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REPRESENTATIONS OF $\mathfrak{p}$-ADIC GROUPS: A SURVEY

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Introduction. The aim of this article is to (partially) survey the present state of knowledge about the representations (mostly infinite-dimensional) of reductive algebraic groups over a local field. This includes the familar p-adic groups like $GL_n(\boldsymbol{Q}_p)$, $Sp_{2n}(\boldsymbol{Q}_p)$, ...

This theory evolved slowly and lately. The first steps were taken around 1960 by Mautner and his students who concerned themselves with a detailed study of the particular group $GL_2(\boldsymbol{Q}_p)$. The first general results were obtained by Bruhat **[8]** who imitated the 'real' methods of his thesis [7] and by Satake who determined the spherical functions **[38]**. But the next developments had to await the deep results of Bruhat and Tits **[10]**, **[11]**, **[12]** and **[13]** about the structure of $\mathfrak{p}$-adic reductive groups.

In their reference work in which they were basically concerned with the group GL_2, Jacquet and Langlands **[34]** introduced the important notion of an admissible representation. They thus opened the way towards a purely algebraic theory of these representations. The basic results about induced representations were soon after obtained by Jacquet **[32]**, who considered the case of GL_n only, but used perfectly general methods. These results have been generalized by Casselman and Harish-Chandra.

The main goal of this article will be the description and study of the principal series and the spherical functions. There shall be almost no mention of two important lines of research which are still actively pursued today:

(a) *Plancherel theorem* and detailed harmonic analysis on $\mathfrak{p}$-adic Lie groups. Here Harish-Chandra is the uncontested leader. We refer the reader to Harish-Chandra's own description of his results **[26]**, **[27]** and also to my Bourbaki lecture **[14]** for more recent results due to Harish-Chandra and Roger Howe.

(b) *Explicit construction of absolutely cuspidal representations* (the so-called 'discrete series'). Here important progress has been made by Shintani **[40]**, Gérardin **[21]** and Howe (forthcoming papers in the Pacific J. Math.). One can expect to meet here difficult and deep arithmetical questions which are barely uncovered.

Let us give a brief description of the contents of these notes. In §I, we describe the various classes of representations in a very general framework. Following Harish-Chandra **[25]**, we give the definitions for totally disconnected locally com-

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pact groups. Number theory and automorphic functions provide us with a host of such groups and their representations. Special attention is paid to various forms of *Frobenius reciprocity* and various notions of *induced representations*. Our exposition is based on an apparently novel method using tensor products (over rings without unit element, alas!).

In §II, we build up the machinery which enables us, after Jacquet, to reduce the classification of the irreducible admissible representations for a $\mathfrak{p}$-adic group G to the two following problems:

(a) *Construct the absolutely cuspidal irreducible representations for G and the Levi components of its parabolic subgroups as well.*

(b) *Study the representations induced from parabolic subgroups to the whole group.*

Our presentation of Jacquet's fundamental construction (the two 'Jacquet's functors') is based on our previous description of the induced representations and is slightly more symmetrical than usual. After specializing the previous results to the classical case of GL_n, we turn to the relation between unitary representations and admissible representations. Here the basic results are due to Harish-Chandra **[28]** (generalizing earlier results of R. Howe **[29]** for GL_n) and Bernshtein **[1]**. They show that any reductive algebraic group over a local field is of type I in the sense of von Neumann-Murray classification. The foundations are thus laid down for Plancherel theorem.

§III is devoted to the *unramified principal series*. These representations are parametrized by the so-called *unramified characters* of the Levi component of a minimal parabolic subgroup. Let us mention that those characters provide the crux of the applications to Langlands theory of L-groups (see Borel lectures in these PROCEEDINGS). In general, the representations $I(\chi)$ in this series are irreducible. They have always a nonzero vector invariant under a special maximal compact subgroup. The main result concerns the explicit construction of an equivalence between $I(\chi)$ and $I(w\chi)$ where w is an element of the (relative) Weyl group. In the case of real Lie groups, this question led to the introduction of singular integral operators (Stein, Knapp). The deep analytical problems involved in the construction of these operators are bypassed by a very ingenious trick of Casselman, making full use of Jacquet's construction $V \Rightarrow V_N$. An important role is played by the so-called Iwahori subgroups and their geometrical interpretation via buildings.

In §IV, we culminate with the theory of spherical functions. Using the results expounded in §III, one recovers Macdonald's formula **[36]**, **[37]** for these spherical functions. This theory has been highly developed in the case of real Lie groups. From the point of view of representation theory, the spherical principal series is quite special, but plays a prominent role in the applications to L-functions à la Langlands.

The whole approach leading to spherical functions through §§III and IV has been developed by Casselman in a still unpublished paper **[18]**. I borrowed extensively from this paper as well as from a preliminary paper **[17]** by the same author developing the foundations of representation theory in the $\mathfrak{p}$-adic case. The reader will have to consult these papers for the details of proofs and for numerous generalizations. It is hoped that they shall appear soon.

I have to thank several friends. Anna Helversen-Pasotto tape-recorded my lectures and made out of her tapes a transcript of the spoken words. This ungra-

tifying task proved very helpful to me when transforming the sketchy notes distributed during the conference into the present report.

Serge Lang and John Tate allowed me to use freely their notes about spherical transforms. My treatment in §IV has been influenced by them. Also John Tate corrected my misinterpretations about Frobenius reciprocity (see §1.7).

Notations and conventions. 1. Let A be a ring with unit element 1. Let $A^\times$ be the set of elements a in A for which there exists b in A with $ab = 1$. Endowed with ring multiplication, $A^\times$ is a group, the "*multiplicative group*" of A.

2. Let M be a module over a ring A without unit element. One says that M is *nondegenerate* if every element in M can be written in the form $a_1 \cdot m_1 + \cdots + a_k \cdot m_k$ with $a_1, \cdots, a_k$ in A and $m_1, \cdots, m_k$ in M.[1]

One says a sequence of elements $m_1, \cdots, m_k$ in M generates M if every element in M is of the form $a_1 \cdot m_1 + \cdots + a_k \cdot m_k + n_1 \cdot m_1 + \cdots + n_k \cdot m_k$ with $a_1, \cdots, a_k$ in A and integers $n_1, \cdots, n_k$ (of either sign). If M is nondegenerate, we can omit the terms $n_1 \cdot m_1, \cdots, n_k \cdot m_k$. The module M is *finitely generated* if there exists a finite sequence generating M.

3. By a *local field* we mean a nonarchimedean local field with finite residue field. Such a field F comes equipped with a subring $\mathfrak{O}_F$, whose elements are called the *integers* in F. There exists in $\mathfrak{O}_F$ a unique nonzero prime ideal $\mathfrak{p}_F$. Moreover there is in $\mathfrak{O}_F$ a *prime element* $\tilde{\omega}_F$ such that $\mathfrak{p}_F = \mathfrak{O}_F \cdot \tilde{\omega}_F$. Every element x in $F^\times$ can be uniquely written as $\tilde{\omega}_F^n \cdot u$ with some integer n and some u in $\mathfrak{O}_F^\times = \mathfrak{O}_F \backslash \mathfrak{p}_F$. We set $\mathrm{ord}_F(x) = n$ in this case. By convention, $\mathrm{ord}_F(0)$ is put equal to $+\infty$. The index $q = (\mathfrak{O}_F : \mathfrak{p}_F)$ is finite and a power of a prime number p. We set $|x|_F = q^{-\mathrm{ord}_F(x)}$ for x in $F^\times$ and $|0|_F = 0$. The index 'F' may be omitted in $\mathfrak{O}_F$, $\mathfrak{p}_F$, $\mathfrak{O}_F^\times$, $\tilde{\omega}_F$, $\mathrm{ord}_F(x)$ and $|x|_F$ when no confusion can arise.

There are two possibilities:

(a) If F is of characteristic 0, then F is a finite algebraic extension of the p-adic field $\boldsymbol{Q}_p$ and $\mathfrak{O}_F$ consists of the elements integrally dependent on the ring $\boldsymbol{Z}_p$ of p-adic integers.

(b) If F is of characteristic p, then F is the quotient field of the ring $\mathfrak{O}_F = \boldsymbol{F}_q[[t]]$ of formal power series in one indeterminant t with coefficients in the Galois field $\boldsymbol{F}_q$ (also denoted as $\mathrm{GF}(q)$ by various authors).

4. We use the standard notations:

$\boldsymbol{Z}$ for the ring of integers,

$\boldsymbol{Q}$ for the field of rational numbers,

$\boldsymbol{R}$ for the field of real numbers,

$\boldsymbol{C}$ for the field of complex numbers.

5. Let G be a topological group, whose unit element shall be denoted by 1. By a *character* of G we mean any continuous homomorphism $\chi: G \to \boldsymbol{C}^\times$. We say χ is *unitary* in case $\chi(g)$ is a complex number of modulus 1 for every g in G, that is [2]

$$\overline{\chi(g)} = \chi(g)^{-1} = \chi(g^{-1}).$$

[1]We make the convention that any unitary module over a ring with unit element is nondegenerate.

[2]Some authors call 'quasi-character' what we call 'character' and 'character' what we call 'unitary character'.

Assume that G is locally compact. We use the symbol $C_c(G)$ to denote the space of continuous compactly supported complex-valued functions on G. If μ is a (left invariant) Haar measure on G, we denote usually by $\int_G f(g)dg$ the integral of a function f in $C_c(G)$ w.r.t. μ. The *modular function* Δ_G is characterized by the integration rule

$$\int_G f(g)\, dg = \Delta_G(g_0)\int_G f(gg_0)\, dg \quad \text{for } f \text{ in } C_c\,(G).$$

The group G is called *unimodular* in case $\Delta_G = 1$.

One has sometimes to integrate over homogeneous spaces G/H. Suppose that f is a continuous function on G and there exists a function f_1 in $C_c(G/H)$ such that $f(g) = f_1(gH)$ for any g in G. Choose an invariant measure μ on G/H. The integral of f_1 w.r.t. μ shall often be denoted by $\int_{G/H} f(g)\, d\bar{g}$.

If G is a Lie group (real or $\mathfrak{p}$-adic) we use the corresponding German letter $\mathfrak{g}$ to denote its Lie algebra.

6. Let X be any set. The *identity map* in X shall be denoted by 1_X. If A is any subset of X, the *characteristic function* $I_A\colon X \to \{0, 1\}$ is defined by

$$\begin{aligned} I_A(x) &= 1 \quad \text{for } x \text{ in } A, \\ &= 0 \quad \text{for } x \text{ in } X\setminus A, \end{aligned}$$

where we denote by $X\setminus A$ the set-theoretic difference.

If X is finite, its cardinality shall be denoted by $|X|$.

7. We use the abbreviations 'iff' for 'if and only if' and 'w.r.t.' for 'with respect to'.

I. Totally disconnected groups and their representations.

1.1. *Groups of td-type*. Let G be a topological group. We say G is of *td-type* if every neighborhood of its unit element 1 contains a compact open subgroup. Such a group is a locally compact Hausdorff space. Moreover it is totally disconnected (hence *td*), that is there is no connected subset of G with more than one element.

Let G be a group of td-type. If X_1 and X_2 are nonempty compact open sets in G, there exists a compact open subgroup K of G such that X_1 and X_2 are unions of finitely many cosets $x_{1,1}\, K, \cdots, x_{1,n_1}\, K$ for X_1 and $x_{2,1}\, K, \cdots, x_{2,n_2}\, K$ for X_2. We set

$$(X_1 : X_2) = n_1/n_2. \tag{1}$$

For instance, if X_1 and X_2 are compact open subgroups and X_2 is contained in X_1, then $(X_1 : X_2)$ is the index of X_2 in X_1. The chain rule holds, namely

$$(X_1 : X_3) = (X_1 : X_2) \cdot (X_2 : X_3). \tag{2}$$

Let μ be a left invariant Haar measure on G. The following formula is obvious:

$$(X_1 : X_2) = \mu(X_1)/\mu(X_2). \tag{3}$$

Hence if $\mu(X)$ is rational for some compact open set $X \neq \varnothing$, the same is true for every such set. In this case one calls the Haar measure μ *rational.*

1.2. *Examples of groups of td-type.* (a) Let G be of td-type. Then every open subgroup, every closed subgroup of G is of td-type. A factor group of G by a closed invariant subgroup is of td-type.

(b) If $G_1, \cdots, G_n$ are groups of td-type, so is their direct product $G_1 \times \cdots \times G_n$ endowed with the product topology.

(c) Let $(G_i)_{i\in I}$ be any infinite family of groups of td-type, and let $K_i \subset G_i$ be a compact open subgroup for each i in I. In the direct product $\prod_{i\in I} G_i$, let G be the subgroup consisting of the families $(g_i)_{i\in I}$ such that the set $\{i \in I | g_i \notin K_i\}$ is finite. Then $K = \prod_{i\in I} K_i$ is a subgroup of G. We endow K with the product topology. A set U in G is open iff $gU \cap K$ is open in K for every g in G. Then G with this topology is a group of td-type and K is a compact open subgroup of G. The group G is known as the *restricted product of the groups* G_i *w.r.t. the groups* K_i.

(d) Let F be a local field. Then $|x-y|_F$ defines a distance in F, hence a topology. Then F as an additive group is of td-type, with $\mathfrak{O}_F$ as a compact open subgroup. Moreover $F^\times$ is open in F, hence, as a multiplicative group, it is of td-type with $\mathfrak{O}_F^\times$ as a compact open subgroup.

(e) Let $n \geqq 1$ be an integer. The linear group $\mathrm{GL}_n(F)$ is the open subspace of the n^2-dimensional space over F with coordinates $x_{11}, x_{12}, \cdots, x_{nn}$, defined by $\det(x_{ij}) \neq 0$. It is a group of td-type. A compact open subgroup is $\mathrm{GL}_n(\mathfrak{O}_F)$, the set of n-by-n matrices with entries in $\mathfrak{O}_F$, and determinant in $\mathfrak{O}_F^\times$.

(f) Let G be a subgroup of $\mathrm{GL}_n(F)$ defined as the set of common zeroes of a set of polynomials in the coordinates x_{ij} with coefficients in F. For short, G is an *algebraic subgroup of* $\mathrm{GL}_n(F)$ (more precisely, the set of F-rational points of an algebraic group defined over F). It is a closed subgroup of $\mathrm{GL}_n(F)$, hence a group of td-type on its own merits. A neighborhood basis of the unit matrix I_n in G is given by the subgroups

$$K_m = \{g = (g_{ij}) \text{ in } G \,|\, |g_{ij} - \delta_{ij}|_F \leqq q^{-m} \text{ for } 1 \leqq i, j \leqq n\}.$$

Let G' be an algebraic subgroup of $\mathrm{GL}_{n'}(F)$ for some integer $n' \geqq 1$. Assume that the group homomorphism $\rho\colon G \to G'$ is rational, that is, there exist polynomials ρ_{kl} in $F[X_{11}, \cdots, X_{nn}]$ and an integer $m \geqq 1$ such that

$$\rho(g) = (\rho_{kl}(g_{11}, \cdots, g_{nn})/(\det g)^m)_{1\leq k,\, l\leq n'}$$

for $g = (g_{ij})$ in G. Then ρ is continuous w.r.t. the topologies defined on G and G' by their embedding in the linear groups $\mathrm{GL}_n(F)$ and $\mathrm{GL}_{n'}(F)$ respectively. In particular if ρ is a biregular isomorphism, i.e., ρ is a group isomorphism and ρ, ρ^{-1} are both rational, then ρ is a homeomorphism.

In more intrinsic terms, the algebraic structure on G is defined by the ring $F[G]$ of polynomial functions[3] and the topology defined above is the coarsest for which the elements of $F[G]$ are continuous mappings from G to F (F is given the topology defined in (d)).

(g) By (b) and (f), the product of finitely many algebraic groups defined over the

[3]This ring consists of the functions $g \mapsto u(g_{11}, \cdots, g_{nn})/(\det g)^m$ for a polynomial u in $F[X_{11}, \cdots, X_{nn}]$ and an integer $m \geqq 0$.

same or distinct local fields is of td-type. Similarly, by (c) and (f), adelic groups without archimedean components are of td-type.

(h) Let F be a local field and F_{sep} a separably algebraic closure of F. Let F_{unr} be the maximal unramified extension of F contained in F_{sep} and σ the Frobenius automorphism of F_{unr} over F. Let $G_F = \text{Gal}(F_{\text{sep}}/F)$ be the Galois group of F_{sep} over F endowed with the Krull topology. It is a compact group of td-type. Let $W_F \subset G_F$ be the subgroup of automorphisms φ of F_{sep} which induce some power σ^n of σ in F_{unr}. In a unique way we can consider W_F, the *Weil group* of F, as a group of td-type in which $\text{Gal}(F_{\text{sep}}/F_{\text{unr}})$ (with Krull topology) is a compact open subgroup of W_F.

1.3. *Hecke algebra.* Let G be a group of td-type. If K is any compact open subgroup of G, we denote by $\mathscr{H}(G, K)$ the complex vector space consisting of the complex-valued functions f on G which satisfy the following two conditions:

(a) *f is bi-invariant under K, that is $f(kgk') = f(g)$ for g in G and k, k' in K.*

(b) *f vanishes off a finite union of double cosets KgK.*

Moreover, let us choose a (left invariant) Haar measure μ on G. One defines a bilinear multiplication in the complex vector space $\mathscr{H}(G, K)$ by the customary convolution formula

$$(f_1 * f_2)(g) = \int_G f_1(x) f_2(x^{-1}g)\, dx. \tag{4}$$

This integral makes sense since, as a function of x, the integrand is locally constant and compactly supported. For this multiplication $\mathscr{H}(G, K)$ becomes an associative algebra over the complex field $\boldsymbol{C}$.

Let us choose a set of representatives $\{g_\alpha\}$ for the double cosets of G modulo K, i.e., G is the disjoint union of the sets $Kg_\alpha K$. For any index α, let u_α be defined by

$$\begin{aligned} u_\alpha(g) &= \mu(K)^{-1} \quad \text{if } g \in Kg_\alpha K, \\ &= 0 \qquad\qquad \text{otherwise.} \end{aligned} \tag{5}$$

In particular we may assume that $g_0 = 1$ for some index 0 and the corresponding function u_0 shall be denoted by e_K. Hence

$$\begin{aligned} e_K(g) &= \mu(K)^{-1} \quad \text{if } g \in K, \\ &= 0 \qquad\qquad \text{otherwise.} \end{aligned} \tag{6}$$

The family $\{u_\alpha\}$ is a basis of the vector space $\mathscr{H}(G, K)$. Moreover, e_K is the unit element of this algebra and the multiplication table is given by $u_\alpha * u_\beta = \sum_\gamma c_{\alpha\beta\gamma} u_\gamma$ where the coefficients $c_{\alpha\beta\gamma}$ are computed as follows. The group $K_\alpha = K \cap g_\alpha K g_\alpha^{-1}$ is compact and open, hence of finite index in K. There exist therefore elements $x_1, \cdots, x_m$ of K such that K is the disjoint union of the sets $x_1 K_\alpha, \cdots, x_m K_\alpha$. Then $Kg_\alpha K$ is the disjoint union of the sets $x_1 g_\alpha K, \cdots, x_m g_\alpha K$. Define similarly K_β and the elements $y_1, \cdots, y_n$. Then $c_{\alpha\beta\gamma}$ is the number of pairs (i, j) such that $g_\gamma^{-1} x_i g_\alpha y_j g_\beta$ belong to K (see Shimura [39]).

When K' is a compact open subgroup of K, then $\mathscr{H}(G, K)$ is a subring of $\mathscr{H}(G, K')$ but with a different unit element if $K \neq K'$. Define $\mathscr{H}(G) = \bigcup_K \mathscr{H}(G, K)$ where K runs through a neighborhood basis of 1 consisting of compact open subgroups. Then *$\mathscr{H}(G)$ is the space of locally constant and compactly supported functions on G.* For the convolution product defined by (4), $\mathscr{H}(G)$ is an associative algebra. It has no unit element unless G is discrete, and it is commutative iff G is commutative.

The algebra $\mathscr{H}(G)$ is called the *Hecke algebra* of G and $\mathscr{H}(G, K)$ is called the Hecke algebra of G w.r.t. K.[4]

For later purposes, we need a generalization of $\mathscr{H}(G)$. Namely, let Z be a closed subgroup of the center of G and χ a character of Z. We denote by $\mathscr{H}_\chi(G)$ the set of complex-valued functions which satisfy the following conditions:

(a) *f is locally constant*;

(b) *one has $f(zg) = \chi(z)^{-1}f(g)$ for z in Z and g in G*;

(c) *f is compactly supported modulo Z.*

More explicitly, assertion (c) means that f vanishes off a set of the form $\Omega \cdot Z$ where Ω is compact in G.

Choose a Haar measure ν on G/Z. The convolution product is defined in $\mathscr{H}_\chi(G)$ by

$$(f_1 *_\chi f_2)(g) = \int_{G/Z} f_1(x) f_2(x^{-1}g)\, d\dot{x} \tag{7}$$

(notice that the integrand takes the same value for x and xz if z belongs to Z). This product is bilinear and associative. There is no unit element in the algebra $\mathscr{H}_\chi(G)$ unless Z is open in G. When $Z = \{1\}$, this construction brings us back to $\mathscr{H}(G)$.

1.4. *Smooth representations.* By a *representation* of G, we mean as customary a pair (π, V) where V is a complex vector space and π a homomorphism from G into the group of invertible linear maps in V. If H is a subgroup of G, we denote by V^H the space of vectors v in V such that $\pi(h)\cdot v = v$ for any h in H, that is, vectors whose stabilizer in G contains H.

DEFINITION 1.1. *A representation (π, V) of G is smooth iff the stabilizer of every vector in V is open, equivalently if $V = \bigcup_K V^K$ where K runs over the compact open subgroups of G.*

Let (π, V) be a smooth representation of G and V^* the space of all linear forms on V. The *coefficient* π_{v,v^*} of π (for v in V and v^* in V^*) is defined by

$$\pi_{v,v^*}(g) = \langle v^*, \pi(g)\cdot v\rangle. \tag{8}$$

It is a locally constant function on G.

For f in $\mathscr{H}(G)$ there exists a linear operator $\pi(f)$ acting on V and such that

$$\langle v^*, \pi(f)\cdot v\rangle = \int_G f(g)\pi_{v,v^*}(g)\, dg. \tag{9}$$

It is computed as follows: given v in V there exist a compact open subgroup K of G, constants $c_1, \cdots, c_m$, elements $g_1, \cdots, g_m$ of G such that $v \in V^K$ and $f = \sum_{i=1}^m c_i I_{g_iK}$. Then $\pi(f)\cdot v$ is equal to $\mu(K)\cdot \sum_{i=1}^m c_i\pi(g_i)\cdot v$.

Using standard calculations one checks that $f \mapsto \pi(f)$ is an algebra homomorphism from $\mathscr{H}(G)$ into $\mathrm{End}_C(V)$; hence we may consider V as an $\mathscr{H}(G)$-module. For every compact open subgroup K of G, the operator $\pi(e_K)$ is a projection of V onto V^K; hence every vector v in V satisfies $v = \pi(e_K)\cdot v$ for a suitable K. As a corollary, V is a nondegenerate $\mathscr{H}(G)$-module.

The following facts are easily proved:

[4]For $G = \mathrm{GL}_2(\boldsymbol{Q}_p)$ and $K = \mathrm{GL}_2(\boldsymbol{Z}_p)$, this algebra is just the classical algebra of Hecke operators attached to p.

(a) *A subspace V_1 of V is stable under the operators $\pi(g)$ for g in G iff it is invariant under the operators $\pi(f)$ for f in $\mathscr{H}(G)$.*

Otherwise stated, the subrepresentations (π_1, V_1) of (π, V) correspond to the submodules of the $\mathscr{H}(G)$-module V. In particular, the representation (π, V) is (algebraically) *irreducible* iff V is a simple $\mathscr{H}(G)$-module.

The representation (π, V) is said to be *finitely generated* if there exist finitely many vectors $v_1, \cdots, v_m$ such that the transforms $\pi(g) \cdot v_i$ for g in G and $1 \leqq i \leqq m$ generate the complex vector space V. By (a), this amounts to the assertion that $(v_1, \cdots, v_m)$ generates the $\mathscr{H}(G)$-module V.

(b) *Let (π, V) and (π', V') be smooth representations of G, and let $u: V \to V'$ be a linear map. Then u satisfies $\pi'(g)u = u\pi(g)$ for every g in G iff it satisfies $\pi'(f)u = u\pi(f)$ for every f in $\mathscr{H}(G)$.*

A map $u: V \to V'$ such that $\pi'(g)u = u\pi(g)$ for every g in G is called an *intertwining map* or a *G-homomorphism.* By (b), it is nothing else but a homomorphism of $\mathscr{H}(G)$-modules.

(c) *Let (π, V) be any irreducible smooth representation of G. Assume that the topology of G has a countable*[5] *basis.*[6] *Then every intertwining map $u : V \to V$ is a scalar.*

This version of *Schur's lemma* is proved as follows. Since G has a countable basis, the index $(G: K)$ is countable for every compact open subgroup K of G; hence $\mathscr{H}(G, K)$ has a countable dimension over $\boldsymbol{C}$. Moreover, there exists a countable basis of neighborhoods of 1; hence $\mathscr{H}(G) = \bigcup_K \mathscr{H}(G, K)$ has a countable dimension. For any $v \neq 0$ in V, the map $f \mapsto \pi(f) \cdot v$ from $\mathscr{H}(G)$ to V is surjective; hence the dimension of V is countable. Let A be the algebra of intertwining maps from V into V. For any $v \neq 0$ in V, the map $u \mapsto u(v)$ of A into V is injective because (π, V) is irreducible; hence the dimension of A is countable. But A is a division algebra over the algebraically closed field $\boldsymbol{C}$. If $A \neq \boldsymbol{C}$, there exists a subfield of A isomorphic over $\boldsymbol{C}$ to the field of rational fractions $\boldsymbol{C}(x)$. In this field, the uncountably many elements $(x - \lambda)^{-1}$, for λ running over $\boldsymbol{C}$, are linearly independent. Contradiction! [This proof is due to Jacquet **[33]**.]

For instance, let z belong to the center $Z(G)$ of G. Then $\pi(z)$ commutes to $\pi(g)$ for every g in G. Hence there exists a character ω_π of $Z(G)$ such that $\pi(z) = \omega_\pi(z) \cdot 1_V$ for every z in $Z(G)$. One refers to ω_π as the *central character of π.*

(d) *Any nondegenerate $\mathscr{H}(G)$-module is associated to a unique smooth representation of G.*

To summarize, *the category of nondegenerate $\mathscr{H}(G)$-modules is identical to the category of smooth representations of G and intertwining maps.*

To conclude this section, we define the contragredient representation to a smooth representation (π, V) of G. Let K be a compact open subgroup of G. Denote by $V^*(K)$ the space of linear forms v^* on V such that $\langle v^*, \pi(e_K) \cdot v \rangle = \langle v^*, v \rangle$ for every v in V. The space $\tilde{V} = \bigcup_K V^*(K)$ is called the *smooth dual* to V. In $\tilde{V}$ there exists a smooth representation $\tilde{\pi}$ of G characterized by the relation

$$\langle \tilde{\pi}(g) \cdot \tilde{v}, v \rangle = \langle \tilde{v}, \pi(g^{-1}) \cdot v \rangle \tag{10}$$

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[5] A finite set is countable!

[6] This condition is satisfied for the algebraic groups over a local field.

for g in G, v in V and $\tilde{v}$ in $\tilde{V}$. The representation $(\tilde{\pi}, \tilde{V})$ is called the representation *contragredient* to (π, V).

1.5. *Admissible representations and their characters.* We come now to a more restricted class of representations.

DEFINITION 1.2. *A representation (π, V) of G is called admissible if it is smooth and the space V^K of vectors invariant under K is finite-dimensional for every compact open subgroup K of G.*

Fix a smooth representation (π, V) of G and a compact open subgroup K of G. Let $\mathscr{E}(K)$ be the set of (equivalence classes of) continuous irreducible finite-dimensional representations of K; every neighborhood of 1 in K contains a subgroup L of K which is compact, open and invariant in K. It follows that every representation in $\mathscr{E}(K)$ factors through the finite group K/L for such a subgroup L.

Let v be a vector in V. Since π is smooth, there exists a subgroup L as above fixing v. Let $k_1, \cdots, k_m$ be coset representatives for K modulo L. The subspace spanned by $\pi(k_1) \cdot v, \cdots, \pi(k_m) \cdot v$ is stable under K and affords a representation of the finite group K/L. It is therefore the direct sum of subspaces affording irreducible representations of K/L, hence of K. In other words, the restriction of π to K is a semisimple representation. For any $\mathfrak{d}$ in $\mathscr{E}(K)$, let $V_\mathfrak{d}$ be the subspace of V generated by the minimal K-invariant subspaces of V affording a representation of K of class $\mathfrak{d}$. The space $V_\mathfrak{d}$ is called the *isotypic component of class* $\mathfrak{d}$ of π; if ε is the one-dimensional representation of K given by $\varepsilon(k) = 1$ for k in K, then $V_\varepsilon = V^K$. More generally for any character χ of K, the space V_χ consists of the vectors v in V such that $\pi(k) \cdot v = \chi(k) \cdot v$ for every k in K.

We state now the elementary properties of admissible representations.

(a) *If (π, V) is a smooth representation of G and K is a compact open subgroup of G, then V is the direct sum $\bigoplus_{\mathfrak{d} \in \mathscr{E}(K)} V_\mathfrak{d}$. Moreover (π, V) is admissible iff $V_\mathfrak{d}$ is finite-dimensional for every $\mathfrak{d}$ in $\mathscr{E}(K)$.*

Let now $\mathfrak{d}$ in $\mathscr{E}(K)$ and let $\tilde{\mathfrak{d}}$ be the representation of K contragredient to $\mathfrak{d}$. Since the space E of $\mathfrak{d}$ is finite-dimensional, the space of $\tilde{\mathfrak{d}}$ is the dual E^* of E.

(b) *Assume (π, V) is admissible. The restriction of the pairing between $\tilde{V}$ and V to $\tilde{V}_{\tilde{\mathfrak{d}}} \times V_\mathfrak{d}$ defines a nondegenerate K-invariant bilinear form on $\tilde{V}_{\tilde{\mathfrak{d}}} \times V_\mathfrak{d}$.*

Since $V_\mathfrak{d}$ is finite-dimensional, so is $\tilde{V}_{\tilde{\mathfrak{d}}}$ and each of the spaces $V_\mathfrak{d}$ and $\tilde{V}_{\tilde{\mathfrak{d}}}$ can be identified to the dual of the other. So we get from (b) the following result:

(c) *If (π, V) is admissible, then $(\tilde{\pi}, \tilde{V})$ is admissible and the pairing between $\tilde{V}$ and V enables one to identify (π, V) with the representation contragredient to $(\tilde{\pi}, \tilde{V})$.*

We come now to the characters.

(d) *Let (π, V) be a smooth representation of G. Then π is admissible iff the operator $\pi(f)$ is of finite rank for every f in $\mathscr{H}(G)$.*

By definition a *distribution* (in the sense of Bruhat [9]) on G is a linear form on the space $\mathscr{H}(G)$ of "test functions". According to property (d) above, we can associate to any admissible representation (π, V) of G a distribution Θ_π on G by

$$\Theta_\pi(f) = \operatorname{Tr}(\pi(f)) \quad \text{for } f \text{ in } \mathscr{H}(G). \tag{11}$$

One refers to Θ_π as the *character* of π.

(e) *Let $(\pi_\alpha, V_\alpha)_{\alpha \in I}$ be a family of admissible and irreducible representations of G. Assume that π_α is inequivalent to π_β for $\alpha \neq \beta$. Then the characters Θ_{π_α} are linearly independent, hence mutually distinct.*

For the *calculation* of Θ_π we may proceed as follows. Assume to simplify matters that G has a countable basis of open sets, hence a countable basis $(K_m)_{m\geqq 1}$ for the neighborhoods of 1 consisting of a decreasing sequence of compact open subgroups. Let V_m be the space of vectors in V invariant under $\pi(K_m)$. Since π is admissible, each V_m is finite-dimensional, and $V_1 \subset V_2 \subset \cdots \subset V_m \subset \cdots$ with $V = \bigcup_m V_m$. Let g be in G. For every m, the operator $\pi(e_{K_m})\cdot\pi(g)\cdot\pi(e_{K_m})$ maps V_m into itself. Since V_m is finite-dimensional, this operator has a trace, to be denoted by $\Theta_{\pi,m}(g)$. Notice that as a function of g, $\Theta_{\pi,m}(g)$ is bi-invariant under K, hence locally constant. Let f be in $\mathscr{H}(G)$. There exists an integer $m_0 \geqq 1$ such that f belongs to $\mathscr{H}(G, K_{m_0})$ and then one gets

$$\Theta_\pi(f) = \int_G \Theta_{\pi,m}(g) f(g)\, dg \tag{12}$$

for every integer $m \geqq m_0$. This fact can also be stated as

$$\Theta_\pi = \lim_{m\to\infty} \Theta_{\pi,m} \quad \text{(weak limit in the space dual to } \mathscr{H}(G)). \tag{13}$$

For each m, let B_m be a basis of V_m over $\boldsymbol{C}$; assume that $B_1 \subset B_2 \subset \cdots \subset B_m \subset B_{m+1} \subset \cdots$. Then $B = \bigcup_m B_m$ is a basis of V over $\boldsymbol{C}$. Put $B = \{v_\alpha\}_{\alpha\in I}$; hence $B_m = \{v_\alpha\}_{\alpha\in I_m}$ for some finite subset I_m of I. Define the matrix $(\pi_{\alpha\beta}(g))$ by

$$\pi(g)\cdot v_\beta = \sum_{\alpha\in I} \pi_{\alpha\beta}(g) v_\alpha. \tag{14}$$

Then $\Theta_{\pi,m}$ can be calculated as

$$\Theta_{\pi,m}(g) = \sum_{\alpha\in I_m} \pi_{\alpha,\alpha}(g). \tag{15}$$

Hence we get the series expansion

$$\Theta_\pi = \sum_{\alpha\in I} \pi_{\alpha,\alpha} \tag{16}$$

which converges in the weak sense.

REMARK. The definitions of the convolution product by (4) and of $\pi(f)$ by (9) depend on a Haar measure μ on G. To free them from this dependence, we can proceed as follows. Let $C^\infty(G)$ be the space of locally constant complex-valued functions on G. By $C_c^\infty(G)$ we denote the space of linear forms T on $C^\infty(G)$ which satisfy the following two properties:

(1) *There exists a compact open subgroup K of G such that T is bi-invariant under K.* Namely, for f in $C^\infty(G)$ and k_1, k_2 in K, then $T(f) = T(f')$ where $f'(g) = f(k_1 g k_2)$.

(2) *T is compactly supported*, namely there exists a compact open subset Ω of G such that $T(f) = 0$ for every f in $C^\infty(G)$ which vanishes identically on Ω.

By a generalized function on G we mean a linear form on $C_c^\infty(G)$; they make up a vector space $C^{-\infty}(G)$ over $\boldsymbol{C}$. We can embed in a natural way $C^\infty(G)$ as a subspace of $C^{-\infty}(G)$.

The *convolution product* on $C_c^\infty(G)$ is defined as follows: for T_i $(i = 1, 2)$ in $C_c^\infty(G)$ choose Ω_i as in (2) above and let $\Omega = \Omega_1 \cup \Omega_2$. For f in $C^\infty(G)$ there exist functions $f'_1, \cdots, f'_m, f''_1, \cdots, f''_m$ in $C^\infty(G)$ such that

(17) $$f(g_1g_2) = \sum_{i=1}^{m} f_i'(g_1)\cdot f_i''(g_2) \quad \text{for } g_1, g_2 \text{ in } \Omega.$$

Then $(T_1 * T_2)(f)$ is defined by

(18) $$(T_1 * T_2)(f) = \sum_{i=1}^{m} T_1(f_i')\cdot T_2(f_i'').$$

For T in $C_c^\infty(G)$, the operator $\pi(T)$ in V is defined in such a way that

(19) $$\langle v^*, \pi(T)\cdot v\rangle = T(\pi_{v,v^*}) \quad \text{for } v \text{ in } V,\ v^* \text{ in } V^*$$

(notice that π_{v,v^*} belongs to $C^\infty(G)$). If π is admissible, its character is the generalized function defined by $\Theta_\pi(T) = \mathrm{Tr}(\pi(T))$ for T in $C_c^\infty(G)$.

If we choose a left invariant Haar measure μ on G, we get an isomorphism of $\mathscr{H}(G)$ onto $C_c^\infty(G)$ which associates to $u \in \mathscr{H}(G)$ the linear form $f \mapsto \int_G f(g)u(g)\,dg$ on $C^\infty(G)$. By duality, one gets an isomorphism of $C^{-\infty}(G)$ with the space of distributions. This brings us back to our previous constructions.

1.6. *Absolutely cuspidal representations.* In this section, we denote by Z a closed subgroup of the center of G. We fix a character χ of Z and a Haar measure on G/Z and assume that G is unimodular. A *χ-representation* of G is a representation (π, V) of G such that $\pi(z) = \chi(z)\cdot 1_V$ for every z in Z. If π is smooth and irreducible, this means that the restriction to Z of the central character ω_π is equal to χ (at least, when G has a countable basis of open sets).

Let π be a smooth χ-representation of G. For f in $\mathscr{H}_\chi(G)$, the linear operator $\pi(f)$ in V is defined in such a way that

(20) $$\langle v^*, \pi(f)\cdot v\rangle = \int_{G/Z} \langle v^*, \pi(g)\cdot v\rangle\, f(g)\, d\bar{g}$$

holds for v in V and v^* in V^*.

It is then easily proved that the category of smooth χ-representations of G is isomorphic to the category of nondegenerate $\mathscr{H}_\chi(G)$-modules. Moreover the irreducible smooth representations correspond to the simple $\mathscr{H}_\chi(G)$-modules.

DEFINITION 1.3. *Let χ be a character of Z. A χ-representation (π, V) of G is called absolutely cuspidal* (or supercuspidal, or parabolic according to some authors) *if it is admissible and each coefficient $\pi_{v,\tilde{v}}$* (v in V, $\tilde{v}$ in $\tilde{V}$) *is compactly supported modulo* Z (hence belongs to $\mathscr{H}_{\chi^{-1}}(G)$).

A representation is called absolutely cuspidal (w.r.t. Z) if it is an absolutely cuspidal χ-representation for some character χ of Z.

One of the main properties of absolutely cuspidal representations is embodied in the so-called *Schur orthogonality relations.*

THEOREM 1.1. *Let (π, V) be any irreducible absolutely cuspidal χ-representation of G. Assume that the character $|\chi|$ of Z can be extended to a character of G. Then there exists a constant $d(\pi) > 0$ such that the following identity*

(21) $$\int_{G/Z} \langle \tilde{v}_1, \pi(g)\cdot v_1\rangle \langle \tilde{v}_2, \pi(g^{-1})\cdot v_2\rangle\, d\bar{g} = d(\pi)^{-1} \langle \tilde{v}_1, v_2\rangle \langle \tilde{v}_2, v_1\rangle$$

holds for v_1, v_2 in V and $\tilde{v}_1$, $\tilde{v}_2$ in $\tilde{V}$.

The assumption about χ is satisfied if χ is unitary or else if G is a connected reductive algebraic group over a local field.

The number $d(\pi)$ is called the *formal degree of* π. It depends on the choice of the Haar measure on G/Z. Indeed, multiplying the Haar measure by a constant $c > 0$ amounts to replacing $d(\pi)$ by $d(\pi)/c$. More invariantly, to π is associated a Haar measure ν_π on G/Z such that

$$\int_{G/Z} \langle \tilde{v}_1, \pi(g)\cdot v_1\rangle \langle \tilde{v}_2, \pi(g^{-1})\cdot v_2\rangle \, d_\pi \bar{g} = \langle \tilde{v}_1, v_2\rangle\langle \tilde{v}_2, v_1\rangle \tag{22}$$

holds for v_1, v_2 in V and $\tilde{v}_1$, $\tilde{v}_2$ in $\tilde{V}$. We denote by $d_\pi g$ the integration w.r.t. ν_π. If K is a compact open subgroup of G/Z, the number $\nu_\pi(K)$ is well defined and may be called the formal degree of π w.r.t. K. For instance assume that π is induced from a finite-dimensional representation λ of K. Then $\nu_\pi(K)$ is the degree of λ.

Theorem 1.1 has a number of interesting corollaries. Generally speaking, let (π, V) be any admissible irreducible χ-representation of G. For any linear map $u: V \to V$, the following conditions are equivalent:

(a) There exists a function f in $\mathscr{H}_\chi(G)$ such that $u = \pi(f)$ holds.

(b) There exists a compact open subgroup K of G such that $u = \pi(k)\cdot u \cdot \pi(k')$ holds for k, k' in K.

(c) There exist vectors $v_1, \cdots, v_m$ in V and linear forms $\tilde{v}_1, \cdots, \tilde{v}_m$ in $\tilde{V}$ such that

$$u(v) = \sum_{i=1}^{m} \langle \tilde{v}_i, v\rangle \cdot v_i \tag{23}$$

holds for any v in V. The set $\mathscr{L}(\pi)$ of all such operators is a subalgebra of the algebra of all linear operators in V.

Assume now that π is absolutely cuspidal. Denote by $\mathscr{K}(\pi)$ the two-sided ideal in $\mathscr{H}_\chi(G)$ consisting of the functions f such that $\pi(f) = 0$. The vector subspace $\mathscr{A}(\pi)$ of $\mathscr{H}_\chi(G)$ generated by the functions of the form $g \mapsto \langle \tilde{v}, \pi(g^{-1})\cdot v\rangle$ is then a two-sided ideal in $\mathscr{H}_\chi(G)$ and $\mathscr{H}_\chi(G)$ *is the direct sum of* $\mathscr{A}(\pi)$ and $\mathscr{K}(\pi)$. Hence we get an isomorphism $f \mapsto \pi(f)$ of the algebra $\mathscr{A}(\pi)$ with the algebra $\mathscr{L}(\pi)$. *The inverse isomorphism* is given by $u \mapsto \varphi_{\pi,u}$ where

$$\varphi_{\pi,u}(g) = d(\pi)\cdot \mathrm{Tr}\,(u\cdot \pi(g^{-1})) \tag{24}$$

for g in G and u in $\mathscr{L}(\pi)$. Notice also that we have an isomorphism $\theta: V \otimes \tilde{V} \to \mathscr{L}(\pi)$ given by

$$\theta(v \otimes \tilde{v})\cdot v' = \langle \tilde{v}, v'\rangle \cdot v \tag{25}$$

for v, v' in V and $\tilde{v}$ in $\tilde{V}$. *The three spaces* $V \otimes \tilde{V}$, $\mathscr{L}(\pi)$ *and* $\mathscr{A}(\pi)$ *carry natural representations of* $G \times G$ *and the previous isomorphisms are equivariant w.r.t. these actions of* $G \times G$.

Let Θ_π be the character of π. The projection Λ_π of $\mathscr{H}_\chi(G)$ onto $\mathscr{A}(\pi)$ with kernel $\mathscr{K}(\pi)$ is given by $\Lambda_\pi(f) = \varphi_{\pi,\pi(f)}$; hence more explicitly[7]

[7] We abuse notation by treating Θ_π as a function!

$$\Lambda_\pi(f)(g) = d(\pi) \int_{G/Z} \Theta_\pi(xg^{-1})f(x)\, d\bar{x} \tag{26}$$

for g in G and f in $\mathscr{H}_\chi(G)$. To simplify assume that $Z = \{1\}$. Then we can rewrite formula (26) as a convolution (of a distribution and a compactly supported function!)

$$\Lambda_\pi(f) = d(\pi)\,(\Theta_\pi^\vee * f), \tag{27}$$

where $\Theta_\pi^\vee$ is the distribution on G deduced from Θ_π by the symmetry $g \mapsto g^{-1}$ of G.

From the decomposition $\mathscr{H}_\chi(G) = \mathscr{K}(\pi) \oplus \mathscr{A}(\pi)$ one deduces easily the following theorem, due to Casselman **[16]**:

THEOREM 1.2. *Let (π, V) be an irreducible absolutely cuspidal χ-representation of G. Then V is projective in the category of nondegenerate $\mathscr{H}_\chi(G)$-modules.*

There are two important corollaries.

COROLLARY 1.1. *Any absolutely cuspidal χ-representation of G is a direct sum of irreducible absolutely cuspidal χ-representations, each counted with finite multiplicity.*

The following is a converse of Schur's lemma and follows immediately from Corollary 1.1.

COROLLARY 1.2. *Let (π, V) be any absolutely cuspidal χ-representation of G. Assume every intertwining map $u\colon V \to V$ is a scalar. Then π is irreducible.*

So far we considered one absolutely cuspidal representation at a time. We give now the second half of Schur's orthogonality relations.

THEOREM 1.3. *Assume (π, V) and (π', V') are inequivalent absolutely cuspidal χ-representations of G. Then the following relation holds*

$$\int_{G/Z} \langle \tilde{v}, \pi(g)\cdot v\rangle \langle \tilde{v}', \pi(g^{-1})\cdot v'\rangle\, d\bar{g} = 0 \tag{28}$$

whatever be v in V, v' in V', $\tilde{v}$ in $\tilde{V}$ and $\tilde{v}'$ in $\tilde{V}'$.

Let Λ be a complete set of mutually inequivalent irreducible absolutely cuspidal χ-representations of G. From Theorem 1.3, it follows immediately that the sum of the two-sided ideals $\mathscr{A}(\pi)$ (for π running over Λ) is direct in $\mathscr{H}_\chi(G)$. This sum is called the *cuspidal part of* $\mathscr{H}_\chi(G)$ to be denoted by $\mathscr{H}_\chi(G)^\circ$.

1.7. *Change of groups and Frobenius reciprocity.* Let G and G' be two groups of td-type and φ a continuous homomorphism from G into G'. Denote by $\mathscr{S}_G$ ($\mathscr{S}_{G'}$) the category of smooth representations of G (G').

The *restriction functor* φ^* from $\mathscr{S}_{G'}$ to $\mathscr{S}_G$ takes the smooth representation (π', V') of G' to the smooth representation (π, V) of G, where $V = V'$ and $\pi(g) = \pi'(\varphi(g))$ for g in G.[8]

To define the *extension functor* φ_* from $\mathscr{S}_G$ to $\mathscr{S}_{G'}$, we take the view that $\mathscr{S}_G$ consists of the nondegenerate $\mathscr{H}(G)$-modules and similarly for $\mathscr{S}_{G'}$. We consider $\mathscr{H}(G')$ both as a left $\mathscr{H}(G')$-module in the obvious way and as a right $\mathscr{H}(G)$-module

[8] If φ is the injection of G into a larger group G', we write $\mathrm{Res}_G^{G'}$ instead of φ^*.

as follows: for f' in $\mathscr{H}(G')$ and f in $\mathscr{H}(G)$, we define the convolution $f' *_\varphi f$ as the function in $\mathscr{H}(G')$ whose values are given by

$$(29) \qquad (f' *_\varphi f)(g') = \int_G f'(g' \cdot \varphi(g)^{-1}) f(g) \, \Delta_{G'}(\varphi(g)^{-1}) \, dg.$$

We define now φ_* via tensor products. Indeed for any V in $\mathscr{S}_G$, we put $\varphi_* V = \mathscr{H}(G') \otimes_{\mathscr{H}(G)} V$ and define the action of $\mathscr{H}(G')$ on $\varphi_* V$ by the rule

$$(30) \qquad f_1' \cdot (f_2' \otimes v) = (f_1' * f_2') \otimes v$$

for f_1', f_2' in $\mathscr{H}(G')$ and v in V.

Let V be in $\mathscr{S}_G$ and V' in $\mathscr{S}_{G'}$. Using the universal property of tensor products, we produce a canonical map

$$(31) \qquad \Theta\colon \mathrm{Hom}_G(V, \varphi^* V') \to \mathrm{Hom}_{G'}(\varphi_* V, V')$$

as follows: for $u\colon V \to \varphi^* V'$, the linear map $\Theta(u)$ takes $f' \otimes v$ into $f' \cdot u(v)$ (f' in $\mathscr{H}(G')$, v in V).

Frobenius reciprocity is the assertion that Θ is an isomorphism. Unfortunately, as John Tate pointed out to me, this is not true in general due to the lack of unit elements in our rings. We have to introduce another functor $\varphi^!$. First of all let us define the notion of a *generalized vector* in a representation space (π, V) for G. One defines in $\mathscr{H}(G)$ the translation operators by

$$(32) \qquad L_g f(g_1) = f(g^{-1} g_1), \qquad R_g f(g_1) = \Delta_G(g) \cdot f(g_1 g)$$

for g, g_1 in G and f in $\mathscr{H}(G)$. The space $V^{-\infty}$ of generalized vectors consists of the linear maps $u : \mathscr{H}(G) \to V$ such that $uL_g = \pi(g)u$ for every g in G. We identify any vector v in V to the generalized vector $f \mapsto \pi(f) \cdot v$. The representation π of G into V is extended to $V^{-\infty}$ by $\pi^{-\infty}(g)(u) = uR_{g^{-1}}$. It is easy to show that V consists of the 'smooth' vectors in $V^{-\infty}$, that is the generalized vectors u for which there exists a compact open subgroup K of G such that $\pi^{-\infty}(k)(u) = u$ for every k in K.

We apply this construction to a smooth representation (π', V') of G'. We get a representation $(\pi'^{-\infty}, V'^{-\infty})$ of G', hence a representation of G on the same space by the operators $\pi'^{-\infty}(\varphi(g))$. One defines $\varphi^! V'$ as the set of smooth vectors for G in the space $V'^{-\infty}$. It is clear that $\varphi^* V'$ is a subspace of $\varphi^! V'$ and the carrier of a subrepresentation for G.

Let (π, V) be a smooth representation of G. One establishes easily the following facts:

(a) *The map Θ extends to an isomorphism*

$$(33) \qquad \Theta'\colon \mathrm{Hom}_G(V, \varphi^! V') \to \mathrm{Hom}_{G'}(\varphi_* V, V').$$

Hence the functor $\varphi^!$ from $\mathscr{S}_{G'}$ to $\mathscr{S}_G$ is a right adjoint to φ_*.

By general functorial results it follows that φ_* is a right exact functor and $\varphi^!$ a left exact functor.

(b) *In order that Θ be an isomorphism for every V in $\mathscr{S}_G$ and every V' in $\mathscr{S}_{G'}$, it is necessary and sufficient that $\varphi^* V' = \varphi^! V'$ for every V' in $\mathscr{S}_{G'}$.*

In order to get a counterexample to Frobenius reciprocity, it is enough to find a smooth representation (π', V') of G' such that $V'^{-\infty} \neq V'$ and to consider the group $G = \{1\}$ since $\varphi^! V' = V'^{-\infty}$ in this case. Consider the left translations acting on

$\mathscr{H}(G')$. The identity map is a generalized vector in $\mathscr{H}(G')^{-\infty}$ and is not of the form $f \mapsto f * f_0$ for a fixed f_0 in $\mathscr{H}(G')$ unless G' is discrete. Hence $\mathscr{H}(G') \neq \mathscr{H}(G')^{-\infty}$ in this case.

(c) *Let φ' be a continuous homomorphism from G' into another group G'' of td-type. Then the functors $\varphi'_* \circ \varphi_*$ and $(\varphi' \circ \varphi)_*$ from $\mathscr{S}_G$ to $\mathscr{S}_{G''}$ are naturally isomorphic.*

This follows from the associativity of tensor products.

Assume now that φ is open, that is $\varphi(U)$ is open in G' for every open set U in G. Then, for any module V' in $\mathscr{S}_{G'}$, a generalized vector in $V'^{-\infty}$ is smooth for G iff it is smooth for G', that is belongs to V'. Hence one has $\varphi^! V' = \varphi^* V'$ and by property (b) above, *Frobenius reciprocity holds.*

Let (π, V) be any smooth representation of G. We can describe more explicitly $\varphi_* V$ as follows. Let H be the kernel of φ and let $V(H)$ be the subspace of V generated by the vectors $\pi(h) \cdot v - v$ for h in H and v in V. Moreover, by Frobenius reciprocity, there is a G-homomorphism $\iota \colon V \to \varphi^* \varphi_* V$ such that $\Theta(\iota)$ is the identity map in $\varphi_* V$. Finally, let $\{g'_\alpha\}_{\alpha \in A}$ be a set of representatives for the cosets $g' \cdot \varphi(G)$ in G' (notice that $\varphi(G)$ is open in G').

(d) *With the previous notations, the kernel of the linear map ι from V into $\varphi_* V$ is equal to $V(H)$ and $\varphi_* V$ is the direct sum $\bigoplus_{\alpha \in A} \pi(g'_\alpha) \cdot \iota(V)$ where π' is the representation of G' in $\varphi_* V'$ deduced from its $\mathscr{H}(G')$-module structure.*

(e) *Assume that the kernel H of φ is the union of its open compact subgroups* (for instance H is a unipotent algebraic group over a local field, or φ is injective). *Then the functor φ_* is exact.*

For every compact open subgroup K of G, the operator $\pi(e_K)$ (see §1.3, formula (6)) is a projection of V onto the set V^K of vectors invariant under K, with kernel $V(K)$. Hence $V = V^K \oplus V(K)$. Under the assumptions made in (e) one has

$$V(H) = \bigcup_K V(K) = \bigcup_K \ker \pi(e_K)$$

where K runs over the compact open subgroups of H. Hence, for every G-invariant subspace W of V, one gets $W(H) = W \cap V(H)$, hence the exactness of φ_*.

There are two special instances of the previous results. Assume first that G is an open subgroup of G' and φ is the injection of G into G'. Then we can identify V to its image by ι in $\varphi_* V$ and then $\varphi_* V = \bigoplus_{\alpha \in A} \pi'(g'_\alpha) \cdot V$.

Assume now that $\varphi(G) = G'$. Then φ defines an isomorphism of topological groups from G/H onto G'. Moreover, ι defines an isomorphism of the linear space $V_H = V/V(H)$ onto $\varphi_* V$.

1.8. *Induced representations.* In this section, we denote by G a group of td-type and by H a closed subgroup of G.

Let (π, V) be a smooth representation of H. We denote by $\mathscr{V}_\pi$ the space of functions $f \colon G \to V$ satisfying the following assumptions:

(a) *One has $f(hg) = \pi(h) \cdot f(g)$ for h in H and g in G.*

(b) *There exists a compact open subgroup K of G such that $f(gk) = f(g)$ for g in G and k in K.*

The group G acts on $\mathscr{V}_\pi$ by right translations, namely

$$(34) \qquad (\theta_\pi(g) \cdot f)(g_1) = f(g_1 g) \quad \text{for } g, g_1 \text{ in } G, f \text{ in } \mathscr{V}_\pi.$$

The representation $(\theta_\pi, \mathscr{V}_\pi)$ of G is smooth. It is called the representation induced from π and usually denoted by $\mathrm{Ind}_H^G \pi$. One has a kind of *dual Frobenius reciprocity*, namely an isomorphism

$$\Theta^* : \mathrm{Hom}_H(\mathrm{Res}_H^G \lambda, \pi) \xrightarrow{\sim} \mathrm{Hom}_G(\lambda, \mathrm{Ind}_H^G \pi) \tag{35}$$

for every smooth representation (λ, W) of G. The proof is trivial.

The functions in $\mathscr{V}_\pi$ are locally constant, but assumption (b) is usually stronger than just local constancy. Denote by $\mathscr{V}_\pi^c$ the subspace of $\mathscr{V}_\pi$ consisting of the functions which vanish off a subset of the form $H\Omega$ where $\Omega \subset G$ is compact. For g in G, the translation operator $\theta_\pi(g)$ induces an operator $\theta_\pi^c(g)$ in $\mathscr{V}_\pi^c$. The representation $(\theta_\pi^c, \mathscr{V}_\pi^c)$ is called the c-induced representation from π and is usually denoted by c-$\mathrm{Ind}_H^G \pi$. If G/H is compact, there is no need to distinguish between $\mathscr{V}_\pi$ and $\mathscr{V}_\pi^c$, and c-$\mathrm{Ind}_H^G \pi = \mathrm{Ind}_H^G \pi$.

The adjoint of a composite functor being the composite of the adjoints in reverse order, one deduces from Frobenius reciprocity the possibility of *inducing by stage*. Namely, if L is a closed subgroup of H, there is a canonical isomorphism

$$\mathrm{Ind}_H^G \mathrm{Ind}_L^H \lambda \xrightarrow{\sim} \mathrm{Ind}_L^G \lambda \tag{36}$$

for any smooth representation (λ, W) of L. A similar property holds for the c-induced representations.

Let φ be the injection of H into G. We want to compare our functor φ_* to the induced representations. Define a character δ of H by

$$\delta(h) = \Delta_H(h)/\Delta_G(h) \quad \text{for } h \text{ in } H. \tag{35}$$

If χ is any character of H and (π, V) a smooth representation of H, the *twisted representation* $(\pi \otimes \chi, V)$ acts on the same space as π via the operators

$$(\pi \otimes \chi)(h) = \chi(h)\,\pi(h) \quad \text{for } h \text{ in } H. \tag{36}$$

THEOREM 1.4. *Let H be a closed subgroup of G and φ the injection of H into G. For every smooth representation (π, V) of H, $\varphi_* V$ is isomorphic to the c-induced representation c-$\mathrm{Ind}_H^G(\pi \otimes \delta^{-1})$.*

COROLLARY 1.3. *The functor c-Ind_H^G from $\mathscr{S}_H$ to $\mathscr{S}_G$ is exact.*

We know that φ_* is right exact (see above, p. 124). It is clear that c-Ind_H^G is a left exact functor, hence the corollary.

An explicit isomorphism $P_\pi: \varphi_* V \to \mathscr{V}^c_{\pi\otimes\delta^{-1}}$ is given as follows:

$$P_\pi(f \otimes v)(g) = \int_H f(g^{-1} h)\, \delta(h)^{-1}\, \pi(h)\cdot v\, dh \tag{37}$$

for f in $\mathscr{H}(G)$, v in V and g in G.

II. The structure of representations of $\mathfrak{p}$-adic reductive groups.

2.1. *Properties of algebraic groups.* We summarize here a few properties of algebraic groups. For a more complete exposition we refer the reader to the lectures by Springer **[41]** in these PROCEEDINGS. As usual, F is a local field.

Let $n \geqq 1$ be an integer and let G be an algebraic subgroup of $\mathrm{GL}_n(F)$. We say:

G is *unipotent* if it consists of unipotent matrices (all eigenvalues in some algebraic closure of F equal to 1);

G is a *torus* if it is connected, commutative and any element of G can be put in diagonal form in some extension of F;

G is a *split torus* if it is a torus and the eigenvalues of every element of G belong to F;

G is *reductive* if there exists no invariant connected unipotent algebraic subgroup of G with more than one element;

G is *semisimple* if it is reductive and its center is finite.

A connected reductive algebraic group G is called *split* if there exists in G a maximal torus which is split. Then every maximal split torus in G is a maximal torus.

From now on, we assume G is reductive and connected. Any split torus in G is contained in some maximal split torus of G. Any two maximal split tori in G are conjugate by an element of G, their common dimension is called the *split rank* of G. There exists in the center of G a largest split torus Z.

We do not repeat the definitions of a parabolic subgroup of G, a Borel subgroup and a quasi-split group (see **[41]**). A *parabolic pair* (P, A) consists of a parabolic subgroup P of G and a split torus A subjected to the following assumption:

If N is the unipotent radical of P (its largest unipotent invariant algebraic subgroup), there exists a connected reductive algebraic subgroup M of G such that $P = M \cdot N$ (semidirect product)[9] and A is the largest split torus contained in the center of M.

Any parabolic subgroup P can be embedded into a parabolic pair. Given P, the split torus A is unique up to conjugation by an element of N. Given (P, A), the group M is the centralizer of A in G.

One says the parabolic pair (P, A) *dominates* the parabolic pair (P', A') in case $P \supset P'$ and $A \subset A'$ hold. There exists then a parabolic subgroup P_1 of M such that $P' = P_1 \cdot N$ and (P_1, A') is a parabolic pair in M. This result is used very often in proofs by induction on the dimension of G.

2.2. *Jacquet's functors.* Let be given G, P, A, M and N as above. Define two homomorphisms

$$\begin{array}{ccc} & P & \\ {}^{\iota}\swarrow & & \searrow^{\rho} \\ G & & M \end{array}$$

were ι is the injection of P into G and $\rho(mn) = m$ for m in M and n in N. The group P is thus a kind of link between the groups G and M, and will be used to define functors relating the categories of G-modules and M-modules.

The *first Jacquet's functor* is $J_{G,M} = \rho_* \iota^* : \mathscr{S}_G \to \mathscr{S}_M$. Let (π, V) be a smooth representation of G. Since ι^* is simply the restriction Res_P^G, the space of the representation $J_{G,M}\,\pi$ is $V_N = V/V(N)$ where $V(N)$ is generated as a vector space by the elements $\pi(n) \cdot v - v$ for n in N and v in V. The representation of M on V_N is obtained from the restriction of π to M, which leaves $V(N)$ invariant since M normalizes N.

[9] The equation $P = M \cdot N$ is called the *Levi decomposition of P*.

THEOREM 2.1. *Suppose* (π, V) *is an admissible finitely generated representation of* G. *Then* $J_{G,M}(\pi, V) = (\pi_N, V_N)$ *is an admissible finitely generated representation of* M.

This theorem is a deep result essentially due to Jacquet [**32**] (see also [**17**]).

COROLLARY 2.1. *Assume* G *is quasi-split and* P *is a Borel subgroup of* G. *Then* V_N *is finite-dimensional.*

If G is quasi-split and P a Borel subgroup, M is a maximal torus in G, hence is commutative. It is then easy to check that any admissible finitely generated representation of M is finite-dimensional, hence the corollary.

The *second Jacquet's functor* is $J_{M,G} = \mathrm{Ind}_P^G \rho^* : \mathscr{S}_M \to \mathscr{S}_G$. More explicitly, this functor takes a smooth representation (λ, W) of M into $(\pi_\lambda, \mathscr{V}_\lambda)$ where $\mathscr{V}_\lambda$ is the space of functions $f : G \to W$ such that

(1) $$f(mng) = \lambda(m)\cdot f(g) \quad \text{for } g \text{ in } G,\ m \text{ in } M \text{ and } n \text{ in } N.$$

The group G acts via right translations, namely

(2) $$(\pi_\lambda(g)\cdot f)(g_1) = f(g_1 g)$$

for g, g_1 in G and f in $\mathscr{V}_\lambda$.

The following result follows easily from the compactness of G/P.

THEOREM 2.2. *Assume that* (λ, W) *is an admissible finitely generated representation of* M. *Then* $(\pi_\lambda, \mathscr{V}_\lambda)$ *is an admissible finitely generated representation of* G.

This construction is especially interesting when G is quasi-split and P is a Borel subgroup of G. We may take for (λ, W) a one-dimensional representation corresponding to a character of the maximal torus M. The corresponding representations of G comprise the *principal series* (see also §III).

The Jacquet's functors are *exact* by the results quoted in §§1.7 and 1.8. They are adjoint to each other, giving rise to a canonical isomorphism

$$\mathrm{Hom}_M(J_{G,M}\, V, W) \xrightarrow{\sim} \mathrm{Hom}_G(V, J_{M,G}\, W)$$

for any smooth representation (λ, W) of M and any smooth representation (π, V) of G.

2.3. *The main theorems.* Let G, P, A, M and N be as before. The parabolic pair (P, A) dominates a parabolic pair (P', A') where P' is a minimal parabolic subgroup of G; hence A' is a maximal split torus in G. Let Φ be the root system of G w.r.t. A' and Δ be the basis of Φ associated to P'. There exists a subset Θ of Δ such that A is the largest torus contained in the intersection of the kernels of the elements of Θ (we view the roots α in Φ as homomorphisms from A' to $F^\times$). For any real number $\varepsilon > 0$, let $A^-(\varepsilon)$ consist of the elements a in A such that $|\alpha(a)|_F \leqq \varepsilon$ for every root α in $\Delta\backslash\Theta$. Moreover, let $\bar{P}$ be the parabolic subgroup of G opposite to P. The roots associated to $\bar{P}$ are the roots $-\alpha$ where α runs over the roots associated to P. Let $\bar{N}$ be the unipotent radical of $\bar{P}$.

Let now (π, V) be any admissible finitely generated representation of G. Let $(\tilde{\pi}, \tilde{V})$ be the representation contragredient to (π, V). Using first Jacquet's functor we get representations (π_N, V_N) and $(\pi_{\bar{N}}, V_{\bar{N}})$ of M.

The following theorem is due to Casselman [**17**]. It plays a crucial role in the representation theory of reductive 𝔭-adic groups.

THEOREM 2.3. *There exists a unique M-invariant nondegenerate pairing* $\langle \cdot, \cdot \rangle_N$ *between* V_N *and* $\tilde{V}_{\bar{N}}$ *with the following property*:

Given v *in* V *and* $\tilde{v}$ *in* $\tilde{V}$, *with canonical images* u *in* V_N *and* $\tilde{u}$ *in* $\tilde{V}_{\bar{N}}$ *respectively, there exists a real number* $\varepsilon > 0$ *such that*

$$\langle \tilde{v}, \pi(a)\cdot v\rangle = \langle \tilde{u}, \pi_N(a)\cdot u\rangle_N \tag{4}$$

holds for every a *in* $A^-(\varepsilon)$.

As a matter of fact, the previous pairing identifies $(\tilde{\pi}_N, \tilde{V}_{\bar{N}})$ to the representation of M contragredient to (π_N, V_N).

The following criterion is due to Jacquet [**32**] and results easily from Theorem 2.3.

THEOREM 2.4. *Let* (π, V) *be any irreducible admissible representation of* G. *The following assertions are equivalent*:

(1) (π, V) *is absolutely cuspidal.*

(2) *For every parabolic subgroup* $P \neq G$ *of* G, *with unipotent radical* N, *we have* $V_N = 0$.

Note that 'absolutely cuspidal' is with reference to the maximal split torus Z in the center $Z(G)$ of G.[10]

An alternate formulation of Theorem 2.4 is as follows. Choose a character χ of Z and recall that $\mathscr{H}_\chi(G)^\circ$ is the subspace of $\mathscr{H}_\chi(G)$ generated by the coefficients of the irreducible absolutely cuspidal χ-representations of G. Then the following conditions are equivalent:

(a) f *belongs to* $\mathscr{H}_\chi(G)^\circ$;

(b) f *belongs to* $\mathscr{H}_\chi(G)$ *and*

$$\int_N f(gn)\, dn = 0 \tag{5}$$

holds for g *in* G *and every subgroup* N *as in Theorem* 2.4.

The next result is again due to Jacquet [**32**]. It is easily proved by induction on the split rank of G/Z.

THEOREM 2.5. *Let* (π, V) *be any irreducible admissible representation of* G. *There exist a parabolic pair* (P, A) *with associated Levi decomposition* $P = M\cdot N$ *and an irreducible absolutely cuspidal representation* (λ, W) *of* M *such that* π *is isomorphic to a subrepresentation of* $\mathrm{Ind}_P^G\, \lambda_1$ (*where* λ_1 *is the extension of* λ *to* $P = M\cdot N$ *given by* $\lambda_1(mn) = \lambda(m)$).

In principle, the classification problem for irreducible admissible representations of G is split into two problems:

(a) Find all irreducible absolutely cuspidal representations for G and for the groups M which occur as centralizer of A for some parabolic pair (P, A) in G.

[10] It is well known that $Z(G)/Z$ is a compact group. It makes no essential difference to take Z or $Z(G)$ as central subgroup. The choice of Z is quite convenient however.

(b) Study the decomposition of the induced representations $\mathrm{Ind}_P^G\ \lambda_1$ as above, in particular look for irreducibility criteria.

Needless to say, a general answer to these problems is not yet in sight. The construction of some absolutely cuspidal representations has been given by Shintani **[40]** for $\mathrm{GL}_n(F)$ and in more general cases by Gérardin in his thesis **[21]**. The idea is to induce from some compact open subgroups.

As to problem (b), let us mention the notion of *associated parabolic subgroups*. Two parabolic pairs (P, A) and (P', A') are associated iff A and A' are conjugate by some element in G. Let $\mathscr{T}(P, A)$ be the set of irreducible admissible representations of G which occur in a composition series of some induced representation $\mathrm{Ind}_P^G\ \lambda_1$ where λ is an irreducible absolutely cuspidal representation of M and M is the centralizer of A in G. If (P, A) and (P', A') are associated, then $\mathscr{T}(P, A) = \mathscr{T}(P', A')$.

2.4. *An example*. Take for instance the group $G = \mathrm{GL}_n(F)$. For any (ordered) partition $n = n_1 + \cdots + n_r$ of n, let $P_{n_1, \cdots, n_r}$ consist of the matrices in block form $g = (G_{kl})_{1 \leq k, l \leq r}$ where G_{kl} is an $n_k \times n_l$ -matrix and $G_{kl} = 0$ if $k > l$. Let $A_{n_1, \cdots, n_r}$ be the set of matrices which in block form are such that $G_{kl} = 0$ for $k \neq l$ and G_{kk} is a scalar matrix $a \cdot I_{n_k}$. For $M_{n_1, \cdots, n_r}$ take the matrices in diagonal block form, i.e., $G_{kl} = 0$ for $k \neq l$ and let finally $N_{n_1, \cdots, n_r}$ be the subgroup of $P_{n_1, \cdots, n_r}$ defined by the conditions $G_{11} = I_{n_1}, \cdots, G_{rr} = I_{n_r}$. Then $(P_{n_1, \cdots, n_r}, A_{n_1, \cdots, n_r})$ is a parabolic pair, with associated Levi decomposition $P_{n_1, \cdots, n_r} = M_{n_1, \cdots, n_r} \cdot N_{n_1, \cdots, n_r}$. Up to conjugation, the pairs $(P_{n_1, \cdots, n_r}, A_{n_1, \cdots, n_r})$ comprise all parabolic pairs in $\mathrm{GL}_n(F)$. The pairs corresponding to partitions $n = n_1 + \cdots + n_r$ and $n = m_1 + \cdots + m_s$ are associated iff $r = s$ and $n_1, \cdots, n_r$ is a permutation of $m_1, \cdots, m_s$. Notice that the group $M_{n_1, \cdots, n_r}$ is isomorphic to $\mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_r}(F)$.

The well-known operation $\alpha \circ \beta$ on characters of the various finite groups $\mathrm{GL}_n(q)$, introduced by Green **[24]**, may now be generalized to the case of a local field. Namely, let $n = n' + n''$, let (π', V') be an admissible representation of $\mathrm{GL}_{n'}(F)$ and (π'', V'') an admissible representation of $\mathrm{GL}_{n''}(F)$. Define a representation λ_1 of $P_{n', n''}$ acting on the space $V' \otimes V''$ by

$$\lambda_1 \begin{pmatrix} G_{11} & G_{12} \\ 0 & G_{22} \end{pmatrix} = \pi'(G_{11}) \otimes \pi''(G_{22}).$$

By induction from $P_{n', n''}$ to $\mathrm{GL}_n(F)$ one gets a representation $\pi' \circ \pi''$ of $\mathrm{GL}_n(F)$. This product is commutative and associative. In particular, given characters $\alpha_1, \cdots, \alpha_n$ of $F^\times$ one gets a representation $\alpha_1 \circ \cdots \circ \alpha_n$ of $\mathrm{GL}_n(F)$; these representations comprise the *principal series*. Given a partition $n = n_1 + \cdots + n_r$, the *intermediate series* associated to the parabolic group $P_{n_1, \cdots, n_r}$ consists of the representations of the form $\pi_1 \circ \cdots \circ \pi_r$ where π_j is an irreducible absolutely cuspidal representation of $\mathrm{GL}_{n_j}(F)$ for $j = 1, \cdots, r$. Finally the *discrete series* is the family of irreducible absolutely cuspidal representations of $\mathrm{GL}_n(F)$.

From the general results summarized in §2.3, one infers that any irreducible admissible representation of $\mathrm{GL}_n(F)$ is contained in some representation of the form $\pi_1 \circ \cdots \circ \pi_r$ where $n = n_1 + \cdots + n_r$ and π_j belongs to the discrete series of $\mathrm{GL}_{n_j}(F)$. The question of the irreducibility of the representations $\pi_1 \circ \cdots \circ \pi_r$ has now been completely settled by Bernshtein and Zhelevinski **[3]**.

2.5. *Square-integrable representations.* For the results expounded in this section, see Harish-Chandra and van Dijk [**28**, especially part I].

Let again Z be the maximal split torus contained in the center of G. Fix a unitary character χ of Z. We let $\mathscr{E}_2(G, \chi)$ denote the (equivalence classes of) irreducible unitary representations (π, V) of G which satisfy the following two conditions:

$$\pi(gz) = \chi(z)\,\pi(g) \quad \text{for } z \in Z, g \in G, \tag{6}$$

$$\int_{G/Z} |(u|\pi(g)\cdot v)|^2 \, d\bar{g} < +\infty. \tag{7}$$

Here $(u|v)$ denotes the scalar product in the Hilbert space V.[11] Due to assumption (6), one gets

$$|(u|\pi(gz)\cdot v)|^2 = |(u|\pi(g)\cdot v)|^2;$$

hence we can integrate over G/Z in formula (7). The integral (7) is equal to $d(\pi)\|u\|^2\|v\|^2$ where the constant $d(\pi) > 0$, the *formal degree of* π, is independent from u and v. These representations are called as usual *square-integrable*.

Fix now a compact open subgroup K of G and a class $\mathfrak{d}$ of irreducible continuous representations of K. Every representation of class $\mathfrak{d}$ acts on a space of finite dimension, to be denoted by deg $\mathfrak{d}$. Moreover, for any unitary representation (π, V) of G, denote by $(\pi : \mathfrak{d})$ the multiplicity of $\mathfrak{d}$ in the restriction of π to K.

The following theorem is easy to prove (see [**28**, p. 6]).

THEOREM 2.6. *Given K, $\mathfrak{d}$ and χ as above, one gets*

$$\sum_{\pi \in \mathscr{E}_2(G, \chi)} d(\pi)\,(\pi : \mathfrak{d}) \leqq \deg \mathfrak{d}/\mathrm{meas}(K/(Z \cap K)) \tag{8}$$

and in particular

$$(\pi : \mathfrak{d}) \leqq \frac{\deg \mathfrak{d}}{d(\pi)\cdot \mathrm{meas}(K/(Z \cap K))} \tag{9}$$

for every square-integrable irreducible representation π of G.

Let (π, V) be in $\mathscr{E}_2(G, \chi)$. If $\mathfrak{d}$ is the unit representation of K, the integer $(\pi : \mathfrak{d})$ is the dimension of the space V^K of vectors in V invariant under $\pi(K)$. Let $V_\infty = \bigcup_K V^K$ where K runs over the compact open subgroups of G. It is easy to check that V_∞ is dense in V and stable under $\pi(G)$. By the previous theorem we get an admissible χ-representation (π_∞, V_∞) of G. Moreover for any function f in $\mathscr{H}(G)$, the operator $\pi(f) = \int_G f(g)\pi(g)\,dg$ in the Hilbert space V has a finite-dimensional range (contained in V^K if f belongs to $\mathscr{H}(G, K)$), hence a trace $\Theta_\pi(f) = \mathrm{Tr}(\pi(f))$. To sum up, *every square-integrable irreducible representation has a character Θ_π, a distribution on G.*

2.6. *The fundamental estimate.* Fix a parabolic pair (P, A) where P is a minimal parabolic subgroup of G; hence A is a maximal split torus in G. Let N be the unipotent radical of P and $\tilde{N}$ the unipotent radical of the parabolic subgroup $\tilde{P}$ op-

[11] The scalar product $(u \mid v)$ is assumed to be linear in the *second* argument v.

posite to P. Hence there exists a basis Δ of the root system of G w.r.t. A such that N (resp. $\tilde{N}$) is associated with the roots α with positive (resp. negative) coefficients when expressed in terms of Δ.

Let A^- be the set of elements a in A such that $|\alpha(a)|_F \leqq 1$ for every root α in Δ. According to Bruhat and Tits [13], there exist a compact subgroup L of A and a finitely generated semigroup S in A such that $A^- = ZLS$. Moreover there exist finitely many elements $g_1, \cdots, g_m$ in G and a compact open subgroup K_0 of G such that $G = \bigcup_{1 \leqq i \leqq m} K_0 SZg_iK_0$ ('Cartan decomposition').

Let K be a compact open subgroup of G. We make the following assumptions:

(a) *K is invariant in K_0;*

(b) *one has $K = (K \cap P)\cdot(K \cap \tilde{N})$ and*

$$a^{-1}(K \cap \tilde{N})a \subset K \cap \tilde{N}, \qquad a(K \cap P)a^{-1} \subset K \cap P$$

for every a in S (hence the inner automorphism $g \mapsto aga^{-1}$ of G expands $K \cap \tilde{N}$ and contracts $K \cap P$).

It is known that every neighborhood of the unit element in G contains such a subgroup K.

The following estimate is due to Bernshtein [1].

THEOREM 2.7. *The compact open subgroup K of G is as above. Then there exists a constant $N = N(G, K) > 0$ such that every simple module over $\mathscr{H}(G, K)$ either is infinite-dimensional (over $\mathbf{C}$) or else has a dimension bounded by N.*

Let χ be a unitary character of Z and $\mathscr{H}_\chi(G, K)^\circ$ be the subalgebra of $\mathscr{H}_\chi(G)^\circ$ consisting of the functions invariant under right and left translation by an element of K. If (π, V) is an irreducible absolutely cuspidal χ-representation of G, then V^K is a simple module over $\mathscr{H}(G, K)$; hence $\dim V^K \leqq N$ by Theorem 2.7. By a well-known argument due to Godement [23], one infers from this bound the following corollary:

COROLLARY 2.2. *There is an integer $p \geqq 2$ such that the higher commutator*

$$[f_1, \cdots, f_p] = \sum_{\sigma \in S_p} \operatorname{sgn}(\sigma) f_{\sigma(1)} \cdots f_{\sigma(p)} \tag{10}$$

vanishes for arbitrary elements $f_1, \cdots, f_p$ in $\mathscr{H}_\chi(G, K)^\circ$.

2.7. *Properties of unitary representations.* In his lectures [28], Harish-Chandra was unable to prove Corollary 2.2, and had to assume it[12] in order to establish the following result:

THEOREM 2.8. *Let K be a compact open subgroup of G and $\mathfrak{d}$ be a class of irreducible continuous representations of K. There exists a constant $N = N(G, K, \mathfrak{d}) > 0$ such that $(\pi : \mathfrak{d}) \leqq N$ for every irreducible unitary representation (π, V) of G.*

In the proof, one may assume that K is as in Bernshtein's Theorem 2.7 and that $\mathfrak{d}$ is the unit class of representations of K.[12]

As before (see end of §2.5), one deduces from Theorem 2.8 the following corollaries:

[12] Harish-Chandra assumes apparently the stronger 'Conjecture I' (p. 16 of [28]). But the proof (p. 18, end of first paragraph) uses only Corollary 2.2 above.

COROLLARY 2.3. *Let (π, V) be any irreducible unitary representation of G. Let V_∞ be the space of vectors in V stable by $\pi(K)$ for some compact open subgroup K of G. Then V_∞ is dense in V and stable under G, hence affords a smooth representation π_∞ of G. The representation (π_∞, V_∞) is admissible.*

COROLLARY 2.4. *Let (π, V) be any irreducible unitary representation of G. For any function f in $\mathscr{H}(G)$ the operator $\pi(f) = \int_G f(g)\pi(g)\, dg$ in V has a finite-dimensional range, hence a trace $\Theta_\pi(f)$.*

The distribution Θ_π is called the *character* of π.

By an easy limiting process one deduces from Corollary 2.4 that $\pi(f)$ is a compact (= completely continuous) operator in the Hilbert space V for every integrable function f on G. Hence the group G belongs to the category *CCR* of Kaplansky. In particular (see Dixmier **[19]**) every factor unitary representation of G is a multiple of an irreducible representation, there exists a Plancherel formula, ...

2.8. *Some other results.* Much more is now known about the characters of the irreducible unitary representations of G. The character Θ_π is for instance represented by a locally integrable function on G, which is locally constant on the set of regular elements (see Harish-Chandra **[26]** and my Bourbaki report **[14]**).

Moreover it is now known that Conjecture II in Harish-Chandra's lectures holds true. More precisely, if the Haar measure on G/Z is suitably normalized, the formal degree of any irreducible absolutely cuspidal representation of G is an integer. As a corollary (see **[28]**, part III]), the algebra $\mathscr{H}_\chi(G, K)^\circ$ is finite-dimensional and its dimension is bounded by a constant depending on G and K, but not on χ. Also for any character χ of Z, any compact open subgroup K of G and any class $\mathfrak{d}$ of irreducible continuous representations of K, there exist only finitely many irreducible absolutely cuspidal representations π of G such that $\omega_\pi = \chi$ and $(\pi : \mathfrak{d}) \neq 0$.

Let (π, V) be a smooth representation of G. We say (π, V) is *preunitary* if there exists a hermitian form Φ on V such that $\Phi(v, v) > 0$ for $v \neq 0$ in V and

$$\Phi(\pi(g)\cdot v, \pi(g)\cdot v') = \Phi(v, v') \tag{11}$$

for v, v' in V and g in G. We can then complete V to a Hilbert space $\hat{V}$ and extend by continuity $\pi(g)$ to a unitary operator $\hat{\pi}(g)$ in $\hat{V}$. Then $(\hat{\pi}, \hat{V})$ is a unitary representation of G. If (π, V) is admissible, V is exactly the set of smooth vectors in $\hat{V}$, that is $V = \bigcup_K \hat{V}^K$ where K runs over the compact open subgroups in G.

It follows from Theorem 2.8 and these remarks that *the classification of the irreducible unitary representations of G amounts to the search of the preunitary representations among the irreducible admissible representations of G.*

III. Unramified principal series of representations.

3.1. *Preliminaries about tori.* For this section only, we denote by k a (commutative) infinite field. Let $\boldsymbol{G}_m$ be the *multiplicative group in one variable*, considered as an algebraic group defined over k. To a connected algebraic group H defined over k, we associate two finitely generated free $\boldsymbol{Z}$-modules, namely

$$X^*(H) = \mathrm{Hom}_{k\text{-gr}}(H, \boldsymbol{G}_m), \qquad X_*(H) = \mathrm{Hom}_{\boldsymbol{Z}}(X^*(H), \boldsymbol{Z}).$$

In these formulas, $\mathrm{Hom}_{k\text{-gr}}$ (resp. $\mathrm{Hom}_{\boldsymbol{Z}}$) means the group of homomorphisms of algebraic groups defined over k (resp. of $\boldsymbol{Z}$-modules). We denote by $\langle\varphi, \lambda\rangle$ (for

φ in $X_*(H)$ and λ in $X^*(H)$) the pairing between $X_*(H)$ and $X^*(H)$. We can as well use this pairing to identify $X^*(H)$ to $\mathrm{Hom}_{\boldsymbol{Z}}(X_*(H), \boldsymbol{Z})$.

Let now S be a split torus defined over k. A sequence $(\lambda_1, \cdots, \lambda_n)$ is a basis of the free $\boldsymbol{Z}$-module $X^*(S)$ iff the mapping $s \mapsto (\lambda_1(s), \cdots, \lambda_n(s))$ is an isomorphism from S onto the product $(\boldsymbol{G}_m)^n = \boldsymbol{G}_m \times \cdots \times \boldsymbol{G}_m$ (n factors). Moreover we may identify $X_*(S)$ to $\mathrm{Hom}_{k\text{-gr}}(\boldsymbol{G}_m, S)$ in such a way that the following relation holds

$$\lambda(\varphi(t)) = t^{\langle \varphi, \lambda \rangle} \tag{1}$$

for φ in $X_*(S)$, λ in $X^*(S)$ and t in k.

By construction, the elements of $X^*(S)$ are polynomial functions on S and it is easily shown that they form a basis of the k-algebra A of such functions. Otherwise stated, S is the spectrum of the group algebra $A = k[X^*(S)]$ of the group $X^*(S)$ with coefficients in k. For any commutative k-algebra L, the L-points of S correspond therefore to the k-algebra homomorphisms from A into L, hence an isomorphism $S(L) \simeq \mathrm{Hom}(X^*(S), L^\times)$. From the duality between $X^*(S)$ and $X_*(S)$ we get another isomorphism $S(L) \simeq X_*(S) \otimes_{\boldsymbol{Z}} L^\times$.

3.2. *Unramified characters*. Let H be a connected algebraic group defined over our local field F. There exists a homomorphism $\mathrm{ord}_H : H \to X_*(H)$ characterized by

$$\langle \mathrm{ord}_H(h), \lambda \rangle = \mathrm{ord}_F(\lambda(h)) \tag{2}$$

for h in H and λ in $X^*(H)$. In the right-hand side of this formula, $\mathrm{ord}_F(\lambda(h))$ is the valuation of the element $\lambda(h)$ of $F^\times$. We denote by ${}^\circ H$ the kernel and by $\Lambda(H)$ the image of the homomorphism ord_H. By construction, one gets an exact sequence

$$1 \longrightarrow {}^\circ H \longrightarrow H \xrightarrow{\mathrm{ord}_H} \Lambda(H) \longrightarrow 1. \tag{S}$$

We can also describe ${}^\circ H$ as the set of elements h in H such that $\lambda(h) \in \mathfrak{O}_F^\times$ for any rational homomorphism λ from H into $F^\times$. Therefore ${}^\circ H$ is an open subgroup of H.

A character χ of H is called *unramified* if it is trivial on ${}^\circ H$. Otherwise stated, an unramified character is of the form $u \circ \mathrm{ord}_H$ where u is a homomorphism from $\Lambda(H)$ into $\boldsymbol{C}^\times$. Introduce the complex algebraic torus $T = \mathrm{Spec}\, \boldsymbol{C}[\Lambda(H)]$. By definition, one has $\Lambda(H) = X^*(T)$ and $T(\boldsymbol{C}) = \mathrm{Hom}(\Lambda(H), \boldsymbol{C}^\times)$. Thus, there exists a well-defined isomorphism $t \mapsto \chi_t$ between the group $T(\boldsymbol{C})$ of complex points of the torus T and the group of unramified characters of H. If H is a torus, one has

$$\chi_t(\varphi(\tilde{\omega}_F)) = \varphi(t) \tag{3}$$

for t in $T(\boldsymbol{C})$, φ in $X^*(H) = X_*(T)$ and any prime element $\tilde{\omega}_F$ of the field F.

Let again G be a connected reductive algebraic group defined over the local field F. We fix a *maximal split torus* A in G and denote by M its centralizer in G. We let $N(A)$ be the normalizer of A in G and $W = N(A)/M$ be the corresponding Weyl group. We choose also a parabolic group P such that (P, A) is a parabolic pair. Hence P is a *minimal parabolic subgroup* of G and $P = M \cdot N$ where N is the unipotent radical of P.

We denote by Φ the set of roots of G w.r.t. A and by Λ the group $\Lambda(M)$. Hence Φ is a subset of $X^*(A)$ and P defines a basis Δ of Φ. Let Δ^- be the set of roots opposite to the roots in Δ. The basis Δ^- of Φ corresponds to a parabolic subgroup

$P^- = M \cdot N^-$ of G. Let $\mathfrak{n}$ be the Lie algebra of N. For any m in M, the adjoint representation defines an automorphism $\mathrm{Ad}_{\mathfrak{n}}(m)$ of $\mathfrak{n}$. We set

$$\delta(m) = |\det \mathrm{Ad}_{\mathfrak{n}}(m)|_F \quad \text{for } m \text{ in } M. \tag{4}$$

The group ${}^\circ A$ (resp. ${}^\circ M$) is the largest compact subgroup of A (resp. M) and ${}^\circ A$ is equal to ${}^\circ M \cap A$. Thus the inclusion of A into M gives rise to an injective homomorphism of $A/{}^\circ A$ into $M/{}^\circ M$. We do not know in general if this map is surjective (see however Borel [**5**, 9.5]). More precisely, the inclusion of A into M gives rise to a commutative diagram with exact lines

$$\text{(D)} \qquad \begin{array}{ccccccccc} 1 & \longrightarrow & {}^\circ A & \longrightarrow & A & \xrightarrow{\mathrm{ord}_A} & X_*(A) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & {}^\circ M & \longrightarrow & M & \xrightarrow{\mathrm{ord}_M} & \Lambda & \longrightarrow & 1 \end{array}$$

Indeed, since A is a split torus, ord_A is surjective. The inclusion of A into M enables us to identify $X_*(A)$ to a subgroup of finite index in $X_*(M)$, hence the relation $X_*(A) \subset \Lambda \subset X_*(M)$.

We denote by X the group of unramified characters of M. We may (and shall) introduce as before a complex torus T such that $X^*(T) = \Lambda$ and an isomorphism $t \mapsto \chi_t$ of $T(\boldsymbol{C})$ onto X. This isomorphism enables us to consider X as a complex Lie group.

The subgroup $N(A)$ of G acts on M, ${}^\circ M$, A, ${}^\circ A$ via inner automorphisms. Using diagram (D) above, we may let the Weyl group $W = N(A)/M$ operate on $X_*(M)$ so as to leave invariant the subgroups $X_*(A)$ and Λ of $X_*(M)$. The group W acts therefore on X and T by automorphisms of complex Lie groups. For instance, if χ is any unramified character of M and w any element of the Weyl group W, the transformed character $w\chi$ is given by

$$(w\chi)(m) = \chi(x_w^{-1} m x_w) \quad \text{for } m \text{ in } M, \tag{5}$$

where x_w is any representative of w in $N(A)$. The unramified character χ of M is called *regular* if $w\chi \neq \chi$ for every element $w \neq 1$ of W.

For the applications to automorphic functions, one has to examine the case where G is *unramified over F*, that is the following hypotheses are fulfilled:

(a) *G is quasi-split over F.*

(b) *There exists an unramified extension F' of F, of finite degree d, such that G splits over F'.*

Let σ denote the Frobenius transformation of F' over F. In this situation, the L-group associated to G is defined. It is a complex connected reductive algebraic group ${}^LG^\circ$ endowed (at least) with a complex torus T', an automorphism $g' \mapsto g'^\sigma$ such that $T'^\sigma = T'$ and a homomorphism $t' \mapsto \chi'_{t'}$ of T' onto X. We say two elements g'_1 and g'_2 of ${}^LG^\circ$ are *σ-conjugate* if there exists h in ${}^LG^\circ$ such that $g'_2 = h^{-1} g'_1 h^\sigma$.

The following theorem has been proved by Gantmacher [**20**] and Langlands [**35**].

THEOREM 3.1. (a) *Any semisimple element in ${}^LG^\circ$ is σ-conjugate to an element of T'.*

(b) *Two elements* t'_1 *and* t'_2 *of* T' *are* σ*-conjugate iff the unramified characters* $\chi'_{t'_1}$ *and* $\chi'_{t'_2}$ *of* M *are conjugate under the action of the Weyl group* W.

Otherwise stated, the orbits of W in the group X of unramified characters of M are in a bijective correspondence to the σ-conjugacy classes of semisimple elements in ${}^LG^\circ$.

For more details, we refer the reader to Borel's lectures [**5**, §6, 9.5] in these PROCEEDINGS.

3.3. *The unramified principal series.* This series shall presently be defined via induction from P with the slight adjustment of $\delta^{1/2}$.

DEFINITION 3.1. *Let* χ *in* X *be any unramified character of* M. *We define the representation* $(\nu_\chi, I(\chi))$ *of* G *as follows*:

(a) *The space* $I(\chi)$ *consists of the locally constant functions* $f: G \to \mathbf{C}$ *such that*

$$f(mng) = \delta(m)^{1/2}\,\chi(m)f(g) \quad \text{for } m \text{ in } M,\ n \text{ in } N,\ g \text{ in } G. \tag{6}$$

(b) *The group* G *acts by right translations on* $I(\chi)$, *namely*

$$(\nu_\chi(g)\cdot f)(g') = f(g'g) \quad \text{for } f \text{ in } I(\chi),\ g,\ g' \text{ in } G. \tag{7}$$

It is important to give an alternate description of $I(\chi)$ as a factor space of $\mathscr{H}(G)$. Indeed one defines a surjective linear map $P_\chi : \mathscr{H}(G) \to I(\chi)$ by

$$P_\chi f(g) = \int_M \int_N \delta^{1/2}(m)\chi^{-1}(m) f(mng)\, dm\, dn. \tag{8}$$

(See formula (37) in §1.8.) The groups M and N are unimodular, hence the Haar measures dm on M and dn on N are left *and* right invariant. The map P_χ intertwines the right translations on $\mathscr{H}(G)$ with the representation ν_χ acting on the space $I(\chi)$.

From the general results described in §II, one gets immediately the following theorem.

THEOREM 3.2. (a) *For every* χ *in* X, *the representation* $(\nu_\chi, I(\chi))$ *of* G *is admissible.*

(b) *The representation* $I(\chi^{-1})$ *is isomorphic to the contragredient* $(I(\chi))^\sim$ *of* $I(\chi)$.

(c) *If* χ *is a unitary unramified character of* M, *the representation* $(\nu_\chi, I(\chi))$ *of* G *is preunitary.*

One of the reasons for inserting the factor $\delta^{1/2}$ in the definition of $I(\chi)$ is to get assertions (b) and (c) above. We state them more precisely: there exists a linear form J on $I(\delta^{1/2})$ invariant under the right translations by the elements of G and characterized by

$$J(P_{\delta^{1/2}} f) = \int_G f(g)\, dg \quad \text{for } f \text{ in } \mathscr{H}(G) \tag{9}$$

(see Bourbaki [**6**, p. 41] for similar calculations). For f in $I(\chi)$ and f' in $I(\chi^{-1})$, the function ff' belongs to $I(\delta^{1/2})$ and the pairing is given by

$$\langle f, f' \rangle = J(ff'). \tag{10}$$

Similarly, for χ unitary and f_1, f_2 in $I(\chi)$ the function $\bar{f}_1 f_2$ belongs to $I(\delta^{1/2})$ and the unitary scalar product in $I(\chi)$ is given by

$$(f_1 | f_2) = J(\bar{f}_1 f_2). \tag{11}$$

We now state one of the main results about irreducibility and equivalence (see also Theorem 3.10 below).

THEOREM 3.3. *Let χ be any unramified character of M.*

(a) *If χ is unitary and regular, the representation $(\nu_\chi, I(\chi))$ is irreducible.*

(b) *Let w be in W. The representations $(\nu_\chi, I(\chi))$ and $(\nu_{w\chi}, I(w\chi))$ have the same character, hence are equivalent if they are irreducible.*

(c) *The $\mathscr{H}(G)$-module $I(\chi)$ is of finite length.*

In general, if V is a module of finite length over any ring, with a Jordan-Hölder series $0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V$, the semisimple module $V_{(s)} = \bigoplus_{i=1}^{n} V_i/V_{i-1}$ is called the *semisimplified* form of V. According to Jordan-Hölder theorem, it is uniquely defined by V up to isomorphism.

Let $I(\chi)_{(s)}$ be the semisimplified form of $I(\chi)$. It exists by Theorem 3.3(c) above. It is clear that $I(\chi)$ and $I(\chi)_{(s)}$ have the same character. Hence for any w in W, the representations $I(\chi)_{(s)}$ and $I(w\chi)_{(s)}$ are semisimple and have the same character. By the linear independence of characters, they are therefore isomorphic.

3.4. *Structure of Jacquet's module $I(\chi)_N$.* For any unramified character χ of M, let $\boldsymbol{C}_\chi$ denote the one-dimensional complex space $\boldsymbol{C}^1$ on which M acts via χ, viz. by $(m, z) \mapsto \chi(m)\cdot z$. Frobenius reciprocity takes here a simple form, namely (see §2.2):

THEOREM 3.4. *Let (π, V) be any smooth representation of G. For any unramified character χ of M, one gets an isomorphism* $\mathrm{Hom}_G(V, I(\chi)) \xrightarrow{\sim} \mathrm{Hom}_M(V_N, \boldsymbol{C}_{\chi\delta^{1/2}})$.

The proof is obvious. Indeed the relation

$$\Phi(v)(g) = \langle \varphi, \pi(g)\cdot v\rangle \quad (\text{for } g \text{ in } G,\ v \text{ in } V) \tag{12}$$

expresses an isomorphism $\Phi \leftrightarrow \varphi$ of $\mathrm{Hom}_G(V, I(\chi))$ with the space of linear forms φ on V such that

$$\langle \varphi, \pi(mn)\cdot v\rangle = \delta^{1/2}(m)\chi(m)\langle \varphi, v\rangle \tag{13}$$

for v in V, m in M and n in N. Recall that $V_N = V/V(N)$ where $V(N)$ is generated by the vectors $\pi(n)\cdot v - v$ for n in N and v in V. Any solution φ of (13) vanishes on $V(N)$, hence factors through V_N.

The previous theorem exemplifies the relevance of Jacquet's module $I(\chi)_N$ in the study of the intertwining operators between representations of the unramified principal series. We know by Theorems 2.1, 3.2 and 3.3(c) that $I(\chi)_N$ is a finite-dimensional complex vector space.

The following basic result is due to Casselman **[17]**.

THEOREM 3.5. *For any unramified character χ of M, the semisimplified form of the M-module $I(\chi)_N$ is $\bigoplus_{w\in W} \boldsymbol{C}_{(w\chi)\cdot\delta^{1/2}}$. Moreover, the group ${}^\circ M$ acts trivially on $I(\chi)_N$.*

COROLLARY 3.1. *The dimension of $I(\chi)_N$ over $\boldsymbol{C}$ is equal to the order $|W|$ of the Weyl group W.*

COROLLARY 3.2. *Assume χ regular. Then $I(\chi)_N$ as an M-module is isomorphic to $\bigoplus_{w\in W} \boldsymbol{C}_{(w\chi)\cdot\delta^{1/2}}$.*

For the proof of Corollary 3.2, notice that M acts on $I(\chi)_N$ through the commutative group $M/{}^\circ M$ isomorphic to Λ. By Schur's lemma, the semisimplified form of $I(\chi)_N$ is therefore of the form $\bigoplus_{i=1}^s C_{\chi_i}$ for some sequence of unramified characters $\chi_1, \cdots, \chi_s$ of M; by a well-known lemma, each representation C_{χ_i} occurs as a subrepresentation of $I(\chi)_N$. Hence for w in W, $C_{(w\chi)\cdot\delta^{1/2}}$ occurs as a subrepresentation of $I(\chi)_N$ by Theorem 3.5. Corollary 3.2 follows at once from this remark as well as the following corollary (use Frobenius reciprocity in the form of Theorem 3.4):

COROLLARY 3.3. *Let χ be any unramified character of M and w be any element of the Weyl group W. There exists a nonzero intertwining operator T_w: $I(\chi) \to I(w\chi)$. If χ is regular this operator T_w is unique up to a scalar.*

A similar argument shows that if (π, V) is an irreducible subquotient of $I(\chi)$, then there exists $w \in W$ such that (π, V) is isomorphic to a subrepresentation of $I(w \cdot \chi)$ (see 6.3.9 in **[17]**).

We shall describe more explicitly the operators T_w in §3.7.

We sketch now a proof of Theorem 3.5, a streamlined version of Casselman's proof of more general results in **[17]**. Basically, it is Mackey's double coset technique extended from finite groups to $\mathfrak{p}$-adic groups. The case of real Lie groups has been considered by Bruhat in his thesis **[7]**, it is much more elaborate.

(A) We remind the reader of Bruhat's decomposition $G = \bigcup_w PwP$, where PwP means PgP for any g in $N(A)$ representing the element w in $W = N(A)/M$. For w in W, the set PwP is irreducible and locally closed in the Zariski topology, hence has a well-defined dimension. Set $d(w) = \dim(PwP) - \dim(P)$. For any integer $r \geqq 0$, let F_r be the union of the double cosets PwP such that $d(w) < r$. The Zariski closure of any double coset PwP is a union of double cosets $Pw'P$ of smaller dimension, hence F_r is Zariski closed, hence closed in the $\mathfrak{p}$-adic topology. We let I_r be the subspace of $I(\chi)$ consisting of the functions vanishing identically on F_r. We have then a decreasing filtration

$$I(\chi) = I_0 \supset I_1 \supset \cdots \supset I_r \supset I_{r+1} \supset \cdots \tag{14}$$

of $I(\chi)$ by *P-stable subspaces.*

(B) *The next step is to prove that any function on F_{r+1} which satisfies the relation*

$$f(mng) = \delta^{1/2}(m)\chi(m)f(g) \quad \text{for } m \text{ in } M,\ n \text{ in } N,\ g \text{ in } F_{r+1} \tag{15}$$

is the restriction of some function belonging to $I(\chi)$. Indeed, one proves easily (using local cross-sections of G fibered over $P\backslash G$) that such a function is of the form

$$f(g) = \int_M \int_N \delta^{1/2}(m)\chi^{-1}(m)\varphi(mng)\, dm\, dn$$

for a suitable locally constant and compactly supported function φ on F_{r+1}. Extend φ to a function φ' in $\mathscr{H}(G)$. Then $P_\chi\varphi'$ belongs to $I(\chi)$ and restricts to f in F_{r+1}.

(C) From this it follows that I_r/I_{r+1} is the space of functions f on F_{r+1} which satisfy the following conditions:

(a) *f is locally constant*;

(b) *f vanishes on F_r*;

(c) *relation* (15).

Moreover, F_{r+1} is the union of F_r and the various double cosets PwP such that $d(w) = r$, which are *open* in F_{r+1}; hence one gets an isomorphism

$$I_r/I_{r+1} \simeq \bigoplus_{d(w)=r} J_w. \tag{16}$$

Here J_w is the space of functions f on PwP such that

$$f(mng) = \delta^{1/2}(m)\chi(m)f(g) \quad \text{for } m \text{ in } M,\ n \text{ in } N, g \text{ in } PwP,$$

and which vanish outside a set of the form $P\Omega$ where Ω is compact. Since Jacquet's functor $V \Rightarrow V_N$ is exact, one infers from (16) that the *$\mathscr{H}(M)$-modules $I(\chi)_N$ and $\bigoplus_{w\in W}(J_w)_N$ have isomorphic semisimplified forms.*

(D) It remains to identify the representation of M on the space $(J_w)_N$. Here we are paid off the dividends of our approach to induced representations via tensor products.

Choose a representative x_w of w in $N(A)$ and put $P(w) = P \cap x_w^{-1}Px_w$; hence $P(w) = M\cdot N(w)$ with a suitable subgroup $N(w)$ of N. It is then easy to show that, as a P-module, J_w carries the representation $c\text{-Ind}_{P(w)}^{P}\,\sigma_w$ where the character σ_w of $P(w)$ is defined by

$$\sigma_w(mn) = w^{-1}(\delta^{1/2}\chi)\,(m) \quad \text{for } m \text{ in } M,\ n \text{ in } N(w). \tag{17}$$

Consider the group homomorphisms $P(w) \xrightarrow{\alpha} P \xrightarrow{\beta} M$ where α is the injection and $\beta(mn) = m$ for m in M and n in N. By Theorem 1.4, one gets $c\text{-Ind}_{P(w)}^{P}\,\sigma_w = \alpha_*(\sigma_w\cdot\delta_w)$ where the character δ_w of $P(w)$ is defined by

$$\delta_w(p) = \Delta_{P(w)}(p)/\Delta_P(p) \quad \text{for } p \text{ in } P(w). \tag{18}$$

Since β^* is Jacquet's functor $V \Rightarrow V_N$ and $\beta_* \circ \alpha_* = (\beta\circ\alpha)_*$ one gets that $(J_w)_N$ is the carrier of the representation $(\beta\circ\alpha)_*(\sigma_w\cdot\delta_w)$. Since $\beta\circ\alpha$ is the projection of $P(w)$ onto M with kernel $N(w)$ and the characters σ_w and δ_w are trivial on $N(w)$ one gets an isomorphism $(J_w)_N \simeq \boldsymbol{C}_\lambda$ where $\lambda = \sigma_w\delta_w|M$. It remains to prove the formula (see formula (24_c) below) $\delta_w^{-1}|M = \delta^{1/2}(w^{-1}\delta)^{-1/2}$ to be able to conclude

$$(J_w)_N \simeq \boldsymbol{C}_{(w^{-1}\chi)\cdot\delta^{1/2}}. \tag{19}$$

(E) From (C) and (D) above, we know that $I(\chi)_N$ and $\bigoplus_{w\in W}\boldsymbol{C}_{(w\chi)\cdot\delta^{1/2}}$ have isomorphic semisimplified forms. It remains to show that *$^\circ M$ acts trivially on $I(\chi)_N$.* But $^\circ M$ is a *compact* subgroup of M. Hence its action on $I(\chi)_N$ and its semisimplified form are equivalent. Since any unramified character of M (including δ) is trivial on $^\circ M$, this group acts trivially on the semisimplified form of $I(\chi)_N$, hence on $I(\chi)_N$.

This concludes our proof of Theorem 3.5.

3.5. *Buildings and Iwahori subgroups.* Our aim in this section is mainly to fix notations. For more details, we refer the reader to the lectures by Tits in these PROCEEDINGS **[42]** or to the book by Bruhat and Tits **[13]**.

Let $\mathscr{B}$ be the building associated to G and let $\mathscr{A}$ be the apartment in $\mathscr{B}$ associated to the split torus A. We choose once for all a special vertex x_0 in $\mathscr{A}$. Among the conical chambers in $\mathscr{A}$ with apex at x_0 there is a unique one, $\mathscr{C}$ say, enjoying the following property: for every n in N, the intersection $\mathscr{C} \cap n\mathscr{C}$ contains a translate of $\mathscr{C}$. There is then a unique chamber C contained in $\mathscr{C}$ having x_0 for one of its vertices (see Figure 1). The stabilizer of x_0 in G shall be denoted by K; it is called by Bruhat and Tits a *special, good, maximal compact subgroup* of G. The interior points of C all have the same stabilizer B in G, called the *Iwahori subgroup of G* attached to C.

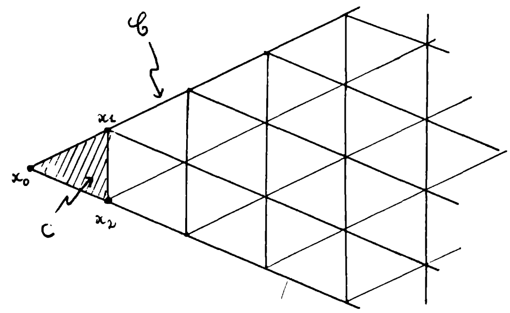

FIGURE 1

Any element g in $N(A)$ takes the apartment $\mathscr{A}$ to itself, it fixes every point of $\mathscr{A}$ iff it belongs to $^\circ M$. We may therefore identify the group $W_1 = N(A)/{}^\circ M$, called the *modified Weyl group*, to a group of affine linear transformations in $\mathscr{A}$. The group W_1 can be represented in two different ways as a semidirect product:

(a) Let Ω be the subgroup of W_1 consisting of the w's taking the chamber C to itself. The walls in $\mathscr{A}$ are certain hyperplanes and to each of them is associated a certain reflection. These reflections generate the invariant subgroup W_{aff} of W_1. Since W_{aff} acts simply transitively on the set of chambers contained in $\mathscr{A}$, the group W_1 is the semidirect product $\Omega \cdot W_{\text{aff}}$.

(b) Any element w of W has a representative $\omega(w)$ in $K \cap N(A)$ and $^\circ M = K \cap M$. We may therefore identify the Weyl group $W = N(A)/M$ to the stabilizer $(K \cap N(A))/(K \cap M)$ of x_0 in W_1. The intersection of W_1 with the group of translations in $\mathscr{A}$ is $M/{}^\circ M$ which we identify to Λ by means of the exact sequence (D), p. 135. Then W_1 is the semidirect product $W \cdot \Lambda$ where Λ is an invariant subgroup.

The fundamental structure theorems may now be formulated as follows.

IWASAWA DECOMPOSITION. *$G = PK$ and, more precisely, G is the disjoint union of the sets $P\omega(w)B$ for w running over the Weyl group W.*

BRUHAT-TITS DECOMPOSITION. *G is the disjoint union of the sets Bw_1B for w_1 running over the modified Weyl group W_1.*

CARTAN DECOMPOSITION. *Let Λ^- be the subset of Λ consisting of the elements of Λ taking the conical chamber $\mathscr{C}^-$ of $\mathscr{A}$ opposite to $\mathscr{C}$ into itself. Then G is the disjoint union of the sets $K \cdot \operatorname{ord}_M^{-1}(\lambda) \cdot K$ for λ running over Λ.*

IWAHORI DECOMPOSITION. *$B = (B \cap N^-) \cdot (B \cap M) \cdot (B \cap N)$ (unique factorization). Moreover one has $B \cap M = {}^\circ M$ and*

$$m(B \cap N)m^{-1} \subset B \cap N, \qquad m^{-1}(B \cap N^-)m \subset B \cap N^-$$

for any m in M such that $\operatorname{ord}_M(m) \in \Lambda^-$.

As before, we denote by Φ the system of roots of G w.r.t. A. We identify the elements of Φ to affine linear functions on $\mathscr{A}$ in such a way that $\alpha(x_0 + \lambda) = \langle \lambda, \alpha \rangle$

for α in Φ and λ in Λ. The conical chamber $\mathscr{C}$ is then defined as the set of points x in $\mathscr{A}$ such that $\alpha(x) > 0$ for every root α in the basis Δ of Φ associated to the parabolic group P.

We denote by Φ_0 the set of affine linear functions α on $\mathscr{A}$, vanishing at x_0 and such that the hyperplane $\alpha^{-1}(r)$ is a wall in $\mathscr{A}$ iff the real number r is an integer. An *affine root* is a function on $\mathscr{A}$ of the form $\alpha + k$ where α belongs to Φ_0 and k is an integer; their set is denoted by Φ_{aff}. For any affine root α, the reflection in the wall $\alpha^{-1}(0)$ is denoted by S_α. The reflections S_α for α running over Φ_0 (resp. Φ_{aff}) generate the group W (resp. W_{aff}). For α in Φ_0, there exists a unique vector t_α in Λ such that

$$S_\alpha(x) = x - \alpha(x)\cdot t_\alpha \quad \text{for any } x \text{ in } \mathscr{A}. \tag{20}$$

We denote by a_α any element in M such that $t_\alpha = \mathrm{ord}_M(a_\alpha)$. The set Φ_1 is obtained by adjoining to Φ_0 the set of functions $\alpha/2$ for α in Φ_0 such that $(B\omega(S_\alpha)B\colon B) \neq \delta(a_\alpha)^{-1/2}$.

The sets Φ, Φ_0 and Φ_1 are root systems in the customary sense (see for instance [**37**, p. 14 sqq.]). When the group G is split, the sets Φ, Φ_0 and Φ_1 are identical. In the nonsplit case, all we can assert is that, for any α in Φ, there exists a unique root $\lambda(\alpha)$ in Φ_0 proportional to α, and that any element of the reduced root system Φ_0 is of the form $\lambda(\alpha)$ for a suitable α in Φ.

Let w_1 be any element of W_1. Since w_1 is a coset modulo the subgroup $^\circ M$ of B the set Bw_1B is a double coset modulo B. We put

$$q(w_1) = (Bw_1B\colon B). \tag{21}$$

Since $K = BWB$, one gets

$$(K\colon B) = \sum_{w \in W} q(w). \tag{22}$$

Let Δ_1 be the set of affine roots in $\mathscr{A}$ which are positive on the chamber C and whose null set is a wall of C. The group W_1 is generated by Ω and the reflections S_α for α in Δ_1. The value of $q(w_1)$ is given by

$$q(w_1) = q(S_{\alpha_1}) \cdots q(S_{\alpha_m}) \tag{23}$$

where $w_1 = \omega S_{\alpha_1} \cdots S_{\alpha_m}$ is a decomposition of minimal length m (ω in Ω, $\alpha_1, \cdots, \alpha_m$ in Δ_1).

To each root β in Φ_1 is associated a real number $q_\beta > 0$. This association is characterized by the following set of properties

$$q_{w\cdot\beta} = q_\beta \quad \text{for } \beta \text{ in } \Phi_1 \text{ and } w \text{ in } W, \tag{24$_\mathrm{a}$}$$

$$q(w) = \prod_{\beta>0;\, w^{-1}\cdot\beta<0} q_\beta, \tag{24$_\mathrm{b}$}$$

$$\delta(m) = \prod_{\beta>0} q_\beta^{-\beta(m\cdot x_0)} \quad \text{for } m \text{ in } M. \tag{24$_\mathrm{c}$}$$

In the previous formulas β is a variable element in Φ_1 and the notation $\beta > 0$ means that β takes only positive values on $\mathscr{C}$. We make the convention that $q_{\alpha/2} = 1$ for α in Φ_0 if $\alpha/2$ does not belong to Φ_1. Two corollaries of the previous relations are worth mentioning

$$q(S_\alpha) = q_\alpha q_{\alpha/2}, \tag{24$_\mathrm{d}$}$$

$$(24_e) \qquad \delta(a_\alpha) = q_\alpha^{-2}\, q_{\alpha/2}^{-1},$$

for any $\alpha > 0$ in Φ_0. When G is split, q_β is equal to the order q of the residue field $\mathfrak{O}_F/\mathfrak{p}_F$ for any root β in $\Phi = \Phi_0 = \Phi_1$.

The structure of the Hecke algebra $\mathscr{H}(G, B)$ has been described by Iwahori and Matsumoto [**30**], [**31**].

THEOREM 3.6. *For w_1 in W_1, let $C(w_1)$ be the characteristic function of the double coset Bw_1B.*

(a) *The family $\{C(w_1)\}_{w_1 \in W_1}$ is a basis of the complex vector space $\mathscr{H}(G, B)$ (Bruhat-Tits decomposition!).*

(b) *Let w_1 be any element of W_1 and let $\omega S_{\alpha_1} \cdots S_{\alpha_m}$ (ω in Ω, $\alpha_1, \cdots, \alpha_m$ in Δ_1) be a decomposition of w_1 of minimal length m. Then*

$$(25) \qquad C(w_1) = C(\omega)C(S_{\alpha_1})\cdots C(S_{\alpha_m}).$$

(c) *For each α in Δ_1, one has*

$$(26) \qquad (C(S_\alpha) - 1)\cdot(C(S_\alpha) + q(S_\alpha)) = 0,$$

where $q(S_\alpha)$ has been defined by formula (21) *above.*

(d) *If α and β are distinct elements in Δ_1, there exists an integer $m_{\alpha\beta} \geqq 2$ such that*

$$(27) \qquad \underbrace{C(S_\alpha)C(S_\beta)C(S_\alpha)\cdots}_{m_{\alpha\beta}\ factors} = \underbrace{C(S_\beta)C(S_\alpha)C(S_\beta)\cdots}_{m_{\alpha\beta}\ factors}.$$

Moreover the relations (26) *and* (27) *are a complete set of relations among the $C(S_\alpha)$'s.*

3.6. *Action of the Iwahori subgroups on the representations.* Here is the main result, due to Casselman [**18**] and Borel [**4**].

THEOREM 3.7. *Let (π, V) be any admissible representation of G. The natural projection of V onto V_N defines an isomorphism of V^B onto $(V_N)^{\circ M}$.*

One proves first that V^B maps onto $(V_N)^{\circ M}$ using Iwahori decomposition of B and the methods used in the proof of Theorem 2.3. There are some simplifications due to the fact that P is a minimal parabolic subgroup of G.

To prove that V^B maps injectively in $(V_N)^{\circ M}$, one first proves that, for any given vector v in $V(N) \cap V^B$, there exists a real number $\varepsilon > 0$ such that $\pi(C(a))\cdot v = 0$ for every a in A satisfying $|\alpha(a)|_F \leqq \varepsilon$ whenever the root $\alpha \in \Phi$ is positive on P. But this relation implies $v = 0$ by Theorem 3.6, since one has $q(S_\alpha) > 0$ there.

COROLLARY 3.4. *Given any unramified character χ of M, one has a direct sum decomposition $I(\chi) = I(\chi)^B \oplus I(\chi)(N)$.*

This follows from Theorem 3.7 since $^\circ M$ acts trivially on $I(\chi)_N$. A direct proof can also be obtained using the methods used in the proof of Theorem 3.5.

Using the decomposition of G into the pairwise disjoint *open* subsets $P\omega(w)B$ (w in W), one gets easily a basis for $I(\chi)^B$. Indeed, $\omega(w)$ normalizes M and $B \cap M = {}^\circ M$ lies in the kernel of $\chi\delta^{1/2}$. Hence the following function is well defined on G

$$(28) \qquad \begin{aligned} \Phi_{w,\chi}(g) &= \delta^{1/2}(m)\chi(m) && \text{if } g = mn\omega(w)b, \\ &= 0 && \text{if } g \notin P\omega(w)B. \end{aligned}$$

The family $\{\Phi_{\omega,\chi}\}_{w \in W}$ is the sought-for basis.

The next result is due to Casselman **[18]** (see also Borel **[4]**).

THEOREM 3.8. *Let (π, V) be any admissible irreducible representation of G. The following assertions are equivalent:*

(a) *There are in V nonzero vectors invariant under B (that is $V^B \neq 0$).*

(b) *There exists some unramified character χ of M such that (π, V) is isomorphic to a subrepresentation of $(\nu_\chi, I(\chi))$.*

By Theorem 3.7, assertion (a) means that $(V_N)^{\circ M} \neq 0$. By Frobenius reciprocity (Theorem 3.4), assertion (b) means that there exists in the space dual to V_N a nonzero vector invariant under $^\circ M$ which is an eigenvector for the group M. Since $^\circ M$ is compact, acts continuously on V_N and $M/^\circ M$ is commutative, the equivalence follows immediately.

COROLLARY 3.5. *Let χ be any unramified character of M. The space $I(\chi)^B$ generates $I(\chi)$ as a G-module.*

We prove Corollary 3.5 by reductio ad absurdum. Assume that $I(\chi)^B$ does not generate $I(\chi)$. Since $I(\chi)$ is finitely generated, there exist an irreducible admissible representation (π, V) of G and a G-homomorphism $u: I(\chi) \to V$ which is nonzero and contains $I(\chi)^B$ in its kernel. Since B is compact, one gets $V^B = u(I(\chi)^B) = 0$. By duality, one gets an injective G-homomorphism $\tilde{u}: \tilde{V} \to (I(\chi))^\sim$. Since $(I(\chi))^\sim$ is isomorphic to $I(\chi^{-1})$ by Theorem 3.2, it follows from Theorem 3.8 that $\tilde{V}^B \neq 0$. But the finite-dimensional spaces V^B and $\tilde{V}^B$ are dual to each other and this is clearly impossible.

3.7. *Intertwining operators.* In this section, *we assume the unramified character χ to be regular*, that is the characters $w\chi$, for w running over W, are all distinct.

Corollary 3.2 may be reformulated as follows: given w in W, there exists a linear form $L_w \neq 0$ on $I(\chi)$, such that

$$L_w(\nu_\chi(mn)\cdot f) = \delta^{1/2}(m) w\chi(m) L_w(f) \tag{29}$$

for m in M, n in N and f in $I(\chi)$, unique up to multiplication by a constant. We normalize L_w by

$$L_w(f) = \int_{N(w)\backslash N} f(\omega(w)^{-1}n)\, d\bar{n} \tag{30}$$

for a function f whose support does not meet F_r $(r = d(w^{-1}))$. The Haar measure is chosen in such a way that $N(w)\backslash N(w)\cdot(N \cap B)$ be of measure 1.

To L_w we associate an intertwining operator $T_w: I(\chi) \to I(w\chi)$ defined by

$$T_w f(g) = L_w(\nu_\chi(g)\cdot f) \quad \text{for } f \text{ in } I(\chi) \text{ and } g \text{ in } G. \tag{31}$$

It is easily checked that L_w, hence T_w, depends only on w, not on the representative $\omega(w)$ for w.

Since $G = PK$ (Iwasawa decomposition) and the character $\delta^{1/2}\chi$ of M is trivial on $M \cap K = {}^\circ M$, there exists in $I(\chi)$ a unique function $\Phi_{K,\chi}$ invariant under K and normalized by $\Phi_{K,\chi}(1) = 1$. Explicitly, one has

$$\Phi_{K,\chi}(mnk) = \delta^{1/2}(m)\chi(m) \quad \text{for } m \text{ in } M,\ n \text{ in } N \text{ and } k \text{ in } K. \tag{32}$$

If the Haar measures on M and N are normalized by $\int_{M\cap K} dm = \int_{N\cap K} dn = 1$, one may also define $\Phi_{K,\chi}$ as $P_\chi(I_K)$ where P_χ is defined as on p. 136 and I_K is the characteristic function of K. The space $I(\chi)^K$ consists of the constant multiples of $\Phi_{K,\chi}$.

The next theorem again is due to Casselman **[18]**.

THEOREM 3.9. *The operator T_w takes $I(\chi)^K$ into $I(w\chi)^K$. More precisely, one has*

$$T_w(\Phi_{K,\chi}) = c_w(\chi)\cdot\Phi_{K,w\chi}, \tag{33}$$

where the constant $c_w(\chi)$ is defined by

$$c_w(\chi) = \prod_\alpha c_\alpha(\chi), \tag{34}$$

$$c_\alpha(\chi) = \frac{(1 - q_{\alpha/2}^{-1/2}\, q_\alpha^{-1}\, \chi(a_\alpha))(1 + q_{\alpha/2}^{-1/2}\, \chi(a_\alpha))}{1 - \chi(a_\alpha)^2}. \tag{35}$$

The product in (34) *is extended over the affine roots α in Φ_0 which are positive over $\mathscr{C}$, but such that $w\alpha$ is negative over $\mathscr{C}$.*

When G is split, formula (35) takes the simpler form

$$c_\alpha(\chi) = \frac{1 - q^{-1}\chi(a_\alpha)}{1 - \chi(a_\alpha)}. \tag{36}$$

The bulk of the proof of Theorem 3.9 rests with the case where w is the reflection associated to a simple root β in Φ_0. In this case, there exists exactly one positive root α in Φ_0 taken by w into a negative root, namely $\alpha = \beta$. The general case follows then since $T_{w_1 w_2}$ is equal to $T_{w_1}T_{w_2}$ when the lengths of w_1 and w_2 add to the length of $w_1 w_2$.

COROLLARY 3.6. *The intertwining map T_w is an isomorphism from $I(\chi)$ onto $I(w\chi)$ iff $c_w(\chi)$ and $c_{w^{-1}}(w\chi)$ are nonzero.*

Indeed $T_{w^{-1}}T_w$ is multiplication by $c_w(\chi)c_{w^{-1}}(w\chi)$ by Theorem 3.9.

One may strengthen the irreducibility criterion (Theorem 3.3).

THEOREM 3.10. *Assume χ is any unramified regular character of M. Then the representation $(\nu_\chi, I(\chi))$ of G is irreducible iff $c_\alpha(\chi) \neq 0$, $c_\alpha(w_0\chi) \neq 0$ for every positive root α in Φ_0 positive over $\mathscr{C}$, where w_0 is the unique element in W which takes $\mathscr{C}$ into $\mathscr{C}^-$.*

IV. Spherical functions.

4.1. *Some integration formulas.* We keep the notation of the previous part.

If Γ is any of the groups G, M, N, K, then Γ is unimodular. We normalize the (left and right invariant) Haar measure on Γ by $\int_{\Gamma\cap K} d\gamma = 1$. The group $P = M\cdot N$ is not unimodular. One checks immediately that the formulas

$$\int_P f(p)\, d_l p = \int_M \int_N f(mn)\, dm\, dn, \tag{1}$$

$$\int_P f(p)\, d_r p = \int_M \int_N f(nm)\, dm\, dn \tag{2}$$

define a left invariant Haar measure $d_l p$ and a right invariant Haar measure

$d_r p$ on P. Since ${}^\circ M = M \cap K$ is the largest compact subgroup of M, one gets $(P \cap K) = (M \cap K) \cdot (N \cap K)$, hence the normalization

$$\int_{P \cap K} d_l p = \int_{P \cap K} d_r p = 1. \tag{3}$$

The Haar measures are related to the Iwasawa decomposition by the formulas

$$\int_G f(g)\, dg = \int_K \int_P f(pk)\, dk\, d_l p, \tag{4}$$

$$\int_G f(g)\, dg = \int_K \int_P f(kp)\, dk\, d_r p \tag{5}$$

for f in $C_c(G)$. Let us prove for instance formula (4). One defines a linear map $h \mapsto u_h$ from $C_c(K \times P)$ into $C_c(G)$ by the rule

$$u_h(pk^{-1}) = \int_{P \cap K} h(kp_1, pp_1)\, d_l p_1. \tag{6}$$

The linear form $h \mapsto \int_G u_h(g)\, dg$ on $C_c(K \times P)$ is then a left invariant Haar measure on $K \times P$; hence by our normalizations, one gets

$$\int_G u_h(g)\, dg = \int_K \int_P h(k, p)\, dk\, d_l p. \tag{7}$$

It suffices to substitute $f(pk^{-1})$ for $h(k, p)$ in formulas (6) and (7) to get formula (4)!

From the definition of δ (see p. 135), one gets

$$\int_N f(mnm^{-1})\, dn = \delta(m)^{-1} \int_N f(n)\, dn \tag{8}$$

for any function f in $C_c(N)$. As a corollary, we get the alternate expressions for the Haar measures on P

$$\int_P f(p)\, d_l p = \int_M \int_N \delta(m)^{-1} f(nm)\, dm\, dn, \tag{9}$$

$$\int_P f(p)\, d_r p = \int_M \int_N \delta(m) f(mn)\, dm\, dn. \tag{10}$$

Otherwise stated, the modular function of P is given by

$$\Delta_P(mn) = \delta(m)^{-1} \quad \text{for } m \text{ in } M,\ n \text{ in } N. \tag{11}$$

The group N is unipotent and M acts on N via inner automorphisms. It is then easy to construct a sequence of subgroups of N, say $N = N_0 \supset N_1 \supset \cdots \supset N_{r-1} \supset N_r = 1$, which are invariant under M and such that N_{j-1}/N_j is isomorphic (for $j = 1, \cdots, r$) to a vector space over F on which M acts linearly. Putting

$$\Delta(m) = \left|\det(\mathrm{Ad}_{\mathfrak{n}}(m) - 1_{\mathfrak{n}})\right|_F \tag{12}$$

for m in M, one gets by induction on r the integration formula

$$\int_N f(n)\, dn = \Delta(m) \int_N f(nmn^{-1} m^{-1})\, dn \tag{13}$$

for f in $C_c(N)$ and any m in M such that $\Delta(m) \neq 0$ (see [**28**, Lemma 22]).

Let us define now the so-called *orbital integrals*. Let m be any element of M

such that $\Delta(m) \neq 0$, let $Z(m)$ be the centralizer of m in G and G_m the conjugacy class of m in G. Then $A \subset M$ and $Z(m) \cap M$ has finite index in $Z(m)$; since M/A is a compact group, the same is true of $Z(m)/A$. Moreover, the conjugacy class G_m is closed in G; hence the mapping $gZ(m) \mapsto gmg^{-1}$ is a homeomorphism from $G/Z(m)$ onto G_m. Therefore, the mapping $gA \mapsto gmg^{-1}$ from G/A into G is proper and we may set after Harish-Chandra

$$F_f(m) = \Delta(m) \int_{G/A} f(gmg^{-1})\, d\bar{g} \tag{14}$$

for any function f in $C_c(G)$. The Haar measure on A is normalized in such a way that $\int_{M/A} d\bar{m} = 1$.

LEMMA 4.1. *Let f be any function in $\mathscr{H}(G, K)$ and m an element of M such that $\Delta(m) \neq 0$. Then $F_f(m)$ is equal to $\int_N f(nm)\, dn$.*

From formulas (2) and (5), one gets

$$\int_G u(g)\, dg = \int_M \int_N u(nm_1)\, dm_1\, dn \tag{15}$$

for any function u in $C_c(G)$ which is invariant under left translation by the elements in K. Putting $u(g) = f(gmg^{-1})$, one gets the following representation for $F_f(m)$

$$F_f(m) = \int_{M/A} h(m_1)\, d\bar{m}_1, \tag{16}$$

$$h(m_1) = \Delta(m) \int_N f(nm_1mm_1^{-1}\, n^{-1})\, dn. \tag{17}$$

Fix m_1 and set $m_2 = m_1mm_1^{-1}$. From the definition (12) of Δ, one gets $\Delta(m) = \Delta(m_2)$. We get therefore

$$\begin{aligned} h(m_1) &= \Delta(m_2) \int_N f((nm_2\, n^{-1}\, m_2^{-1})m_2)\, dn && \text{by (17)} \\ &= \int_N f(nm_2)\, dn && \text{by (13).} \end{aligned}$$

Notice that the group $M/{}^\circ M = M/(M \cap K)$ is commutative. Hence we get $m_2 \in mK$ and since the function f is invariant under right translation by the elements of K, one gets $f(nm_2) = f(nm)$ for any n in N. It follows that $h(m_1)$ is equal to $\int_N f(nm)\, dn$ for any m_1 in M. The contention of Lemma 4.1 follows from formula (16) since M/A is of measure 1.

4.2. *Satake isomorphism.* The construction we are going to expound now is due to Satake [**38**]. It is the $\mathfrak{p}$-adic counterpart of a well-known construction of Harish-Chandra in the set-up of real Lie groups.

For any λ in Λ, let $\mathrm{ch}(\lambda)$ be the characteristic function of the subset $\mathrm{ord}_M^{-1}(\lambda)$ of M. Since $\int_{M\cap K} dm = 1$, one gets

$$\mathrm{ch}(\lambda) * \mathrm{ch}(\lambda') = \mathrm{ch}(\lambda + \lambda') \tag{18}$$

for λ, λ' in Λ. Moreover the elements $\mathrm{ch}(\lambda)$ (for λ in Λ) form a basis of the complex algebra $\mathscr{H}(M, {}^\circ M)$, which may be therefore identified to the group algebra $\mathbf{C}[\Lambda]$.

We define now a linear map $S: \mathscr{H}(G, K) \to \mathscr{H}(M, {}^\circ M)$ by the formula

$$Sf(m) = \delta(m)^{1/2} \int_N f(mn)\, dn = \delta(m)^{-1/2} \int_N f(nm)\, dn \tag{19}$$

for f in $\mathscr{H}(G, K)$ and m in M. The two integrals are equal by formula (8). It is immediate that the function Sf on M belongs to $C_c(M)$. That it is bi-invariant under $^\circ M$ follows from the fact that f is bi-invariant under K and that $^\circ M = M \cap K$.

The following fundamental theorem is due to Satake **[38]**.

THEOREM 4.1. *The Satake transformation S is an algebra isomorphism from $\mathscr{H}(G, K)$ onto the subalgebra $\mathbf{C}[\Lambda]^W$ of $\mathbf{C}[\Lambda]$ consisting of the invariants of the Weyl group W.*

Here are the main steps in the proof.

(A) *S is a homomorphism of algebras.* By construction, S is the composition of three linear maps

$$\mathscr{H}(G, K) \xrightarrow{\alpha} \mathscr{H}(P) \xrightarrow{\beta} \mathscr{H}(M) \xrightarrow{\gamma} \mathscr{H}(M).$$

Here α is simply the restriction of functions from G to P. It is compatible with convolution by an easy corollary of (4). The map β is given by $(\beta u)(m) = \int_N u(mn)\, dn$ and one checks easily that it is compatible with convolution. The map γ is given by $(\gamma f)(m) = f(m)\delta(m)^{1/2}$; since δ is a character of M, it is compatible with convolution.

(B) *The image of S is contained in $\mathbf{C}[\Lambda]^W$.* Since $W = (N(A) \cap K)/^\circ M$, this property is equivalent to

$$Sf(xmx^{-1}) = Sf(m) \tag{20}$$

for m in M and x in $N(A) \cap K$.

The function $m \mapsto \det(\mathrm{Ad}_\mathfrak{n}(m) - 1_\mathfrak{n})$ from M to F is polynomial and nonzero. The elements of M which do not annihilate this function are therefore dense in M; they are called the regular elements. Hence by continuity it suffices to prove (20) for m regular.

From Lemma 4.1, one gets

$$Sf(m) = D(m) \int_{G/A} f(gmg^{-1})\, d\bar{g} \tag{21}$$

for m regular in M. Here $D(m)$ is equal to $\Delta(m)\delta(m)^{-1/2}$; hence

$$\begin{aligned} D(m)^2 &= |\det(\mathrm{Ad}_\mathfrak{n}(m) - 1_\mathfrak{n})|_F^2 \cdot |\det \mathrm{Ad}_\mathfrak{n}(m)|_F^{-1} \\ &= |\det(\mathrm{Ad}_\mathfrak{n}(m) - 1_\mathfrak{n})|_F \cdot |\det(\mathrm{Ad}_\mathfrak{n}(m^{-1}) - 1_\mathfrak{n})|_F \\ &= |\det(\mathrm{Ad}_\mathfrak{n}(m) - 1_\mathfrak{n})|_F \cdot |\det(\mathrm{Ad}_{\mathfrak{n}^-}(m) - 1_{\mathfrak{n}^-})|_F. \end{aligned}$$

The last equality follows for instance from the fact that the weights in $\mathfrak{n} \otimes_F \bar{F}$ ($\bar{F}$ an algebraic closure of F) are the inverses of the weights in $\mathfrak{n}^- \otimes_F \bar{F}$ of any maximal torus of M. Since $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{m} \oplus \mathfrak{n}^-$, one gets

$$D(m) = |\det(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{m}}(m) - 1_{\mathfrak{g}/\mathfrak{m}})|_F^{1/2} \tag{22}$$

and therefore

$$D(xmx^{-1}) = D(m) \quad \text{for } m \text{ in } M,\ x \text{ in } N(A). \tag{23}$$

On the other hand, the compact group $N(A) \cap K$ acts by inner automorphisms on G and A. It leaves therefore invariant the Haar measures on G and A, hence the G-invariant measure on G/A. Let m in M be regular, x in $N(A) \cap K$ and f in $\mathscr{H}(G, K)$. One has $f(xgx^{-1}) = f(g)$ for any g in G; hence

$$\int_{G/A} f(g(xmx^{-1})g^{-1})\, d\bar{g} = \int_{G/A} f((x^{-1}gx)m(x^{-1}gx)^{-1})\, d\bar{g}$$
$$= \int_{G/A} f(gmg^{-1})\, d\bar{g}.$$

The invariance property (20) follows from the representation (21), the invariance property (23) and the just-established invariance.

(C) *The linear map* $S\colon \mathscr{H}(G, K) \to \boldsymbol{C}[\Lambda]^W$ *is bijective.* We let as before Λ^- denote the subset of Λ consisting of the translations in $\mathscr{A}$ which take $\mathscr{C}^-$ into itself. For λ in Λ^-, let φ_λ be the characteristic function of the double coset $K \cdot \operatorname{ord}_M^{-1}(\lambda) \cdot K$; by Cartan decomposition, the family $\{\varphi_\lambda\}_{\lambda\in\Lambda^-}$ is a basis of $\mathscr{H}(G, K)$. Moreover, any element of Λ is conjugate under W to a unique element in Λ^-. We get therefore a basis $\{\mathrm{ch}'(\lambda)\}_{\lambda\in\Lambda^-}$ of $\boldsymbol{C}[\Lambda]^W$ by putting

$$\mathrm{ch}'(\lambda) = \frac{1}{|W(\lambda)|} \sum_{w\in W} \mathrm{ch}(w\cdot\lambda), \tag{24}$$

where $W(\lambda)$ is the stabilizer of λ in W. Define the matrix $\{c(\lambda, \lambda')\}$ by

$$S\varphi_{\lambda'} = \sum_{\lambda} c(\lambda, \lambda') \cdot \mathrm{ch}'(\lambda) \tag{25}$$

where λ, λ' run over Λ^-.

To calculate $c(\lambda, \lambda')$, choose representatives m and m' respectively of λ, λ' in M. Then we get (μ is the Haar measure on G)

$$c(\lambda, \lambda') = S\varphi_{\lambda'}(m) = \delta(m)^{-1/2}\, \mu(Km'K \cap NmK). \tag{26}$$

It is clear that $KmK \cap NmK \supset mK$; hence

$$c(\lambda, \lambda') \geqq \delta(m)^{-1/2}. \tag{27}$$

Moreover $Km'K \cap NmK$ *is empty unless* $\lambda' - \lambda$ *is a linear combination with non-negative real coefficients of the positive roots.* Using a suitable lexicographic ordering $\geqq$, we conclude that $c(\lambda, \lambda') = 0$, unless $0 \geqq \lambda' \geqq \lambda$. Since $c(\lambda, \lambda) \neq 0$ by (27), this remark shows that the elements $S\varphi_{\lambda'}$, for λ' in Λ, form a basis of $\boldsymbol{C}[\Lambda]^W$, hence our contention (C).

COROLLARY 4.1. *The algebra* $\mathscr{H}(G, K)$ *is commutative and finitely generated over* $\boldsymbol{C}$.

The algebra $\boldsymbol{C}[\Lambda]$ is commutative and generated by $\mathrm{ch}(\lambda_1), \cdots, \mathrm{ch}(\lambda_m), \mathrm{ch}(\lambda_1^{-1}), \cdots, \mathrm{ch}(\lambda_m^{-1})$ if $\{\lambda_1, \cdots, \lambda_m\}$ is a basis of Λ over $\boldsymbol{Z}$. Since the group W is finite, it follows from well-known results in commutative algebra[13] that $\boldsymbol{C}[\Lambda]^W$ is finitely generated as an algebra over $\boldsymbol{C}$, and that $\boldsymbol{C}[\Lambda]$ is finitely generated as a module over $\boldsymbol{C}[\Lambda]^W$.

We determine now the algebra homomorphisms from $\mathscr{H}(G, K)$ to $\boldsymbol{C}$. Let χ be any unramified character of M. Since $\int_{{}^\circ M} dm = 1$, the map $f \mapsto \int_M f(m)\chi(m)\, dm$ is a unitary homomorphism from $\mathscr{H}(M, {}^\circ M)$ to $\boldsymbol{C}$, and we get in this way all such homomorphisms. Define a linear map $\omega_\chi\colon \mathscr{H}(G, K) \to \boldsymbol{C}$ by

[13] See for instance N. Bourbaki, *Commutative algebra*, Chapter V, p. 323, Addison-Wesley, 1972.

$$\omega_\chi(f) = \int_M Sf(m)\cdot\chi(m)\,dm. \tag{28}$$

COROLLARY 4.2. *Any unitary homomorphism from $\mathscr{H}(G, K)$ into $\mathbf{C}$ is of the form ω_χ for some unramified character χ of M. Moreover, one has $\omega_\chi = \omega_{\chi'}$ iff there exists an element w in W such that $\chi' = w\cdot\chi$.*

This corollary follows from Theorem 4.1 and the classical properties of invariants of finite groups acting on polynomial algebras (see previous footnote).

Otherwise stated, the set of unitary homomorphisms from $\mathscr{H}(G, K)$ to $\mathbf{C}$ is in a bijective correspondence with the set X/W of orbits of W in the set X of unramified characters of M. We refer the reader to §3.2 for a discussion of this set X/W. Notice that since X is isomorphic to a complex torus T such that $X^*(T) = \Lambda$, then X/W is a complex algebraic affine variety and Satake isomorphism defines an isomorphism of $\mathscr{H}(G, K)$ with the algebra of polynomial functions on X/W.

4.3. *Determination of the spherical functions.* Since the characteristic functions of the double cosets KgK form a basis of the complex vector space $\mathscr{H}(G, K)$, one defines as follows an isomorphism of the dual to the space $\mathscr{H}(G, K)$ onto the space of functions on G, bi-invariant under K:

$$\omega(f) = \int_G f(g)\Gamma(g)\,dg \qquad \text{for } f \text{ in } \mathscr{H}(G, K), \tag{29}$$

$$\Gamma(g) = \omega(I_{KgK})\Big/\int_{KgK} dg_1 \qquad \text{for } g \text{ in } G. \tag{30}$$

I claim that the following conditions are equivalent:

(a) *ω is a homomorphism of algebras from $\mathscr{H}(G, K)$ to $\mathbf{C}$.*

(b) *One has*

$$\Gamma(g_1)\Gamma(g_2) = \int_K \Gamma(g_1kg_2)\,dk \tag{31}$$

for g_1, g_2 in G.

(c) *For any function f in $\mathscr{H}(G, K)$, there exists a constant $\lambda(f)$ such that*

$$f*\Gamma = \Gamma*f = \lambda(f)\cdot\Gamma. \tag{32}$$

The equivalence of (a) and (b) follows from the following calculation

$$\begin{aligned}\omega(f_1*f_2) &= \iint \Gamma(g_1g_2)f_1(g_1)f_2(g_2)\,dg_1\,dg_2\\ &= \int_G f_1(g_1)\,dg_1\int_G f_2(g_2)\,dg_2\int_K \Gamma(g_1kg_2)\,dk\end{aligned}$$

and the fact that the function $(g_1, g_2) \mapsto \int_K\Gamma(g_1kg_2)\,dk$ on $G\times G$ is invariant under left and right translations by elements of $K\times K$.

The equivalence of (a) and (c) follows from the following formula:

$$\omega(f_1*f_2) = \int f_2(g)(\check{f}_1*\Gamma)(g)\,dg = \int f_1(g)(\Gamma*\check{f}_2)(g)\,dg$$

for f_1, f_2 in $\mathscr{H}(G, K)$ where $\check{f}(g) = f(g^{-1})$. Hence $\lambda(f) = \omega(\check{f})$.

DEFINITION 4.1. *A* (zonal) *spherical function on G w.r.t. K is a function Γ on G, bi-invariant under K, such that* $\Gamma(1) = 1$ *and enjoying the equivalent properties* (a), (b) *and* (c) *above.*

We may now translate our previous results in terms of spherical functions. Let χ be any unramified character of M. Recall that the function $\Phi_{K,\chi}$ is defined by

$$\Phi_{K,\chi}(mnk) = \chi(m)\delta^{1/2}(m) \quad \text{for } m \text{ in } M,\ n \text{ in } N \text{ and } k \text{ in } K. \tag{33}$$

We put

$$\Gamma_\chi(g) = \int_K \Phi_{K,\chi}(kg)\, dk \quad \text{for } g \text{ in } G. \tag{34}$$

THEOREM 4.2. (a) *The spherical functions on G w.r.t. K are the functions* Γ_χ.

(b) *Let* χ *and* χ' *be unramified characters of M. The spherical functions* Γ_χ *and* $\Gamma_{\chi'}$ *are equal iff there exists an element w in the Weyl group W such that* $\chi' = w\cdot\chi$.

It is clear that Γ_χ is bi-invariant under K, and $\Gamma_\chi(1) = 1$. Theorem 4.2 follows from Corollary 4.2 and the formula

$$\omega_\chi(f) = \int_G \Gamma_\chi(g)\cdot f(g)\, dg \quad \text{for } f \text{ in } \mathscr{H}(G, K). \tag{35}$$

This in turn is proved as follows:

$$\begin{aligned}\int_G \Gamma_\chi(g)f(g)\,dg &= \int_G \Phi_{K,\chi}(g)f(g)\,dg \\ &= \int_K\int_M\int_N \Phi_{K,\chi}(mnk)f(mnk)\,dk\,dm\,dn \\ &= \int_M \chi(m)\delta(m)^{1/2}\,dm \int_N f(mn)\,dn \\ &= \int_M \chi(m)Sf(m)\,dm = \omega_\chi(f).\end{aligned}$$

We used the integration relations (1) and (4).

4.4. *The spherical principal series of representations.*

DEFINITION 4.2. *A representation of G is called spherical* (w.r.t. K) *if it is smooth, irreducible and contains a nonzero vector invariant under K.*

Let Γ be a spherical function on G (w.r.t. K). We denote by V_Γ the space of functions f on G of the form $f(g) = \sum_{i=1}^n c_i\Gamma(gg_i)$ for $c_1, \cdots, c_n$ in $\mathbf{C}$ and $g_1, \cdots, g_n$ in G. From the functional equation of the spherical functions (formula (31)), one deduces

$$\int_K f(gkg')\, dk = \Gamma(g)\cdot f(g') \quad \text{for } f \text{ in } V_\Gamma \text{ and } g, g' \text{ in } G. \tag{36}$$

We let G operate on V_Γ by right translations, namely

$$(\pi_\Gamma(g)\cdot f)(g_1) = f(g_1 g) \quad \text{for } f \text{ in } V_\Gamma \text{ and } g, g_1 \text{ in } G. \tag{37}$$

I claim that *the representation* (π_Γ, V_Γ) *is spherical and that the elements of* V_Γ *invariant under* $\pi_\Gamma(K)$ *are the constant multiples of* Γ. Indeed, it is clear that for any function f in V_Γ there exists a compact open subgroup K_f of G such that f is invariant under right translation by the elements of K_f; hence the representation (π_Γ, V_Γ) is smooth. Let $f \neq 0$ in V_Γ and choose an element g' in G such that $f(g') \neq 0$. The functional equation (36) may be rewritten as

$$\Gamma = f(g')^{-1} \int_K \pi_\Gamma(kg')\cdot f\, dk = f(g')^{-1}\pi_\Gamma(I_{Kg'})f. \tag{38}$$

Any vector subspace of V_Γ containing f and invariant under $\pi_\Gamma(G)$ contains therefore Γ, hence is identical to V_Γ. Finally, if a function f in V_Γ is invariant under $\pi_\Gamma(K)$, one gets $f = f(1)\cdot\Gamma$ by substituting $g' = 1$ in the functional equation (36).

THEOREM 4.3. *Let (π, V) be any spherical representation of G. There exists a unique spherical function Γ such that (π, V) is isomorphic to (π_Γ, V_Γ).*

As usual, let V^K denote the subspace of V consisting of the vectors invariant under $\pi(K)$. If f is any function in $\mathscr{H}(G, K)$, the operator $\pi(f)$ takes V^K into itself; hence V^K is a module over $\mathscr{H}(G, K)$. I claim that this module is *simple*: indeed, let $v \neq 0$ and v' be two elements of V^K. Since the $\mathscr{H}(G)$-module V is simple, there exists a function f in $\mathscr{H}(G)$ such that $v' = \pi(f)\cdot v$. The function $f_K = I_K * f * I_K$ belongs to $\mathscr{H}(G, K)$ and $v' = \pi(f_K)\cdot v$, substantiating our claim.

The algebra $\mathscr{H}(G, K)$ over the field $\boldsymbol{C}$ is commutative and of countable dimension. By the reasoning used to prove Schur's lemma (see p. 118) (or by Hilbert's Zero Theorem), one concludes that any simple module over $\mathscr{H}(G, K)$ is of dimension 1 over $\boldsymbol{C}$. Hence V^K is of dimension 1 over $\boldsymbol{C}$ and there exists a unitary homomorphism $\omega\colon \mathscr{H}(G, K) \to \boldsymbol{C}$ such that

$$\pi(f)\cdot v = \omega(f)v \quad \text{for any } f \text{ in } \mathscr{H}(G, K) \text{ and any } v \text{ in } V^K. \tag{39}$$

Let $(\tilde\pi, \tilde V)$ be the representation of G contragredient to (π, V). The space $\tilde V^K$ of vectors in $\tilde V$ invariant under $\tilde\pi(K)$ is dual to V^K, hence of dimension 1. Choose a vector v in V^K and a vector $\tilde v$ in $\tilde V^K$ such that $\langle \tilde v, v\rangle = 1$ and define the function Γ on G by

$$\Gamma(g) = \langle \tilde v, \pi(g)\cdot v\rangle \quad \text{for } g \text{ in } G. \tag{40}$$

From (39) and (40) one deduces $\omega(f) = \int_G \Gamma(g)\cdot f(g)\, dg$ for any f in $\mathscr{H}(G, K)$. It is obvious that $\Gamma(1) = 1$ and that Γ is bi-invariant under K. Hence Γ is a spherical function.

The map which associates to any vector v' in V the coefficient $\pi_{v',\tilde v}$ defines an isomorphism of (π, V) with (π_Γ, V_Γ). Moreover, for any spherical function Γ' on G, the representation (π, V) is isomorphic to $(\pi_{\Gamma'}, V_{\Gamma'})$ iff the following relation holds

$$\pi(I_K)\pi(g)\pi(I_K) = \Gamma'(g)\cdot\pi(I_K) \quad \text{for } g \text{ in } G. \tag{41}$$

This holds for $\Gamma' = \Gamma$ only. Q.E.D.

The definition of spherical functions as well as the results obtained so far in this section depend only on the fact that K is a compact open subgroup of G and that the Hecke algebra $\mathscr{H}(G, K)$ is commutative. We use now the classification of spherical functions on G afforded by Theorem 4.2. Let χ be any unramified character of M; when the spherical function Γ is set equal to Γ_χ, we write (π_χ, V_χ) instead of

(π_Γ, V_Γ). The family of representations $\{(\pi_\chi, V_\chi)\}_{\chi \in X}$ is called the *spherical principal series of representations of G.*

We summarize now the main properties of the spherical principal series; they are immediate corollaries of the results obtained so far.

(a) *Any representation (π_χ, V_χ) is irreducible, admissible, and the only functions in V_χ invariant under $\pi_\chi(K)$ are the constant multiples of Γ_χ.*

(b) *Let χ and χ' be unramified characters of M. The representations (π_χ, V_χ) and $(\pi_{\chi'}, V_{\chi'})$ are isomorphic iff there exists an element w in the Weyl group W such that $\chi' = w \cdot \chi$.*

(c) *Assume that the representation $(\nu_\chi, I(\chi))$ in the unramified principal series is irreducible. For any function f in $I(\chi)$ define the function $f^\natural$ by $f^\natural(g) = \int_K f(kg)\, dk$. Then the map $f \mapsto f^\natural$ is an isomorphism of the representation $(\nu_\chi, I(\chi))$ with the representation (π_χ, V_χ).*

(d) *In general, let $0 = V_0 \subset V_1 \subset \cdots \subset V_{r-1} \subset V_r = I(\chi)$ be a Jordan-Hölder series of the $\mathscr{H}(G)$-module $I(\chi)$. There exists a unique index j such that $1 \leqq j \leqq r$ and that the representation of G in V_j/V_{j-1} is spherical. Then this representation is isomorphic to (π_χ, V_χ).*

The last two statements come from the fact that $\Phi_{K,\chi}$ is, up to constant multiples, the unique function in $I(\chi)$ invariant under $\nu_\chi(K)$ and from the relation (34) which can be expressed as $\Gamma_\chi = \Phi^\natural_{K,\chi}$.

REMARKS. (1) It is easy to show without recourse to Satake's Theorem 4.1 that the representation $(\nu_\chi, I(\chi))$ is spherical *provided it is irreducible.* Since then the representations $(\nu_\chi, I(\chi))$ and $(\nu_{w\cdot\chi}, I(w \cdot \chi))$ are equivalent for any w in W, this provides another proof of step (B) in Satake's theorem (see criterion 3.10, p. 144).

(2) We refer the reader to Macdonald [**37**, p. 63] for a characterization of the *bounded* spherical functions, that is the spectrum of the Banach algebra of integrable functions on G which are bi-invariant under K. It does not seem to be known which spherical functions are positive-definite, or stated in other terms, which spherical representations are preunitary.

4.5. *The explicit formula for the spherical functions.* The following result is due to Macdonald [**36**], [**37**].

THEOREM 4.4. *Suppose that the unramified character χ of M is regular. For any m in M^-, the value of the spherical function Γ_χ is given as follows:*

$$\Gamma_\chi(m) = Q^{-1}\, \delta(m)^{1/2} \sum_{w \in W} \mathbf{c}(w \cdot \chi)\, w \cdot \chi(m) \tag{42}$$

where

$$Q = \sum_{w \in W} q(w)^{-1}, \tag{43}$$

$$\mathbf{c}(\lambda) = \prod_{\alpha > 0} (1 - q_{\alpha/2}^{-1/2}\, q_\alpha^{-1} \lambda(a_\alpha)^{-1})(1 + q_{\alpha/2}^{-1/2}\, \lambda(a_\alpha)^{-1})(1 - \lambda(a_\alpha)^{-2})^{-1} \tag{44}$$

for any regular unramified character λ of M (product extended over the roots α in Φ_0 which are positive on the conical chamber $\mathscr{C}$).

We sketch the proof given by Casselman [**18**], which rests on the properties of the intertwining operators. We use the notations of §3.7.

For each w in W, the linear form L_w on $I(\chi)$ transforms by M according to the character $\delta^{1/2}(w \cdot \chi)$ and is trivial on $I(\chi)(N)$. Since χ is regular, these characters of M are distinct; hence the linear forms L_w are linearly independent. The vector space $I(\chi)^B$ is supplementary to $I(\chi)(N)$ in $I(\chi)$ and its dimension is equal to $|W|$. Hence there exists in $I(\chi)^B$ a basis $\{f_{w,\chi}\}_{w \in W}$ characterized by

$$(45) \qquad L_w(f_{w',\chi}) = 1 \quad \text{if } w' = w, \qquad = 0 \quad \text{if } w' \neq w.$$

As a corollary of Theorem 3.9 one gets $\Phi_{K,\chi} = \sum_{w \in W} c_w(\chi) \cdot f_{w,\chi}$. On the other hand, one has $\Gamma_\chi(m) = \int_K \Phi_{K,\chi}(km)\, dk$. Since $f_{w,\chi}$ is invariant under $\nu_\chi(B)$ and B is a subgroup of K, one gets from these remarks the relation

$$(46) \qquad \Gamma_\chi(m) = \sum_{w \in W} c_w(\chi) \int_K g_w(k)\, dk$$

where[14] $g_w = \mu(BmB)^{-1}\pi(I_{BmB}) \cdot f_{w,\chi}$. By the methods used in Theorem 2.3, one proves, in general, that, for any admissible representation (π, V) of G and any m in M, the operator $\pi(I_{BmB}) - \mu(BmB)\pi(m)$ maps V into $V(N)$. From (45), one infers

$$(47) \qquad g_w = \delta(m)^{1/2}\, w \cdot \chi(m)\, f_{w,\chi}.$$

From (46) and (47), one deduces that, on M^-, the spherical function Γ_χ agrees with a linear combination of the characters $\delta^{1/2}(w \cdot \chi)$ of M. Taking into account the invariance property $\Gamma_{w \cdot \chi} = \Gamma_\chi$, it suffices to calculate one of these coefficients, for instance the coefficient of $\delta^{1/2}(w_0 \cdot \chi)$ where w_0 is the (unique) element of W which takes the conical chamber $\mathscr{C}$ into its opposite $\mathscr{C}^-$ (or any positive root in Φ_0 to a negative root). In this case, one proves without difficulty that $f_{w_0,\chi}$ is equal to the function $\Phi_{w_0,\chi}$ defined by formula (28) in §3.6. The sought-for coefficient is obtained by multiplying $c_{w_0}(w_0 \cdot \chi)$ by

$$\int_K f_{w_0,\chi}(k)\, dk = \int_K \Phi_{w_0,\chi}(k)\, dk = \mu(Bw_0B).$$

It remains to show that the measure of Bw_0B, that is the index $(Bw_0B\colon K)$, is equal to Q^{-1}. This follows easily from the formulas (21) to (24) in §3.5. Q.E.D.

Note added in proof. As I was told by the editors, my conventions about algebraic groups differ slightly from those of other authors in these PROCEEDINGS. The local field F being infinite, the set $G(F)$ of F-points of any of the algebraic groups G used in the previous paper is Zariski-dense in G and I allowed myself to identify G to $G(F)$.

[14] The Haar measure μ on G is normalized by $\mu(K) = 1$.

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