CASIMIRS OF THE GOLDMAN LIE ALGEBRA OF A CLOSED SURFACE

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1. Introduction

Let Σ be a connected closed oriented surface of genus g. In 1986 Goldman [Go] attached to Σ a Lie algebra $L = L(\Sigma)$, later shown by Turaev ([Tu]) to have a natural structure of a Lie bialgebra. It is defined as follows. As a vector space, L has a basis e_γ labeled by conjugacy classes γ in the fundamental group $\pi_1(\Sigma)$, geometrically represented by closed oriented curves on Σ without a base point. To define the commutator $[e_{\gamma_1}, e_{\gamma_2}]$, one needs to bring the two curves γ_1, γ_2 into general position by isotopy, and then for each intersection point p_i of the two curves, define γ_{3i} to be the curve obtained by tracing γ_1 and then γ_2 starting and ending at p_i. Then one defines $[e_{\gamma_1}, e_{\gamma_2}]$ to be $\sum_i \varepsilon_i e_{\gamma_{3i}}$, where $\varepsilon_i = 1$ if γ_1 approaches γ_2 from the right at p_i (with respect to the orientation of Σ), and -1 otherwise.

The combinatorial structure of L has been much studied; see e.g. [C, Tu]. However, many problems about the structure of L remained open. In particular, in 2001, M. Chas and D. Sullivan communicated to me the following conjecture.

Conjecture 1.1. The center of L is spanned by the element e_1, where $1 \in \pi_1(\Sigma)$ is the trivial loop.

In this paper, we will prove this conjecture. In fact, we prove a more general result.

Theorem 1.2. *The Poisson center of the Poisson algebra $S^\bullet L$ is $Z = \mathbb{C}[e_1]$.*

The proof of the theorem occupies the rest of the paper.

Remark. A quiver theoretical analog of Theorem 1.2 is given in [CEG]. It claims that if Π is the preprojective algebra of a quiver Q which is not Dynkin or affine Dynkin, then the Poisson center of $S^\bullet L$ (where $L = \Pi/[\Pi, \Pi]$ is the necklace Lie algebra attached to Π) consists of polynomials in the vertex idempotents.

2. Proof of the theorem

2.1. **Moduli spaces of flat bundles.** We will assume that $g > 1$, since in the case $g \leq 1$ the theorem is easy.

Recall that the fundamental group $\Gamma = \pi_1(\Sigma)$ is generated by $X_1, ..., X_g$, $Y_1, ..., Y_g$ with defining relation

$$\prod_{i=1}^{g} X_i Y_i X_i^{-1} Y_i^{-1} = 1. \tag{1}$$

Thus we can define the scheme of homomorphisms $\widetilde{M}_g(N) = \mathrm{Hom}(\Gamma, GL_N(\mathbb{C}))$ to be the closed subscheme in $GL_N(\mathbb{C})^{2g}$ defined by equation (1). One can also define the moduli scheme of representations (or equivalently, of flat connections on Σ) to be the categorical quotient $M_g(N) = \widetilde{M}_g(N)/PGL_N(\mathbb{C})$.

The schemes $\widetilde{M}_g(N)$ and $M_g(N)$ carry the Atiyah-Bott Poisson structure; its algebraic presentation may be found in [FR] (using r-matrices) and [AMM] (using quasi-Hamiltonian reduction); see also [Go].

Let us recall the following known results about these schemes, which we will use in the sequel.

Theorem 2.1. *(i) $\widetilde{M}_g(N)$ and $M_g(N)$ are reduced.*

(ii) $\widetilde{M}_g(N)$ is a complete intersection in $GL_N(\mathbb{C})^{2g}$.

(iii) $\widetilde{M}_g(N)$ and $M_g(N)$ are irreducible algebraic varieties. Their generic points correspond to irreducible representations of Γ.

(iv) The Poisson structure on $M_g(N)$ is generically symplectic.

Proof. Let $\widetilde{M}'_g(N)$ be the algebraic variety corresponding to the scheme $\widetilde{M}_g(N)$. It is shown in [Li] that this variety is irreducible. Moreover, it is clear that the generic point of this variety corresponds to an irreducible representation of Γ (we can choose X_i, Y_i generically for $i < g$ and then solve for X_g, Y_g). It is easy to show that near such a point the map $\mu : GL(N)^{2g} \to SL(N)$ given by the left hand side of (1) is a submersion. This implies (ii). We also see that $\widetilde{M}_g(N)$ is generically reduced. Since it is a complete intesection, it is Cohen-Macaulay and hence reduced everywhere. Thus we get (i) and (iii). Property (iv) is well known and is readily seen from [FR] or [AMM]. The theorem is proved. □

2.2. **Injectivity of the Goldman homomorphism.** Now let us return to the study of the Lie algebra L. To put ourselves in an algebraic framework, we note that L is naturally identified with $A/[A, A]$, where $A = \mathbb{C}[\Gamma]$ is the group algebra of Γ. Thus, elements of L can be represented by linear combinations of cyclic words in $X_i^{\pm 1}, Y_i^{\pm 1}$.

In [Go], Goldman defined a homomorphism of Poisson algebras

$$\phi_N : S^\bullet L \to \mathbb{C}[M_g(N)]$$

defined by the formula $\phi_N(w)(\rho) = \mathrm{Tr}(\rho(w))$, where ρ is an N-dimensional representation of Γ and w is any cyclic word representing an element of L. It follows from Weyl's fundamental theorem of invariant theory that the Goldman homomorphism is surjective.

Let $L_+ \subset L$ be the linear span of the elements $e_\gamma - e_1$. Obviously, we have $L = L_+ \oplus \mathbb{C}e_1$,

Proposition 2.2. *For any finite dimensional subspace $Y \subset S^\bullet L_+$, there exists an integer $N(Y)$ such that for $N \geq N(Y)$, the map $\phi_N|_Y$ is injective.*

Proof. Let $K(N)$ be the kernel of ϕ_N on $S^\bullet L_+$. It is clear that $K(N+1) \subset K(N)$ (as $\phi_{N+1}(e_\gamma - e_1)(\rho \oplus \mathbb{C}) = \phi_N(e_\gamma - e_1)(\rho)$). Thus it suffices to show that $\cap_{N \geq 1} K(N) = 0$.

Assume the contrary. Then there exists an element $0 \neq f \in S^\bullet L_+$ such that $\phi_N(f) = 0$ for all N.

Recall that according to [FiR], the group Γ is **conjugacy separable**, i.e., if elements $\gamma_0, ..., \gamma_m$ are pairwise not conjugate in Γ then there exists a finite quotient Γ' of Γ such that the images of $\gamma_0, ..., \gamma_m$ are not conjugate in Γ'.

Now let $\gamma_0 = 1$ and $f = P(e_{\gamma_1} - e_1, ..., e_{\gamma_m} - e_1)$, where P is some polynomial. Let Γ' be the finite group as above, $V_1, ..., V_s$ be the irreducible representations of Γ', and $\chi_1, ..., \chi_s$ be their characters. Let $V = \oplus_j N_j V_j$; we regard V as a representation of Γ and let $N = \dim V$. Then $\phi_N(f)(V) = P(w_1, ..., w_m)$, where $w_i = \sum_j N_j(\chi_j(\gamma_i) - \chi_j(1))$. By representation theory of finite groups, the matrix with entries $a_{ij} = \chi_j(\gamma_i) - \chi_j(1)$ has rank m; thus, there exist $N_j \geq 0$ such that $P(w_1, ..., w_m) \neq 0$. For such N_j, $\phi_N(f) \neq 0$, which is a contradiction. □

2.3. **Proof of Theorem 1.2.** Now we are ready to prove Theorem 1.2. Let z be a central element of the Poisson algebra $S^\bullet L$. Consider the element $\phi_N(z)$. This is a regular function on $M_g(N)$ which Poisson commutes with all other functions (since ϕ_N is surjective). Since by Theorem 2.1 the scheme $M_g(N)$ is in fact a variety, which is irreducible and generically symplectic, any Casimir on this variety must be a scalar.

Since $S^\bullet L = S^\bullet L_+ \otimes \mathbb{C}[e_1]$, we can write z as

$$z = \zeta(e_1) + \sum_{j=1}^{m} \zeta_j(e_1) f_j,$$

were f_j are linearly independent elements which belong to the augmentation ideal of $S^\bullet L_+$, and $\zeta, \zeta_j \in \mathbb{C}[t]$. Applying ϕ_N to this equation, and using that $\phi_N(e_1) = N$, we get that

$$\zeta(N) + \sum_{j=1}^{m} \zeta_j(N) \phi_N(f_j) = \gamma_N.$$

Let Y be the linear span of 1 and f_j, $j = 1, ..., m$ in $S^\bullet L_+$. By Proposition 2.2, for $N \geq N(Y)$, we have

$$\zeta(N) + \sum_{j=1}^{m} \zeta_j(N) f_j = \gamma_N.$$

Thus $\zeta_j(N) = 0$ for $N \geq N(Y)$. Hence $\zeta_j = 0$ for all j and $z = \zeta(e_1)$. The theorem is proved.

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