediate decrease in the fox population; in fact, the fox population will continue to grow until the rabbit population

t	x_{20t}	y_{20t}
0	400	50
5	408	95
10	332	158
15	225	175
20	161	137
25	138	91
30	139	58
35	156	38
40	185	28
45	226	23
50	278	23
55	339	29
60	395	47
65	411	89
70	343	152
75	235	177
80	165	142
85	139	95
90	138	61
95	153	40
100	181	28
105	221	23
110	272	23
115	333	28
120	390	44
125	413	83
130	354	145
135	245	177
140	170	147
145	141	100
150	138	64

Table 8.6.1 Predator-prey poplulations

is too small to support its growth, and it is at that point that the fox population begins to decline. The phase curve for this solution is shown in Figure 8.6.5. Here the fact that the phase curve is a closed curve reveals the periodic nature of the solution. Note that the phase curve encloses the nonzero stationary point $(250, 75)$. This point is in fact a center. Figure 8.6.6 shows several phase curves, all of which are closed curves about $(250, 75)$. We have omitted arrows on the phase curves in Figure 8.6.6, but the direction of increasing t is counter-clockwise, as it was in Figure 8.6.5.

Example For our final example in this section, we return to the pendulum problem discussed in Section 8.5. Suppose our pendulum consists of a bob of mass m at the end of a rigid rod of length b. We will assume that the upper end of the rod is attached to

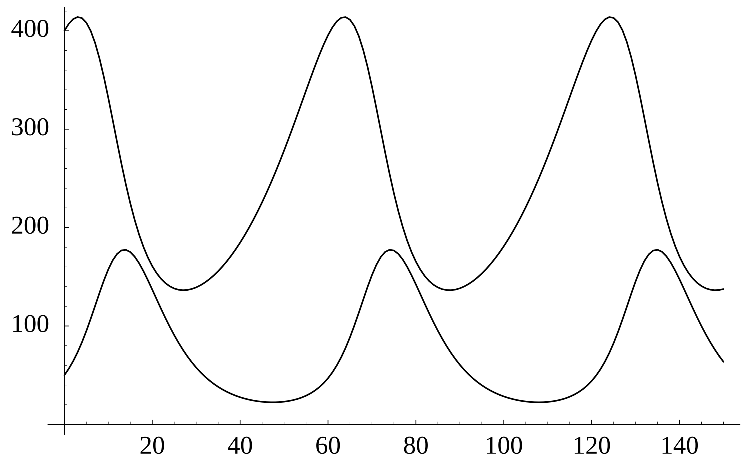


Figure 8.6.4 Predator-prey populations of Table 8.6.1

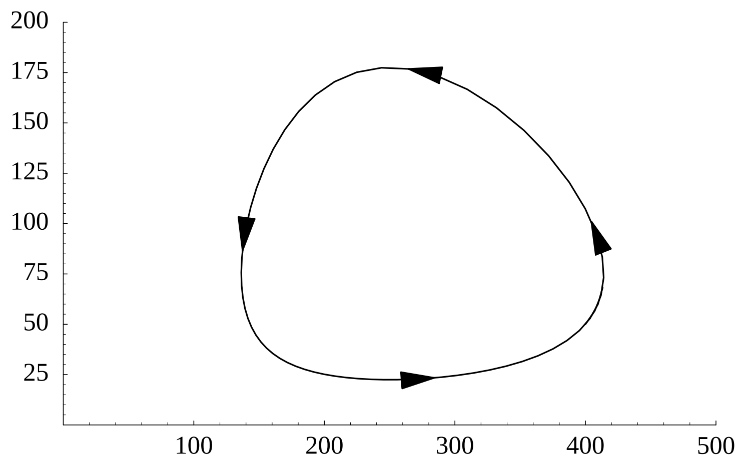


Figure 8.6.5 Phase curve for the predator-prey populations in Table 8.6.1

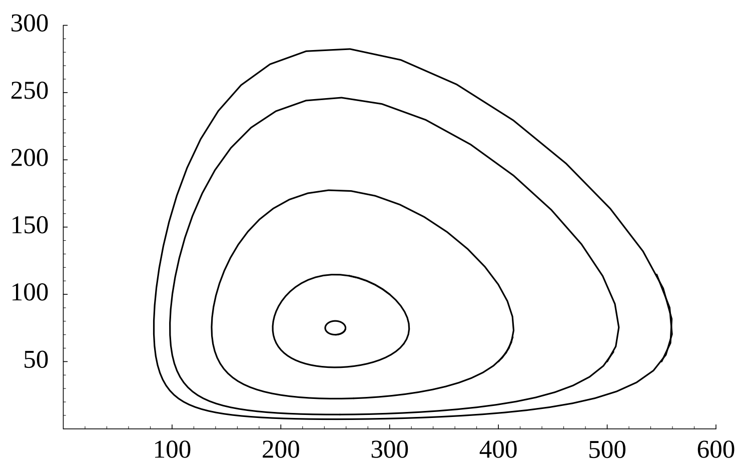


Figure 8.6.6 Phase curves for the predator-prey system (8.6.15)

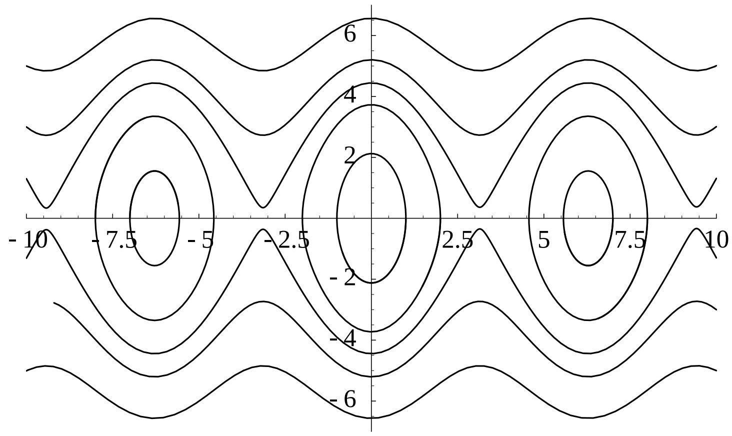


Figure 8.6.7 Phase curves for a pendulum: $\dot{x} = y$, $\dot{y} = -4.9\sin(x)$

another rod, held fixed and perpendicular to the plane of motion of the pendulum, in such a way that the pendulum is free to move through complete circles about this axis. If we let $x(t)$ be the angle between the rod and the vertical at time t, then we showed in Section 8.5 that x must satisfy the equation

$$\ddot{x} = -\frac{g}{b}\sin(x).$$

Equivalently, as a system of first order equations, we have

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\frac{g}{b}\sin(x). \end{aligned} \tag{8.6.16}$$

The stationary points for this system are the points (x, y) where $y = 0$ and $\sin(x) = 0$. Hence there are an infinite number of stationary points, namely, $(n\pi, 0)$ for $n = 0, 1, 2, \ldots$. Note that for even values of n, the stationary points $(n\pi, 0)$ correspond to the pendulum hanging at rest with the bob end down. We should expect these stationary points to be centers since, without friction, any nearby initial conditions would result in the pendulum oscillating about the given stationary point. For odd values of n, the stationary points $(n\pi, 0)$ correspond to the pendulum balancing with the bob end up. We should expect that any initial condition near one of these stationary points would result in motion away from the given stationary point. That is, any slight motion away from the balanced position should cause the pendulum to fall and begin an oscillatory motion. Hence these stationary points should be unstable equilibriums. A look at the phase curves in Figure 8.6.7, shown for a pendulum of length 2 meters, supports these statements: For any integer k, $(2k\pi, 0)$ is a center and $((2k-1)\pi, 0)$ is an unstable equilibrium. We have again omitted arrows on the phase curves in Figure 8.6.7, but the direction of increasing t is from left to right above the x-axis and from right to left below the x-axis.

Problems

1. For each of the following differential equations, find the general solution and then plot the phase curves of the solutions for the given initial conditions over the given time interval I. For each equation decide whether the stationary point $(0,0)$ is a center, a stable equilibrium, or an unstable equilibrium.

(a) $\ddot{x}+x=0$	$I=[0,2\pi]$	Initial conditions:	$x(0)=1,\ \dot{x}(0)=0$ $x(0)=2,\ \dot{x}(0)=0$ $x(0)=3,\ \dot{x}(0)=0$ $x(0)=4,\ \dot{x}(0)=0$ $x(0)=5,\ \dot{x}(0)=0$
(b) $\ddot{x}+3\dot{x}+2x=0$	$I=[-2,2]$	Initial conditions:	$x(0)=-2,\ \dot{x}(0)=0$ $x(0)=-1,\ \dot{x}(0)=0$ $x(0)=-0.5,\ \dot{x}(0)=0$ $x(0)=0.5,\ \dot{x}(0)=0$ $x(0)=1,\ \dot{x}(0)=0$ $x(0)=2,\ \dot{x}(0)=0$
(c) $\ddot{x}-x=0$	$I=[-2,2]$	Initial conditions:	$x(0)=0,\ \dot{x}(0)=-0.2$ $x(0)=0,\ \dot{x}(0)=0.2$ $x(0)=-0.2,\ \dot{x}(0)=0$ $x(0)=0.2,\ \dot{x}(0)=0$
(d) $\ddot{x}+2\dot{x}+2x=0$	$I=[-1,5]$	Initial conditions:	$x(0)=0,\ \dot{x}(0)=1$ $x(0)=0,\ \dot{x}(0)=-1$ $x(0)=-1,\ \dot{x}(0)=0$ $x(0)=1,\ \dot{x}(0)=0$
(e) $\ddot{x}-2\dot{x}+2x=0$	$I=[-5,1]$	Initial conditions:	$x(0)=0,\ \dot{x}(0)=1$ $x(0)=0,\ \dot{x}(0)=-1$ $x(0)=-1,\ \dot{x}(0)=0$ $x(0)=1,\ \dot{x}(0)=0$

2. Plot the phase curves for the examples of overdamped and critically damped mass-spring systems given in Section 8.5.

3. Consider the equation of motion for a mass-spring system

$$\ddot{x}+\frac{c}{m}\dot{x}+\frac{k}{m}x=0$$

with initial conditions $x(0)=10$ and $\dot{x}(0)=0$.

(a) Suppose $k=10$ and $m=10$. Plot the phase curves for the solutions with $c=0$, $c=5$, $c=10$, $c=20$, $c=25$, and $c=30$. Compare your results with your plots of $x(t)$ from Problem 3 of Section 8.5.

(b) Suppose $m=10$ and $c=20$. Plot the phase curves for the solutions with $k=2$, $k=5$, $k=10$, and $k=15$. Compare your results with your plots of $x(t)$ from Problem 4 of Section 8.5.

4. For each of the following second order differential equations, write the equation as a system of first order equations and approximate the solution for the given initial conditions over the interval I using the second order Runge-Kutta method with step size h. Plot both $x(t)$ and the corresponding phase curve.

   (a) $\ddot{x} = -x^2$, $x(0) = -2$, $\dot{x}(0) = 4$, $I = [0, 3]$, $h = 0.01$

   (b) $\ddot{x} + \dot{x} = -\sin(x)$, $x(0) = -3$, $\dot{x}(0) = 2$, $I = [0, 10]$, $h = 0.02$

   (c) $\ddot{x} + x^3 = 0$, $x(0) = 2$, $\dot{x}(0) = 0$, $I = [0, 4]$, $h = 0.02$

   (d) $\ddot{x} + (x^2 - 1)\dot{x} + x = 0$, $x(0) = 0.5$, $\dot{x}(0) = 0$, $I = [0, 20]$, $h = 0.01$

   (e) $\ddot{x} - (x^2 - 1)\dot{x} + x = 0$, $x(0) = -2$, $\dot{x}(0) = 0$, $I = [0, 20]$, $h = 0.01$

   (f) $\ddot{x} + tx = 0$, $x(0) = 2$, $\dot{x}(0) = 0$, $I = [0, 10]$, $h = 0.05$

5. For each of the following systems of first order differential equations, approximate the solution for the given initial conditions over the interval I using the second order Runge-Kutta method with step size h. Plot $x(t)$, $y(t)$, and the corresponding phase curve.

   (a) $\begin{aligned}\dot{x} &= 2xy \\ \dot{y} &= y^2 - x^2\end{aligned}$ $\quad \begin{aligned}x(0) &= 0.05 \\ y(0) &= 0.5\end{aligned}$ $\quad I = [0, 10]$ $\quad h = 0.05$

   (b) $\begin{aligned}\dot{x} &= x(1 - y^2) \\ \dot{y} &= -y(1 - x^2)\end{aligned}$ $\quad \begin{aligned}x(0) &= 3 \\ y(0) &= 2.5\end{aligned}$ $\quad I = [0, 10]$ $\quad h = 0.02$

   (c) $\begin{aligned}\dot{x} &= -y + x(1 - x^2 - y^2) \\ \dot{y} &= -x + y(1 - x^2 - y^2)\end{aligned}$ $\quad \begin{aligned}x(0) &= 0 \\ y(0) &= 4\end{aligned}$ $\quad I = [0, 20]$ $\quad h = 0.04$

6. Consider the predator-prey model

$$\begin{aligned}\dot{x} &= x(\alpha - \beta y) \\ \dot{y} &= -y(\gamma - \delta x)\end{aligned}$$

   where x is the size of the prey population, y is the size of the predator population, and α, β, γ, and δ are nonnegative constants.

   (a) Find explicit solutions for x and y if α and γ are both positive, but $\beta = \delta = 0$. Describe the behavior of the solutions in this case.

   (b) Suppose $\alpha = 0.05$, $\beta = 0.001$, $\gamma = 0.25$, and $\delta = 0.0005$. Using the initial conditions $x(0) = 700$ and $y(0) = 50$, plot x and y over an interval of time long enough to capture at least two periods (use a step size of $h = 0.05$). Plot the corresponding phase curve. What is the nonzero stationary point in this case?

   (c) For the solution found in (b), what are the maximum and minimum predator populations? What are the corresponding prey populations?

   (d) For the solution found in (b), what are the maximum and minimum prey populations? What are the corresponding predator populations?

   (e) Plot four more phase curves using the parameters specified in (b) with varying the initial conditions, plotting two inside and two outside the phase curve plotted in (b). Be sure to plot a complete cycle in each case.

7. Consider the motion of a pendulum as described by the equation

$$\ddot{x} = -\frac{g}{b}\sin(x)$$

as in the final example of the section. Use the second order Runge-Kutta method to approximate x for a pendulum of length 2 meters over the interval $[0, 10]$ using the initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$ and a step size of $h = 0.05$. Graph $x(t)$ and use your results to estimate the period of $x(t)$. How does your estimate compare with the period of the linearized system

$$\ddot{x} = -\frac{g}{b}x$$

that we considered in Section 8.5?

8. For $c > 0$, consider the equation

$$\ddot{x} = -\frac{g}{b}\sin(x) - c\dot{x},$$

the equation for the motion of a pendulum of length b with a damping force proportional to its angular velocity. Suppose $b = 2$ meters and $c = 0.8$.

(a) Write this equation as a system of first order equations. What are the stationary points of this system? Which stationary points do you expect to be stable equilibriums? Which stationary points do you expect to be unstable equilibriums? Which stationary points do you expect to be centers?

(b) Plot phase curves corresponding to the initial conditions $x(0) = 0$ and, in turn, $\dot{x}(0) = -20$, $\dot{x}(0) = -15$, $\dot{x}(0) = -10$, $\dot{x}(0) = -5$, $\dot{x}(0) = 5$, $\dot{x}(0) = 10$, $\dot{x}(0) = 15$, and $\dot{x}(0) = 20$. Describe the behavior of the pendulum for each of these curves.

(c) Plot phase curves corresponding to the initial conditions $x(0) = 0$ and, in turn, $\dot{x}(0) = 6.0$, $\dot{x}(0) = 6.2$, $\dot{x}(0) = 6.4$, $\dot{x}(0) = 6.6$, $\dot{x}(0) = 6.8$, and $\dot{x}(0) = 7.0$. Describe the behavior of the pendulum for each of these curves.

(d) Do your results in (b) and (c) agree with your expectations from (a)?

9. Consider the equation

$$\ddot{x} + \alpha\dot{x} - x(1 - x^2) = 0,$$

where α is a constant.

(a) Write this equation as a system of first order equations and find all the stationary points.

(b) Let $\alpha = 1$. Plot enough phase curves to convince yourself of the proper classification of the stationary points found in (a).

(c) Let $\alpha = -1$. Plot enough phase curves to convince yourself of the proper classification of the stationary points found in (a). How do your answers compare with your results in (b)?

Difference Equations to **D**ifferential Equations

Section 8.7

Power Series Solutions

In this section we consider one more approach to finding solutions, or approximate solutions, to differential equations. Although the method may be applied to first order equations, our discussion will center on second order equations.

The idea is simple: Assuming that the equation

$$\ddot{x} = f(x, \dot{x}, t) \tag{8.7.1}$$

has a solution which is analytic on an interval about $t = t_0$, we express x as a power series

$$x(t) = \sum_{n=0}^{\infty} a_n (t - t_0)^n, \tag{8.7.2}$$

compute $\dot{x}$ and $\ddot{x}$, substitute the results into the equation, solve for the coefficients a_0, a_1, a_2, ..., and verify that the resulting series converges on an interval about t_0. As we shall see, in practice the difficult part is solving for the coefficients. This method will lead us to a closed form solution for the equation only in the rare case that we are able to recognize the resulting power series as the Taylor series of some known function. One advantage of this technique over numerical methods, such as the Runge-Kutta method, is that we are able to work with general solutions and equations involving unspecified parameters, whereas with a numerical method every quantity must be specified as a number. The disadvantage of this technique is that it is not as widely applicable, due to the difficulty of solving for the coefficients, and, when numerical results are needed, one must still approximate the infinite series which results when evaluating x at a point.

To illustrate the procedure, we will begin with an example which we know to be solvable by the techniques of Section 8.4.

Example Consider the equation

$$\ddot{x} = -x. \tag{8.7.3}$$

This is a constant coefficient homogeneous linear equation with characteristic equation $k^2 + 1 = 0$. Since the roots of the characteristic equation are $-i$ and i, we know from our work in Section 8.4 that the general solution of this equation is

$$x = c_1 \cos(t) + c_2 \sin(t),$$

where c_1 and c_2 are arbitrary constants.

We may obtain the same result using power series. If we suppose that x is analytic on an interval about $t = 0$, then we may write

$$x(t) = \sum_{n=0}^{\infty} a_n t^n$$

for some constants a_0, a_1, a_2, ... Differentiating, we have

$$\dot{x}(t) = \sum_{n=1}^{\infty} n a_n t^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} t^n$$

and

$$\ddot{x}(t) = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n.$$

Substituting into (8.7.3) gives us

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n = -\sum_{n=0}^{\infty} a_n t^n.$$

Since power series representations are unique, the coefficient of t^n in the power series on the left must equal the coefficient of t^n in the power series on the right for all values of n. That is, we must have

$$(n+2)(n+1)a_{n+2} = -a_n$$

for $n = 0, 1, 2, \ldots$. Hence the coefficients of the power series representation of x satisfy the difference equation

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)} \tag{8.7.4}$$

for $n = 0, 1, 2, \ldots$. Note that (8.7.4) does not restrict either a_0 or a_1, but determines all of the other coefficients once these values are specified. Thus, given any values for a_0 and a_1,

$$\begin{aligned}
a_2 &= -\frac{a_0}{(2)(1)} = -\frac{a_0}{2}, \\
a_3 &= -\frac{a_1}{(3)(2)} = -\frac{a_1}{3!}, \\
a_4 &= -\frac{a_2}{(4)(3)} = \frac{a_0}{(4)(3)(2)} = \frac{a_0}{4!}, \\
a_5 &= -\frac{a_3}{(5)(4)} = \frac{a_1}{(5)(4)(3)(2)} = \frac{a_1}{5!}, \\
a_6 &= -\frac{a_4}{(6)(5)} = -\frac{a_0}{(6)(5)(4)(3)(2)} = -\frac{a_0}{6!}, \\
a_7 &= -\frac{a_5}{(7)(6)} = -\frac{a_1}{(7)(6)(5)(4)(3)(2)} = -\frac{a_1}{7!},
\end{aligned}$$

and so on. In fact, we see that for $k = 0, 1, 2, \ldots$,

$$a_{2k} = \frac{(-1)^k a_0}{(2k)!}$$

and

$$a_{2k+1} = \frac{(-1)^k a_1}{(2k+1)!}.$$

In most cases, this is as far as we can go; we would now check for the interval of convergence of the resulting power series and conclude that x is a solution of (8.7.3) on that interval. However, in this case we see that

$$\begin{aligned} x &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_0 + a_1 t - \frac{a_0}{2}t^2 - \frac{a_1}{3!}t^3 + \frac{a_0}{4!}t^4 + \frac{a_1}{5!}t^5 - \frac{a_0}{6!}t^6 - \frac{a_1}{7!}t^7 + \cdots \\ &= a_0\left(1 - \frac{t^2}{2} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots\right) + a_1\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots\right) \\ &= a_0 \cos(t) + a_1 \sin(t), \end{aligned}$$

the general solution that we noted above. Hence there is no need to check for the interval of convergence since we recognize our power series representation of x as the Taylor series of a familiar function.

In general, if

$$x = \sum_{n=0}^{\infty} a_n (t - t_0)^n, \tag{8.7.5}$$

then $x(t_0) = a_0$ and $\dot{x}(t_0) = a_1$. Hence if we are seeking the solution of a differential equation in this form, then the values of a_0 and a_1 are determined by any initial conditions which specify $x(t_0)$ and $\dot{x}(t_0)$. Thus we shall see that all of our examples will be of the general form of the previous example. Namely, after substituting x, $\dot{x}$, and $\ddot{x}$ into the equation, we will find a difference equation which determines the coefficients, a_2, a_3, a_4, $\ldots$, in terms of a_0 and a_1. However, unlike the first example, our remaining examples will not result in closed form expressions for our solutions. Nevertheless, we will find power series representations for the solutions which may be used to approximate a specific solution to any desired order on some interval of convergence.

Example Consider the equation

$$\ddot{x} - tx = 0. \tag{8.7.6}$$

Suppose x is analytic on an interval about $t = 0$ and write

$$x = \sum_{n=0}^{\infty} a_n t^n$$

for some constants a_0, a_1, a_2, ... Then, as in the previous example,

$$\dot{x} = \sum_{n=1}^{\infty} n a_n t^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} t^n$$

and

$$\ddot{x} = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n.$$

Substituting into (8.7.6) gives us

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n - t \sum_{n=0}^{\infty} a_n t^n = 0,$$

from which it follows that

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n = \sum_{n=0}^{\infty} a_n t^{n+1}.$$

Since the powers of t in the series on the left begin with 0 while that the powers of t in the series on the right begin with 1, we will move the constant term of the series on the left out of the summation and adjust the index of the sum on the right so that it agrees with the index of the sum on the left. We then have

$$2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} t^n = \sum_{n=1}^{\infty} a_{n-1} t^n.$$

We can now use the uniqueness of power series representations to equate the coefficients on the two sides of this equation, giving us

$$2a_2 = 0$$

and, for $n = 1, 2, 3, \ldots$,

$$(n+2)(n+1) a_{n+2} = a_{n-1}.$$

Hence the coefficients of the power series for x are specified by

$$a_2 = 0$$

and the difference equation

$$a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)} \tag{8.7.7}$$

for $n = 1, 2, 3, \ldots$. As in the previous example, these equations do not restrict the values of a_0 and a_1. However, after specifying a_0 and a_1 by the initial conditions $x(0) = a_0$ and $\dot{x}(0) = a_1$, we may compute

$$\begin{aligned}
a_2 &= 0, \\
a_3 &= \frac{a_0}{(3)(2)} = \frac{a_0}{6}, \\
a_4 &= \frac{a_1}{(4)(3)} = \frac{a_1}{12}, \\
a_5 &= \frac{a_2}{(5)(4)} = 0, \\
a_6 &= \frac{a_3}{(6)(5)} = \frac{a_0}{180}, \\
a_7 &= \frac{a_4}{(7)(6)} = \frac{a_1}{504},
\end{aligned}$$

and so on for as many terms as are desired. We then have

$$\begin{aligned}
x &= a_0 + a_1 t + \frac{a_0}{6}t^3 + \frac{a_1}{12}t^4 + \frac{a_0}{180}t^6 + \frac{a_1}{504}t^7 + \cdots \\
&= a_0\left(1 + \frac{t^3}{6} + \frac{t^6}{180} + \cdots\right) + a_1\left(t + \frac{t^4}{12} + \frac{t^7}{504} + \cdots\right).
\end{aligned}$$

To find the interval of convergence for x, we look at the two series on the right individually. Applying the ratio test to the first series, and making use of the difference equation (8.7.7) to find a_{3n+3} in terms of a_{3n}, we have, for any value of t,

$$\rho = \lim_{n\to\infty}\left|\frac{a_{3n+3}t^{3n+3}}{a_{3n}t^{3n}}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{a_{3n}}{(3n+3)(3n+2)}}{a_{3n}}\right||t|^3 = \lim_{n\to\infty}\frac{|t|^3}{(3n+3)(3n+2)} = 0.$$

Hence $\rho < 1$ for all t and the series converges on $(-\infty, \infty)$. Similarly, for the second series we have, for any value of t,

$$\rho = \lim_{n\to\infty}\left|\frac{a_{3n+4}t^{3n+4}}{a_{3n+1}t^{3n+1}}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{a_{3n+1}}{(3n+4)(3n+3)}}{a_{3n+1}}\right||t|^3 = \lim_{n\to\infty}\frac{|t|^3}{(3n+4)(3n+3)} = 0.$$

Again, $\rho < 1$ for all t and this series also converges on $(-\infty, \infty)$. Thus we have found a solution for (8.7.6) which is analytic on $(-\infty, \infty)$.

The computation of the interval of convergence of a solution found in the manner of the last example can be very involved. Although the justification of the following proposition is itself too involved for us to go into at this point, we will make use of it in our final two examples.

Proposition Suppose $p(t)$ and $q(t)$ are analytic on the interval (t_0-R, t_0+R). Then for any two constants a_0 and a_1, there is a unique function $x(t)$, analytic on (t_0-R, t_0+R), which satisfies the differential equation

$$\ddot{x} + p(t)\dot{x} + q(t)x = 0 \tag{8.7.8}$$

with initial conditions $x(t_0) = a_0$ and $\dot{x}(t_0) = a_1$.

In our previous example, we have, in the notation of the proposition, $p(t) = 0$ and $q(t) = -t$, both of which are analytic on $(-\infty, \infty)$. Hence it follows from the proposition, as we saw by direct computation, that our power series solution converges on $(-\infty, \infty)$.

Note that this proposition also tells us the we analytic solutions to an equation of the form (8.7.8) will exist provided p and q are both analytic. Equation (8.7.8) is similar to the equations we studied in Section 8.4, the difference being that (8.7.8) does not require the coefficients of $\dot{x}$ and x to be constants.

Example Consider the equation

$$(1-t)\ddot{x} + x = 0. \tag{8.7.9}$$

Suppose x is analytic on an interval about $t = 0$ and write

$$x = \sum_{n=0}^{\infty} a_n t^n$$

for some constants a_0, a_1, a_2, ... Then, as before,

$$\dot{x} = \sum_{n=1}^{\infty} n a_n t^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} t^n$$

and

$$\ddot{x} = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n.$$

Substituting into (8.7.9) gives us

$$(1-t)\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n + \sum_{n=0}^{\infty} a_n t^n = 0.$$

Expanding the first term, we have

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n - \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^{n+1} + \sum_{n=0}^{\infty} a_n t^n = 0.$$

To adjust for the fact that the powers of t begin with 1 in the middle series, but with 0 for the other series, we move the constant terms of the latter series out of the summation and adjust the index of the middle series to obtain

$$2a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}t^n - \sum_{n=1}^{\infty}(n+1)na_{n+1}t^n + a_0 + \sum_{n=1}^{\infty}a_nt^n = 0,$$

from which we obtain

$$a_0 + 2a_2 + \sum_{n=1}^{\infty}((n+2)(n+1)a_{n+2} - (n+1)na_{n+1} + a_n)t^n = 0.$$

Using the uniqueness of power series representations, we conclude that all the coefficients on the left-hand side of this equation must be 0. Hence

$$a_0 + 2a_2 = 0$$

and, for $n = 1, 2, 3, \ldots$,

$$(n+2)(n+1)a_{n+2} - (n+1)na_{n+1} + a_n = 0.$$

Thus

$$a_2 = -\frac{a_0}{2} \tag{8.7.10}$$

and

$$a_{n+2} = \frac{(n+1)na_{n+1} - a_n}{(n+2)(n+1)} \tag{8.7.11}$$

for $n = 1, 2, 3, \ldots$. Since (8.7.11) becomes (8.7.10) when $n = 0$, we may combine them into a single difference equation,

$$a_{n+2} = \frac{(n+1)na_{n+1} - a_n}{(n+2)(n+1)} \tag{8.7.12}$$

for $n = 0, 1, 2, \ldots$. As always, a_0 and a_1 are determined by the initial conditions and a_2, a_3, a_4, $\ldots$ may be computed from (8.7.12). For example,

$$\begin{aligned} a_2 &= -\frac{a_0}{2}, \\ a_3 &= \frac{2a_2 - a_1}{(3)(2)} = -\frac{a_0 + a_1}{6}, \\ a_4 &= \frac{(3)(2)a_3 - a_2}{(4)(3)} = \frac{-(a_0 + a_1) + \dfrac{a_0}{2}}{12} = -\frac{a_0 + 2a_1}{24}, \end{aligned}$$

and

$$a_5 = \frac{(4)(3)a_4 - a_3}{(5)(4)} = \frac{-\dfrac{1}{2}(a_0 + 2a_1) + \dfrac{1}{6}(a_0 + a_1)}{20} = -\frac{2a_0 + 5a_1}{120}.$$

Hence

$$\begin{aligned} x &= a_0 + a_1 t - \frac{a_0}{2}t^2 - \frac{(a_0+a_1)}{6}t^3 - \frac{(a_0+2a_1)}{24}t^4 - \frac{(2a_0+5a_1)}{120}t^5 + \cdots \\ &= a_0\left(1 - \frac{t^2}{2} - \frac{t^3}{6} - \frac{t^4}{24} - \frac{t^5}{60} - \cdots\right) + a_1\left(t - \frac{t^3}{6} - \frac{t^4}{12} - \frac{t^5}{24} - \cdots\right). \end{aligned}$$

Finally, if we rewrite (8.7.9) as

$$\ddot{x} + \frac{1}{1-t}\,x = 0,$$

then, in the notation of the previous proposition,

$$p(t) = 0$$

and

$$q(t) = \frac{1}{1-t}.$$

Now p is analytic on $(-\infty,\infty)$, but, considering intervals about 0, q is analytic on only $(-1,1)$. Thus the proposition guarantees only that our solution will be analytic on $(-1,1)$. That is, we know that the two power series in the expression for x converge at least on $(-1,1)$.

Example For an example involving an unspecified parameter, consider the equation

$$\ddot{x} - 2t\dot{x} + 2rx = 0, \tag{8.7.13}$$

where r is a constant. Known as *Hermite's equation*, the solutions to this equation are important in certain areas of mathematics and quantum mechanics. As usual, we suppose x is analytic on an interval about $t = 0$, write

$$x = \sum_{n=0}^{\infty} a_n t^n$$

for some constants a_0, a_1, a_2, $\ldots$, and compute

$$\dot{x} = \sum_{n=1}^{\infty} n a_n t^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}t^n$$

and

$$\ddot{x} = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n.$$

Substituting into (8.7.13), we have

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n - 2t\sum_{n=0}^{\infty}(n+1)a_{n+1}t^n + 2r\sum_{n=0}^{\infty}a_n t^n = 0.$$

Thus

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n - \sum_{n=0}^{\infty} 2(n+1)a_{n+1}t^{n+1} + \sum_{n=0}^{\infty} 2ra_nt^n = 0.$$

Adjusting all these series to start with t raised to the first power gives us

$$2a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}t^n - \sum_{n=1}^{\infty} 2na_nt^n + 2ra_0 + \sum_{n=1}^{\infty} 2ra_nt^n = 0.$$

Hence

$$2ra_0 + 2a_2 + \sum_{n=1}^{\infty}((n+2)(n+1)a_{n+2} + 2(r-n)a_n)t^n = 0.$$

Therefore, by the uniqueness of power series representations, we must have

$$2ra_0 + 2a_2 = 0$$

and, for $n = 1, 2, 3, \ldots$,

$$(n+2)(n+1)a_{n+2} + 2(r-n)a_n = 0.$$

Thus

$$a_2 = -ra_0 \tag{8.7.14}$$

and

$$a_{n+2} = -\frac{2(r-n)a_n}{(n+2)(n+1)} \tag{8.7.15}$$

for $n = 1, 2, 3, \ldots$. Since (8.7.15) becomes (8.7.14) when $n = 0$, we see that, after a_0 and a_1, the coefficients of the solution are determined by the difference equation

$$a_{n+2} = -\frac{2(r-n)a_n}{(n+2)(n+1)}, \tag{8.7.16}$$

$n = 0, 1, 2, \ldots$. For example, we have

$$\begin{aligned}
a_2 &= -ra_0, \\
a_3 &= -\frac{2(r-1)a_1}{(3)(2)} = -\frac{2(r-1)a_1}{3!}, \\
a_4 &= -\frac{2(r-2)a_2}{(4)(3)} = \frac{2^2 r(r-2)a_0}{4!}, \\
a_5 &= -\frac{2(r-3)a_3}{(5)(4)} = \frac{2^2(r-1)(r-3)a_1}{5!}, \\
a_6 &= -\frac{2(r-4)a_4}{(6)(5)} = -\frac{2^3 r(r-2)(r-4)a_0}{6!},
\end{aligned}$$

and

$$a_7 = -\frac{2(r-5)a_5}{(7)(6)} = -\frac{2^3(r-1)(r-3)(r-5)a_1}{7!}.$$

Thus

$$\begin{aligned} x &= a_0 + a_1 t - r a_0 t^2 - \frac{2(r-1)a_1}{3!}t^3 + \frac{2^2 r(r-2)a_0}{4!}t^4 + \frac{2^2(r-1)(r-3)a_1}{5!}t^5 \\ &\quad - \frac{2^3 r(r-2)(r-4)a_0}{6!}t^6 - \frac{2^3(r-1)(r-3)(r-5)a_1}{7!}t^7 + \cdots \\ &= a_0\left(1 - rt^2 + \frac{2^2 r(r-2)}{4!}t^4 - \frac{2^3 r(r-2)(r-4)}{6!}t^6 + \cdots\right) \\ &\quad + a_1\left(t - \frac{2(r-1)}{3!}t^3 + \frac{2^2(r-1)(r-3)}{5!}t^5 - \frac{2^3(r-1)(r-3)(r-5)}{7!}t^7 + \cdots\right). \end{aligned}$$

In the notation of the previous proposition, we have $p(t) = 2t$ and $q(t) = 2r$, both of which are analytic on $(-\infty, \infty)$. Hence it follows that the two series in our solution converge for all values of t.

Moreover, note that if we let

$$x_1(t) = 1 - rt^2 + \frac{2^2 r(r-2)}{4!}t^4 - \frac{2^3 r(r-2)(r-4)}{6!}t^6 + \cdots$$

and

$$x_2(t) = t - \frac{2(r-1)}{3!}t^3 + \frac{2^2(r-1)(r-3)}{5!}t^5 - \frac{2^3(r-1)(r-3)(r-5)}{7!}t^7 + \cdots,$$

so that

$$x(t) = a_0 x_1(t) + a_1 x_2(t),$$

then x_1 is a polynomial when r is an nonnegative even integer and x_2 is a polynomial when r is a positive odd integer. That is, when r is a nonnegative integer, Hermite's equation will have a polynomial solution. When suitably normalized, as described in Problem 6 below, these polynomials are called *Hermite polynomials*.

Our final example shows the strength of the power series method of solving differential equations. Through one computation we have found analytic solutions to an entire family of equations parametrized by the real number r. As an added consequence, we have discovered that the equation has polynomial solutions for certain values of the parameter r. If we were only interested in numerical values of a solution of Hermite's equation for one value of r and one set of initial conditions, then using a numerical method, such as the Runge-Kutta method of Section 8.6, would be the proper approach; however, we can see that the power series approach leads to a much richer understanding of the solutions to the general form of the equation.

Problems

1. Solve the following first order differential equations using power series with the initial condition $x(0) = a_0$. Verify your answer by finding a closed form solution for the equation using the techniques of Sections 8.2 and 8.3

   (a) $\dot{x} = 3x$ (b) $\dot{x} = 2tx$

   (c) $\dot{x} = x - 1$ (d) $\dot{x} = -x$

2. Solve the following second order differential equations using power series with the initial conditions $x(0) = a_0$ and $\dot{x}(0) = a_1$. Write the solution out through the first six nonzero terms and give an interval of convergence for each solution.

   (a) $\ddot{x} + tx = 0$ (b) $\ddot{x} + \dot{x} - tx = 0$

   (c) $\ddot{x} + t\dot{x} + x = 0$ (d) $\ddot{x} - (1 + t^2)x = 0$

   (e) $(1 - t^2)\ddot{x} - 2t\dot{x} - x = 0$ (f) $(1 + t)\ddot{x} - x = 0$

3. (a) Use power series to show that the solution of

   $$\ddot{x} = x$$

   satisfying $x(0) = a_0$ and $\dot{x}(0) = a_1$ is given by $x = a_0 \cosh(t) + a_1 \sinh(t)$.

   (b) Solve the equation in (a) using the techniques of Section 8.4 and show that your answer agrees with the answer in (a).

4. Use the ratio test to verify that the solutions x_1 and x_2 of Hermite's equation found in the last example of this section converge for all t in $(-\infty, \infty)$.

5. Find polynomial solutions of Hermite's equation for $r = 0$, $r = 1$, $r = 2$, $r = 3$, $r = 4$, and $r = 5$.

6. A polynomial solution of Hermite's equation with highest degree term of the form $2^n t^n$ is called a *Hermite polynomial* and is denoted $H_n(t)$.

   (a) Show that $H_0(t) = 1$, $H_1(t) = 2t$, $H_2(t) = 4t^2 - 2$, and $H_3(t) = 8t^3 - 12t$.

   (b) Find $H_4(t)$ and $H_5(t)$.

7. The equation

   $$(1 - t^2)\ddot{x} - 2t\dot{x} + r(r + 1)x = 0,$$

   where r is a constant, is known as *Legendre's equation*.

   (a) Show that the general solution to Legendre's equation may be written as

   $$x(t) = a_0 x_1(t) + a_1 x_2(t),$$

   where

   $$x_1(t) = 1 - \frac{r(r+1)}{2!}t^2 + \frac{r(r-2)(r+1)(r+3)}{4!}t^4 - \frac{r(r-2)(r-4)(r+1)(r+3)(r+5)}{6!}t^6 + \cdots,$$

$$x_2(t) = t - \frac{(r-1)(r+2)}{3!}t^3 + \frac{(r-1)(r-3)(r+2)(r+4)}{5!}t^5 - \frac{(r-1)(r-3)(r-5)(r+2)(r+4)(r+6)}{7!}t^7 + \cdots,$$

and a_0 and a_1 are constants.

(b) Explain why the radius of convergence of each of these series is at least 1.

(c) Note that if r is a nonnegative even integer, then x_1 is a polynomial, and if r is a positive odd integer, then x_2 is a polynomial. If r is an even nonnegative integer, let

$$P_r(t) = \frac{x_1(t)}{x_1(1)}$$

and if r is a positive odd integer let

$$P_r(t) = \frac{x_2(t)}{x_2(1)}.$$

Then $P_r(t)$, $r = 0, 1, 2, \ldots$, is a polynomial solution of Legendre's equation, known as a *Legendre polynomial*, normalized so that $P_r(1) = 1$. Find $P_0(t)$, $P_1(t)$, $P_2(t)$, $P_3(t)$, $P_4(t)$, and $P_5(t)$ and plot their graphs on the interval $[-1, 1]$.

8. Discuss all the interconnections we have seen between difference equations and differential equations.