INTRODUCTION
TO
LINEAR
ALGEBRA
Third Edition

SOLUTION MANUAL

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Solutions to Exercises

Problem Set 1.1, page 6

1 Line through $(1,1,1)$; plane; same plane!

3 $\boldsymbol{v} = (2,2)$ and $\boldsymbol{w} = (1,-1)$.

4 $3\boldsymbol{v}+\boldsymbol{w} = (7,5)$ and $\boldsymbol{v}-3\boldsymbol{w} = (-1,-5)$ and $c\boldsymbol{v}+d\boldsymbol{w} = (2c+d, c+2d)$.

5 $\boldsymbol{u}+\boldsymbol{v} = (-2,3,1)$ and $\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w} = (0,0,0)$ and $2\boldsymbol{u}+2\boldsymbol{v}+\boldsymbol{w} =$ (add first answers) $= (-2,3,1)$.

6 The components of every $c\boldsymbol{v}+d\boldsymbol{w}$ add to zero. Choose $c=4$ and $d=10$ to get $(4,2,-6)$.

8 The other diagonal is $\boldsymbol{v}-\boldsymbol{w}$ (or else $\boldsymbol{w}-\boldsymbol{v}$). Adding diagonals gives $2\boldsymbol{v}$ (or $2\boldsymbol{w}$).

9 The fourth corner can be $(4,4)$ or $(4,0)$ or $(-2,2)$.

10 $\boldsymbol{i}+\boldsymbol{j}$ is the diagonal of the base.

11 Five more corners $(0,0,1),(1,1,0),(1,0,1),(0,1,1),(1,1,1)$. The center point is $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$. The centers of the six faces are $(\frac{1}{2},\frac{1}{2},0),(\frac{1}{2},\frac{1}{2},1)$ and $(0,\frac{1}{2},\frac{1}{2}),(1,\frac{1}{2},\frac{1}{2})$ and $(\frac{1}{2},0,\frac{1}{2}),(\frac{1}{2},1,\frac{1}{2})$.

12 A four-dimensional cube has $2^4 = 16$ corners and $2\cdot 4 = 8$ three-dimensional sides and 24 two-dimensional faces and 32 one-dimensional edges. See Worked Example **2.4 A**.

13 sum = zero vector; sum = -4:00 vector; 1:00 is 60° from horizontal $= (\cos\frac{\pi}{3}, \sin\frac{\pi}{3}) = (\frac{1}{2},\frac{\sqrt{3}}{2})$.

14 Sum $= 12\boldsymbol{j}$ since $\boldsymbol{j} = (0,1)$ is added to every vector.

15 The point $\frac{3}{4}\boldsymbol{v}+\frac{1}{4}\boldsymbol{w}$ is three-fourths of the way to $\boldsymbol{v}$ starting from $\boldsymbol{w}$. The vector $\frac{1}{4}\boldsymbol{v}+\frac{1}{4}\boldsymbol{w}$ is halfway to $\boldsymbol{u} = \frac{1}{2}\boldsymbol{v}+\frac{1}{2}\boldsymbol{w}$, and the vector $\boldsymbol{v}+\boldsymbol{w}$ is $2\boldsymbol{u}$ (the far corner of the parallelogram).

16 All combinations with $c+d=1$ are on the line through $\boldsymbol{v}$ and $\boldsymbol{w}$. The point $\boldsymbol{V} = -\boldsymbol{v}+2\boldsymbol{w}$ is on that line beyond $\boldsymbol{w}$.

17 The vectors $c\boldsymbol{v}+c\boldsymbol{w}$ fill out the line passing through $(0,0)$ and $\boldsymbol{u} = \frac{1}{2}\boldsymbol{v}+\frac{1}{2}\boldsymbol{w}$. It continues beyond $\boldsymbol{v}+\boldsymbol{w}$ and $(0,0)$. With $c \geq 0$, half this line is removed and the "ray" starts at $(0,0)$.

18 The combinations with $0 \leq c \leq 1$ and $0 \leq d \leq 1$ fill the parallelogram with sides $\boldsymbol{v}$ and $\boldsymbol{w}$.

19 With $c \geq 0$ and $d \geq 0$ we get the "cone" or "wedge" between $\boldsymbol{v}$ and $\boldsymbol{w}$.

20 (a) $\frac{1}{3}\boldsymbol{u}+\frac{1}{3}\boldsymbol{v}+\frac{1}{3}\boldsymbol{w}$ is the center of the triangle between $\boldsymbol{u},\boldsymbol{v}$ and $\boldsymbol{w}$; $\frac{1}{2}\boldsymbol{u}+\frac{1}{2}\boldsymbol{w}$ is the center of the edge between $\boldsymbol{u}$ and $\boldsymbol{w}$ (b) To fill in the triangle keep $c \geq 0$, $d \geq 0$, $e \geq 0$, and $c+d+e=1$.

21 The sum is $(\boldsymbol{v}-\boldsymbol{u})+(\boldsymbol{w}-\boldsymbol{v})+(\boldsymbol{u}-\boldsymbol{w})=$ zero vector.

22 The vector $\frac{1}{2}(\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w})$ is *outside* the pyramid because $c+d+e=\frac{1}{2}+\frac{1}{2}+\frac{1}{2}>1$.

23 *All* vectors are combinations of $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$.

24 Vectors $c\boldsymbol{v}$ are in both planes.

25 (a) Choose $\boldsymbol{u}=\boldsymbol{v}=\boldsymbol{w}=$ any nonzero vector (b) Choose $\boldsymbol{u}$ and $\boldsymbol{v}$ in different directions, and $\boldsymbol{w}$ to be a combination like $\boldsymbol{u}+\boldsymbol{v}$.

26 The solution is $c=2$ and $d=4$. Then $2(1,2)+4(3,1)=(14,8)$.

27 The combinations of $(1,0,0)$ and $(0,1,0)$ fill the xy plane in xyz space.

28 An example is $(a,b)=(3,6)$ and $(c,d)=(1,2)$. The ratios a/c and b/d are equal. Then $ad=bc$. Then (divide by bd) the ratios a/b and c/d are equal!

Problem Set 1.2, page 17

1 $\boldsymbol{u}\cdot\boldsymbol{v}=1.4$, $\boldsymbol{u}\cdot\boldsymbol{w}=0$, $\boldsymbol{v}\cdot\boldsymbol{w}=24=\boldsymbol{w}\cdot\boldsymbol{v}$.

2 $\|\boldsymbol{u}\|=1$ and $\|\boldsymbol{v}\|=5=\|\boldsymbol{w}\|$. Then $1.4<(1)(5)$ and $24<(5)(5)$.

3 Unit vectors $\boldsymbol{v}/\|\boldsymbol{v}\|=(\frac{3}{5},\frac{4}{5})=(.6,.8)$ and $\boldsymbol{w}/\|\boldsymbol{w}\|=(\frac{4}{5},\frac{3}{5})=(.8,.6)$. The cosine of θ is $\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}\cdot\frac{\boldsymbol{w}}{\|\boldsymbol{w}\|}=\frac{24}{25}$. The vectors $\boldsymbol{w},\boldsymbol{u},-\boldsymbol{w}$ make $0°, 90°, 180°$ angles with $\boldsymbol{w}$.

4 $\boldsymbol{u}_1=\boldsymbol{v}/\|\boldsymbol{v}\|=\frac{1}{\sqrt{10}}(3,1)$ and $\boldsymbol{u}_2=\boldsymbol{w}/\|\boldsymbol{w}\|=\frac{1}{3}(2,1,2)$. $\boldsymbol{U}_1=\frac{1}{\sqrt{10}}(1,-3)$ or $\frac{1}{\sqrt{10}}(-1,3)$. $\boldsymbol{U}_2$ could be $\frac{1}{\sqrt{5}}(1,-2,0)$.

5 (a) $\boldsymbol{v}\cdot(-\boldsymbol{v})=-1$ (b) $(\boldsymbol{v}+\boldsymbol{w})\cdot(\boldsymbol{v}-\boldsymbol{w})=\boldsymbol{v}\cdot\boldsymbol{v}+\boldsymbol{w}\cdot\boldsymbol{v}-\boldsymbol{v}\cdot\boldsymbol{w}-\boldsymbol{w}\cdot\boldsymbol{w}=1+(\quad)-(\quad)-1=0$ so $\theta=90°$ (c) $(\boldsymbol{v}-2\boldsymbol{w})\cdot(\boldsymbol{v}+2\boldsymbol{w})=\boldsymbol{v}\cdot\boldsymbol{v}-4\boldsymbol{w}\cdot\boldsymbol{w}=-3$

6 (a) $\cos\theta=\frac{1}{(2)(1)}$ so $\theta=60°$ or $\frac{\pi}{3}$ radians (b) $\cos\theta=0$ so $\theta=90°$ or $\frac{\pi}{2}$ radians
(c) $\cos\theta=\frac{-1+3}{(2)(2)}=\frac{1}{2}$ so $\theta=60°$ or $\frac{\pi}{3}$ (d) $\cos\theta=-1/\sqrt{2}$ so $\theta=135°$ or $\frac{3\pi}{4}$.

7 All vectors $\boldsymbol{w}=(c,2c)$; all vectors (x,y,z) with $x+y+z=0$ lie on a *plane*; all vectors perpendicular to $(1,1,1)$ and $(1,2,3)$ lie on a *line*.

8 (a) False (b) True: $\boldsymbol{u}\cdot(c\boldsymbol{v}+d\boldsymbol{w})=c\boldsymbol{u}\cdot\boldsymbol{v}+d\boldsymbol{u}\cdot\boldsymbol{w}=0$ (c) True

9 If $v_2w_2/v_1w_1=-1$ then $v_2w_2=-v_1w_1$ or $v_1w_1+v_2w_2=0$.

10 Slopes $\frac{2}{1}$ and $-\frac{1}{2}$ multiply to give -1: perpendicular.

11 $\boldsymbol{v}\cdot\boldsymbol{w}<0$ means angle $>90°$; this is half of the plane.

12 $(1,1)$ perpendicular to $(1,5)-c(1,1)$ if $6-2c=0$ or $c=3$; $\boldsymbol{v}\cdot(\boldsymbol{w}-c\boldsymbol{v})=0$ if $c=\boldsymbol{v}\cdot\boldsymbol{w}/\boldsymbol{v}\cdot\boldsymbol{v}$.

13 $\boldsymbol{v}=(1,0,-1)$, $\boldsymbol{w}=(0,1,0)$.

14 $\boldsymbol{u}=(1,-1,0,0)$, $\boldsymbol{v}=(0,0,1,-1)$, $\boldsymbol{w}=(1,1,-1,-1)$.

15 $\frac{1}{2}(x+y)=5$; $\cos\theta=2\sqrt{16}/\sqrt{10}\sqrt{10}=.8$.

16 $\|\boldsymbol{v}\|^2=9$ so $\|\boldsymbol{v}\|=3$; $\boldsymbol{u}=\frac{1}{3}\boldsymbol{v}$; $\boldsymbol{w}=(1,-1,0,\ldots,0)$.

17 $\cos\alpha=1/\sqrt{2}$, $\cos\beta=0$, $\cos\gamma=-1/\sqrt{2}$, $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=(v_1^2+v_2^2+v_3^2)/\|\boldsymbol{v}\|^2=1$.

18 $\|\boldsymbol{v}\|^2 = 4^2 + 2^2 = 20$, $\|\boldsymbol{w}\|^2 = (-1)^2 + 2^2 = 5$, $\|(3,4)\|^2 = 25 = 20 + 5$.

19 $\boldsymbol{v} - \boldsymbol{w} = (5,0)$ also has $(\text{length})^2 = 25$. Choose $\boldsymbol{v} = (1,1)$ and $\boldsymbol{w} = (0,1)$ which are not perpendicular; $(\text{length of } \boldsymbol{v})^2 + (\text{length of } \boldsymbol{w})^2 = 1^2 + 1^2 + 1^2$ but $(\text{length of } \boldsymbol{v} - \boldsymbol{w})^2 = 1$.

20 $(\boldsymbol{v}+\boldsymbol{w})\cdot(\boldsymbol{v}+\boldsymbol{w}) = (\boldsymbol{v}+\boldsymbol{w})\cdot\boldsymbol{v} + (\boldsymbol{v}+\boldsymbol{w})\cdot\boldsymbol{w} = \boldsymbol{v}\cdot(\boldsymbol{v}+\boldsymbol{w}) + \boldsymbol{w}\cdot(\boldsymbol{v}+\boldsymbol{w}) = \boldsymbol{v}\cdot\boldsymbol{v} + \boldsymbol{v}\cdot\boldsymbol{w} + \boldsymbol{w}\cdot\boldsymbol{v} + \boldsymbol{w}\cdot\boldsymbol{w} = \boldsymbol{v}\cdot\boldsymbol{v} + 2\boldsymbol{v}\cdot\boldsymbol{w} + \boldsymbol{w}\cdot\boldsymbol{w}$. Notice $\boldsymbol{v}\cdot\boldsymbol{w} = \boldsymbol{w}\cdot\boldsymbol{v}$!

21 $2\boldsymbol{v}\cdot\boldsymbol{w} \le 2\|\boldsymbol{v}\|\|\boldsymbol{w}\|$ leads to $\|\boldsymbol{v}+\boldsymbol{w}\|^2 = \boldsymbol{v}\cdot\boldsymbol{v} + 2\boldsymbol{v}\cdot\boldsymbol{w} + \boldsymbol{w}\cdot\boldsymbol{w} \le \|\boldsymbol{v}\|^2 + 2\|\boldsymbol{v}\|\|\boldsymbol{w}\| + \|\boldsymbol{w}\|^2 = (\|\boldsymbol{v}\| + \|\boldsymbol{w}\|)^2$.

22 Compare $\boldsymbol{v}\cdot\boldsymbol{v} + \boldsymbol{w}\cdot\boldsymbol{w}$ with $(\boldsymbol{v}-\boldsymbol{w})\cdot(\boldsymbol{v}-\boldsymbol{w})$ to find that $-2\boldsymbol{v}\cdot\boldsymbol{w} = 0$. Divide by -2.

23 $\cos\beta = w_1/\|\boldsymbol{w}\|$ and $\sin\beta = w_2/\|\boldsymbol{w}\|$. Then $\cos(\beta - a) = \cos\beta\cos\alpha + \sin\beta\sin\alpha = v_1w_1/\|\boldsymbol{v}\|\|\boldsymbol{w}\| + v_2w_2/\|\boldsymbol{v}\|\|\boldsymbol{w}\| = \boldsymbol{v}\cdot\boldsymbol{w}/\|\boldsymbol{v}\|\|\boldsymbol{w}\|$.

24 We know that $(\boldsymbol{v}-\boldsymbol{w})\cdot(\boldsymbol{v}-\boldsymbol{w}) = \boldsymbol{v}\cdot\boldsymbol{v} - 2\boldsymbol{v}\cdot\boldsymbol{w} + \boldsymbol{w}\cdot\boldsymbol{w}$. The Law of Cosines writes $\|\boldsymbol{v}\|\|\boldsymbol{w}\|\cos\theta$ for $\boldsymbol{v}\cdot\boldsymbol{w}$. When $\theta < 90^\circ$ this is positive and $\boldsymbol{v}\cdot\boldsymbol{v} + \boldsymbol{w}\cdot\boldsymbol{w}$ is larger than $\|\boldsymbol{v}-\boldsymbol{w}\|^2$.

25 (a) $v_1^2w_1^2 + 2v_1w_1v_2w_2 + v_2^2w_2^2 \le v_1^2w_1^2 + v_1^2w_2^2 + v_2^2w_1^2 + v_2^2w_2^2$ is true because the difference is $v_1^2w_2^2 + v_2^2w_1^2 - 2v_1w_1v_2w_2$ which is $(v_1w_2 - v_2w_1)^2 \ge 0$.

26 Example 6 gives $|u_1||U_1| \le \frac{1}{2}(u_1^2 + U_1^2)$ and $|u_2||U_2| \le \frac{1}{2}(u_2^2 + U_2^2)$. The whole line becomes $.96 \le (.6)(.8) + (.8)(.6) \le \frac{1}{2}(.6^2 + .8^2) + \frac{1}{2}(.8^2 + .6^2) = 1$.

27 The cosine of θ is $x/\sqrt{x^2+y^2}$, near side over hypotenuse. Then $|\cos\theta|^2 = x^2/(x^2+y^2) \le 1$.

28 Try $\boldsymbol{v} = (1,2,-3)$ and $\boldsymbol{w} = (-3,1,2)$ with $\cos\theta = \frac{-7}{14}$ and $\theta = 120^\circ$. Write $\boldsymbol{v}\cdot\boldsymbol{w} = xz + yz + xy$ as $\frac{1}{2}(x+y+z)^2 - \frac{1}{2}(x^2+y^2+z^2)$. If $x+y+z = 0$ this is $-\frac{1}{2}(x^2+y^2+z^2)$, so $\boldsymbol{v}\cdot\boldsymbol{w}/\|\boldsymbol{v}\|\|\boldsymbol{w}\| = -\frac{1}{2}$.

29 The length $\|\boldsymbol{v}-\boldsymbol{w}\|$ is between 2 and 8. The dot product $\boldsymbol{v}\cdot\boldsymbol{w}$ is between -15 and 15.

30 The vectors $\boldsymbol{w} = (x,y)$ with $\boldsymbol{v}\cdot\boldsymbol{w} = x + 2y = 5$ lie on a line in the xy plane. The shortest $\boldsymbol{w}$ is $(1,2)$ in the direction of $\boldsymbol{v}$.

31 Three vectors in the plane could make angles $> 90^\circ$ with each other: $(1,0), (-1,4), (-1,-4)$. Four vectors could not do this (360° total angle). How many can do this in $\mathbf{R}^3$ or $\mathbf{R}^n$?

Problem Set 2.1, page 29

1 Row picture: The planes $x = 2$ and $y = 3$ and $z = 4$ are perpendicular to the x, y, z axes.

2 The columns are $\boldsymbol{i} = (1,0,0)$ and $\boldsymbol{j} = (0,1,0)$ and $\boldsymbol{k} = (0,0,1)$ and $\boldsymbol{b} = (2,3,4) = 2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$.

3 The planes are the same: $2x = 4$ is $x = 2$, $3y = 9$ is $y = 3$, and $4z = 16$ is $z = 4$. The solution is the same intersection point. The columns are changed; but same combination $\widehat{\boldsymbol{x}} = \boldsymbol{x}$.

4 The solution is not changed; the second plane and row 2 of the matrix and all columns of the matrix are changed.

5 If $z = 2$ then $x + y = 0$ and $x - y = z$ give the point $(1,-1,2)$. If $z = 0$ then $x + y = 6$ and $x - y = 4$ give the point $(5,1,0)$. Halfway between is $(3,0,1)$.

6 If x, y, z satisfy the first two equations they also satisfy the third equation. The line **L** of solutions contains $\boldsymbol{v} = (1, 1, 0)$ and $\boldsymbol{w} = (\frac{1}{2}, 1, \frac{1}{2})$ and $\boldsymbol{u} = \frac{1}{2}\boldsymbol{v} + \frac{1}{2}\boldsymbol{w}$ and all combinations $c\boldsymbol{v} + d\boldsymbol{w}$ with $c + d = 1$.

7 Equation 1 + equation 2 − equation 3 is now $0 = -4$. Line misses plane; *no solution.*

8 Column 3 = Column 1; solutions $(x, y, z) = (1, 1, 0)$ or $(0, 1, 1)$ and you can add any multiple of $(-1, 0, 1)$; $\boldsymbol{b} = (4, 6, c)$ needs $c = 10$ for solvability.

9 Four planes in 4-dimensional space normally meet at a *point*. The solution to $A\boldsymbol{x} = (3, 3, 3, 2)$ is $\boldsymbol{x} = (0, 0, 1, 2)$ if A has columns $(1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 1, 0), (1, 1, 1, 1)$. The equations are $x + y + z + t = 3, y + z + t = 3, z + t = 3, t = 2$.

10 $A\boldsymbol{x} = (18, 5, 0)$, $A\boldsymbol{x} = (3, 4, 5, 5)$.

11 Nine multiplications for $A\boldsymbol{x} = (18, 5, 0)$.

12 $(14, 22)$ and $(0, 0)$ ($2 \times$ column 1 = column 2) and $(9, 7)$.

13 (z, y, x) and $(0, 0, 0)$ and $(3, 3, 6)$.

14 (a) $\boldsymbol{x}$ has n components, $A\boldsymbol{x}$ has m components (b) Planes in n-dimensional space, but the columns are in m-dimensional space.

15 $2x + 3y + z + 5t = 8$ is $A\boldsymbol{x} = \boldsymbol{b}$ with the 1 by 4 matrix $A = [2\ \ 3\ \ 1\ \ 5]$. The solutions $\boldsymbol{x}$ fill a 3D "plane" in 4 dimensions.

16 $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

17 $R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, 180° rotation from $R^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$.

18 $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ produces (y, z, x) and $Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ recovers (x, y, z).

19 $E = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$, $E = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

20 $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, $E\boldsymbol{v} = (3, 4, 8)$, $E^{-1}E\boldsymbol{v} = (3, 4, 5)$.

21 $P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $P_1\boldsymbol{v} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$, $P_2P_1\boldsymbol{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

22 $R = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$.

23 The dot product $[1\ \ 4\ \ 5]\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ = (1 by 3)(3 by 1) is zero for points (x, y, z) on a plane in three dimensions. The columns of A are one-dimensional vectors.

24 $A = [\,1\ \ 2\ \ ;\ \ 3\ \ 4\,]$ and $\boldsymbol{x} = [\,5\ \ -2\,]'$ and $\boldsymbol{b} = [\,1\ \ 7\,]'$. $\boldsymbol{r} = \boldsymbol{b} - A * \boldsymbol{x}$ prints as zero.

25 $A * \boldsymbol{v} = [\,3\ \ 4\ \ 5\,]'$ and $\boldsymbol{v}' * \boldsymbol{v} = 50$; $\boldsymbol{v} * A$ gives an error message.

26 $\mathsf{ones}(4,4) * \mathsf{ones}(4,1) = [\,4\ \ 4\ \ 4\ \ 4\,]'$; $B * \boldsymbol{w} = [\,10\ \ 10\ \ 10\ \ 10\,]'$.

27 The row picture has two lines meeting at $(4, 2)$. The column picture has $4(1, 1) + 2(-2, 1) =$ $4(\text{column } 1) + 2(\text{column } 2) = \text{right side } (0, 6)$.

28 The row picture shows 2 planes in 3-dimensional space. The column picture is in 2-dimensional space. The solutions normally lie on a *line*.

29 The row picture shows four *lines*. The column picture is in *four*-dimensional space. No solution unless the right side is a combination of *the two columns*.

30 $\boldsymbol{u}_2 = \begin{bmatrix} .7 \\ .3 \end{bmatrix}$, $\boldsymbol{u}_3 = \begin{bmatrix} .65 \\ .35 \end{bmatrix}$. The components always add to 1. They are always positive.

31 $\boldsymbol{u}_7, \boldsymbol{v}_7, \boldsymbol{w}_7$ are all close to $(.6, .4)$. Their components still add to 1.

32 $\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} .6 \\ .4 \end{bmatrix} = \begin{bmatrix} .6 \\ .4 \end{bmatrix} =$ *steady state* $\boldsymbol{s}$. No change when multiplied by $\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$.

34 $M = \begin{bmatrix} 8 & 3 & 4 \\ 1 & 5 & 9 \\ 6 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 5+u & 5-u+v & 5-v \\ 5-u-v & 5 & 5+u+v \\ 5+v & 5+u-v & 5-u \end{bmatrix}$; $M_3(1,1,1) = (15, 15, 15)$;
$M_4(1,1,1,1) = (34, 34, 34, 34)$ because the numbers 1 to 16 add to 136 which is $4(34)$.

Problem Set 2.2, page 40

1 Multiply by $l = \frac{10}{2} = 5$ and subtract to find $2x + 3y = 14$ and $-6y = 6$.

2 $y = -1$ and then $x = 2$. Multiplying the right side by 4 will multiply (x, y) by 4 to give the solution $(x, y) = (8, -4)$.

3 Subtract $-\frac{1}{2}$ times equation 1 (or add $\frac{1}{2}$ times equation 1). The new second equation is $3y = 3$. Then $y = 1$ and $x = 5$. If the right side changes sign, so does the solution: $(x, y) = (-5, -1)$.

4 Subtract $l = \frac{c}{a}$ times equation 1. The new second pivot multiplying y is $d - (cb/a)$ or $(ad - bc)/a$. Then $y = (ag - cf)/(ad - bc)$.

5 $6x + 4y$ is 2 times $3x + 2y$. There is no solution unless the right side is $2 \cdot 10 = 20$. Then all points on the line $3x + 2y = 10$ are solutions, including $(0, 5)$ and $(4, -1)$.

6 Singular system if $b = 4$, because $4x + 8y$ is 2 times $2x + 4y$. Then $g = 2 \cdot 16 = 32$ makes the system solvable. The lines become the *same*: infinitely many solutions like $(8, 0)$ and $(0, 4)$.

7 If $a = 2$ elimination must fail. The equations have no solution. If $a = 0$ elimination stops for a row exchange. Then $3y = -3$ gives $y = -1$ and $4x + 6y = 6$ gives $x = 3$.

8 If $k = 3$ elimination must fail: no solution. If $k = -3$, elimination gives $0 = 0$ in equation 2: infinitely many solutions. If $k = 0$ a row exchange is needed: one solution.

9 $6x - 4y$ is 2 times $(3x - 2y)$. Therefore we need $b_2 = 2b_1$. Then there will be infinitely many solutions.

10 The equation $y = 1$ comes from elimination. Then $x = 4$ and $5x - 4y = c = 16$.

11 $\begin{aligned} 2x + 3y + z &= 8 \\ y + 3z &= 4 \\ 8z &= 8 \end{aligned}$ gives $\begin{aligned} x &= 2 \\ y &= 1 \\ z &= 1 \end{aligned}$ If a zero is at the start of row 2 or 3, that avoids a row operation.

12 $\begin{aligned} 2x - 3y &= 3 \\ y + z &= 1 \\ 2y - 3z &= 2 \end{aligned}$ gives $\begin{aligned} 2x - 3y &= 3 \\ y + z &= 1 \\ -5z &= 0 \end{aligned}$ and $\begin{aligned} x &= 3 \\ y &= 1 \\ z &= 0 \end{aligned}$ Subtract $2 \times$ row 1 from row 2; Subtract $1 \times$ row 1 from row 3; Subtract $2 \times$ row 2 from row 3

13 Subtract 2 times row 1 from row 2 to reach $(d-10)y - z = 2$. Equation (3) is $y - z = 3$. If $d = 10$ exchange rows 2 and 3. If $d = 11$ the system is singular; third pivot is missing.

14 The second pivot position will contain $-2 - b$. If $b = -2$ we exchange with row 3. If $b = -1$ (singular case) the second equation is $-y - z = 0$. A solution is $(1, 1, -1)$.

15 (a) $\begin{aligned} 0x + 0y + 2z &= 4 \\ x + 2y + 2z &= 5 \\ 0x + 3y + 4z &= 6 \end{aligned}$ (exchange 1 and 2, then 2 and 3)

(b) $\begin{aligned} 0x + 3y + 4z &= 4 \\ x + 2y + 2z &= 5 \\ 0x + 3y + 4z &= 6 \end{aligned}$ (rows 1 and 3 are not consistent)

16 If row 1 = row 2, then row 2 is zero after the first step; exchange the zero row with row 3 and there is no *third* pivot. If column 1 = column 2 there is no *second* pivot.

17 $x + 2y + 3z = 0$, $4x + 8y + 12z = 0$, $5x + 10y + 15z = 0$ has infinitely many solutions.

18 Row 2 becomes $3y - 4z = 5$, then row 3 becomes $(q + 4)z = t - 5$. If $q = -4$ the system is singular — no third pivot. Then if $t = 5$ the third equation is $0 = 0$. Choosing $z = 1$ the equation $3y - 4z = 5$ gives $y = 3$ and equation 1 gives $x = -9$.

19 (a) Another solution is $\frac{1}{2}(x + X, y + Y, z + Z)$. (b) If 25 planes meet at two points, they meet along the whole line through those two points.

20 Singular if row 3 is a combination of rows 1 and 2. From the end view, the three planes form a triangle. This happens if rows $1 + 2 =$ row 3 on the left side but not the right side: for example $x + y + z = 0$, $x - 2y - z = 1$, $2x - y = 1$. No parallel planes but still no solution.

21 Pivots 2, $\frac{3}{2}$, $\frac{4}{3}$, $\frac{5}{4}$ in the equations $2x + y = 0$, $\frac{3}{2}y + z = 0$, $\frac{4}{3}z + t = 0$, $\frac{5}{4}t = 5$. Solution $t = 4$, $z = -3$, $y = 2$, $x = -1$.

22 The solution is $(1, 2, 3, 4)$ instead of $(-1, 2, -3, 4)$.

23 The fifth pivot is $\frac{6}{5}$. The nth pivot is $\frac{(n+1)}{n}$.

24 $A = \begin{bmatrix} 1 & 1 & 1 \\ a & a+1 & a+1 \\ b & b+c & b+c+3 \end{bmatrix}$ for any a, b, c leads to $U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$.

25 Elimination fails on $\begin{bmatrix} a & 2 \\ a & a \end{bmatrix}$ if $a = 2$ or $a = 0$.

26 $a = 2$ (equal columns), $a = 4$ (equal rows), $a = 0$ (zero column).

27 Solvable for $s = 10$ (add equations); $\begin{bmatrix} 1 & 3 \\ 1 & 7 \end{bmatrix}$ and $\begin{bmatrix} 0 & 4 \\ 2 & 6 \end{bmatrix}$. $A = [1\ \ 1\ \ 0\ \ 0;\ \ 1\ \ 0\ \ 1\ \ 0;\ \ 0\ \ 0\ \ 1\ \ 1;\ \ 0\ \ 1\ \ 0\ \ 1]$ and $U = [1\ \ 1\ \ 0\ \ 0;\ \ 0\ \ -1\ \ 1\ \ 0;\ \ 0\ \ 0\ \ 1\ \ 1;\ \ 0\ \ 0\ \ 0\ \ 0]$.

28 Elimination leaves the diagonal matrix $\mathsf{diag}(3, 2, 1)$. Then $x = 1, y = 1, z = 4$.

29 $A(2, :) = A(2, :) - 3 * A(1, :)$ Subtracts 3 times row 1 from row 2.

30 The average pivots for $\mathsf{rand}(3)$ *without* row exchanges were $\frac{1}{2}, 5, 10$ in one experiment—but pivots 2 and 3 can be arbitrarily large. Their averages are actually infinite! *With* row exchanges in MATLAB's lu code, the averages .75 and .50 and .365 are much more stable (and should be predictable, also for randn with normal instead of uniform probability distribution).

Problem Set 2.3, page 50

1 $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 7 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

2 $E_{32}E_{21}\boldsymbol{b} = (1, -5, -35)$ but $E_{21}E_{32}\boldsymbol{b} = (1, -5, \mathbf{0})$. Then row 3 feels no effect from row 1.

3 $\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} M = E_{32}E_{31}E_{21} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{bmatrix}$.

4 Elimination on column 4: $\boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix} \to \begin{bmatrix} 1 \\ -4 \\ 10 \end{bmatrix}$. Then back substitution in $U\boldsymbol{x} = \boldsymbol{c} = (1, -4, 10)$ gives $z = -5$, $y = \frac{1}{2}$, $x = \frac{1}{2}$. This solves $A\boldsymbol{x} = \boldsymbol{b} = (1, 0, 0)$.

5 Changing a_{33} from 7 to 11 will change the third pivot from 5 to 9. Changing a_{33} from 7 to 2 will change the pivot from 5 to *no pivot*.

6 If all columns are multiples of column 1, there is no second pivot.

7 To reverse E_{31}, *add* 7 times row 1 to row 3. The matrix is $R_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 0 & 1 \end{bmatrix}$.

8 The same R_{31} from Problem 7 is changed by E_{31} into I. Thus $E_{31}R_{31} = R_{31}E_{31} = I$.

9 $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$. After the exchange, we need E_{31} (not E_{21}) to act on the new row 3.

10 $E_{13} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$; $E_{31}E_{13} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Test on the identity matrix!

11 $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ has pivots $1, -1, -1$.

12 $\begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 2 & -3 \end{bmatrix}$.

13 (a) E times the third column of B is the third column of EB

(b) E could add row 2 to row 3 to give nonzeros.

14 E_{21} has $\ell_{21} = -\frac{1}{2}$, E_{32} has $\ell_{32} = -\frac{2}{3}$, E_{43} has $\ell_{43} = -\frac{3}{4}$. Otherwise the E's match I.

15 $A = \begin{bmatrix} -1 & -4 & -7 \\ 1 & -2 & -5 \\ 3 & 0 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & -4 & -7 \\ 0 & -6 & -12 \\ 0 & -12 & -24 \end{bmatrix}$. $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$.

16 (a) $X - 2Y = 0$ and $X + Y = 33$; $X = 22, Y = 11$ (b) $2m + c = 5$, $3m + c = 7$; $m = 2$, $c = 1$.

17 $\begin{aligned} a + b + c &= 4 \\ a + 2b + 4c &= 8 \\ a + 3b + 9c &= 14 \end{aligned}$ gives $\begin{aligned} a &= 2 \\ b &= 1 \\ c &= 1 \end{aligned}$.

18 $EF = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$, $FE = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b+ac & c & 1 \end{bmatrix}$, $E^2 = \begin{bmatrix} 1 & 0 & 0 \\ 2a & 1 & 0 \\ 2b & 0 & 1 \end{bmatrix}$, $F^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3c & 1 \end{bmatrix}$.

19 $PQ = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $QP = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $P^2 = I, (-P)^2 = I, I^2 = I, (-I)^2 = I$ (many more).

20 (a) Each column is E times a column of B (b) $EB = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}$

Rows of EB are combinations of rows of B, so multiples of $[1 \;\; 2 \;\; 4]$.

21 *No.* $E = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $F = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $EF = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, $FE = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

22 (a) $\sum a_{3j}x_j$ (b) $a_{21} - a_{11}$ (c) $a_{21} - 2a_{11}$ (d) $(EA\boldsymbol{x})_1 = (A\boldsymbol{x})_1 = \sum a_{1j}x_j$.

23 $E(EA)$ subtracts **4** times row 1 from row 2. AE subtracts 2 (**column** 2) of A from **column** 1.

24 $[A \;\; \boldsymbol{b}] = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 17 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 3 & 1 \\ 0 & -5 & 15 \end{bmatrix}$: *Triangular* $\begin{aligned} 2x_1 + 3x_2 &= 1 \\ -5x_2 &= 15 \end{aligned}$ $\begin{aligned} x_1 &= 5 \\ x_2 &= -3. \end{aligned}$

25 The last equation becomes $0 = 3$. Change the original 6 to 3. Then row 1 + row 2 = row 3.

26 (a) Add two extra columns; $\begin{bmatrix} 1 & 4 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} -7 \\ 2 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix}$.

27 (a) No solution if $d = 0$ and $c \neq 0$ (b) Infinitely many solutions if $d = 0$ and $c = 0$. No effect from a and b.

28 $A = AI = A(BC) = (AB)C = IC = C$.

29 Given positive integers with $ad - bc = 1$. Certainly $c < a$ and $b < d$ would be impossible. Also $c > a$ and $b > d$ would be impossible with integers. This leaves row 1 < row 2 OR row 2 < row 1. An example is $M = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$. Multiply by $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ to get $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$, then multiply twice by $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ to get $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. This shows that $M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

30 $E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$. Eventually $M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$ = "inverse of Pascal" reduces Pascal to I.

Problem Set 2.4, page 59

1 $BA = 3I$ is 5 by 5 $AB = 5I$ is 3 by 3 $ABD = 5D$ is 3 by 1. ABD: No $A(B + C)$: No.

2 (a) A (column 3 of B) (b) (Row 1 of A) B (c) (Row 3 of A)(column 4 of B) (d) (Row 1 of C)D(column 1 of E).

3 $AB + AC = A(B + C) = \begin{bmatrix} 3 & 8 \\ 6 & 9 \end{bmatrix}$.

4 $A(BC) = (AB)C =$ zero matrix

5 $A^n = \begin{bmatrix} 1 & bn \\ 0 & 1 \end{bmatrix}$ and $A^n = \begin{bmatrix} 2^n & 2^n \\ 0 & 0 \end{bmatrix}$.

6 $(A + B)^2 = \begin{bmatrix} 10 & 4 \\ 6 & 6 \end{bmatrix} = A^2 + AB + BA + B^2$. But $A^2 + 2AB + B^2 = \begin{bmatrix} 16 & 2 \\ 3 & 0 \end{bmatrix}$.

7 (a) True (b) False (c) True (d) False.

8 Rows of DA are 3 · (row 1 of A) and 5 · (row 2 of A). Both rows of EA are row 2 of A. Columns of AD are 3 · (column 1 of A) and 5 · (column 2 of A). Columns of AE are zero and column 1 of A + column 2 of A.

9 $AF = \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix}$ and $E(AF)$ equals $(EA)F$ because matrix multiplication is *associative*.

10 $FA = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix}$ and then $E(FA) = \begin{bmatrix} a+c & b+d \\ a+2c & b+2d \end{bmatrix}$. $E(FA)$ is not $F(EA)$ because multiplication is not commutative.

11 (a) $B = 4I$ (b) $B = 0$ (c) $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (d) Every row of B is $1, 0, 0$.

12 $AB = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = BA = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ gives $b = c = 0$. Then $AC = CA$ gives $a = d$: $A = aI$.

13 $(A - B)^2 = (B - A)^2 = A(A - B) - B(A - B) = A^2 - AB - BA + B^2$.

14 (a) True (b) False (c) True (d) False (take $B = 0$).

15 (a) mn (every entry) (b) mnp (c) n^3 (this is n^2 dot products).

16 By linearity $(AB)\boldsymbol{c}$ agrees with $A(B\boldsymbol{c})$. Also for all other columns of C.

17 (a) Use only column 2 of B (b) Use only row 2 of A (c)–(d) Use row 2 of first A.

18 $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 1/1 & 1/2 & 1/3 \\ 2/1 & 2/2 & 2/3 \\ 3/1 & 3/2 & 3/3 \end{bmatrix}.$

19 Diagonal matrix, lower triangular, symmetric, all rows equal. Zero matrix.

20 (a) a_{11} (b) $\ell_{31} = a_{31}/a_{11}$ (c) $a_{32} - (\frac{a_{31}}{a_{11}})a_{12}$ (d) $a_{22} - (\frac{a_{21}}{a_{11}})a_{12}$.

21 $A^2 = \begin{bmatrix} 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A\boldsymbol{v} = \begin{bmatrix} 2y \\ 2z \\ 2t \\ 0 \end{bmatrix}, \quad A^2\boldsymbol{v} = \begin{bmatrix} 4z \\ 4t \\ 0 \\ 0 \end{bmatrix}, \quad A^3\boldsymbol{v} = \begin{bmatrix} 8t \\ 0 \\ 0 \\ 0 \end{bmatrix}$

The matrix A^4 is all zeros so $A^4\boldsymbol{v} = \boldsymbol{0}$.

22 $A = A^2 = A^3 = \cdots$ but $AB = \begin{bmatrix} .5 & -.5 \\ .5 & -.5 \end{bmatrix}$ and $(AB)^2 = 0$.

23 $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has $A^2 = -I$; $BC = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$;

$DE = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = -ED.$

24 $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has $A^2 = 0$; $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ has $A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ but $A^3 = 0$.

25 $A_1^n = \begin{bmatrix} 2^n & 2^n - 1 \\ 0 & 1 \end{bmatrix}, \quad A_2^n = 2^{n-1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad A_3^n = \begin{bmatrix} a^n & a^{n-1}b \\ 0 & 0 \end{bmatrix}.$

26 $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 0 \\ 6 & 6 & 0 \\ 6 & 6 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 4 & 8 & 4 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 0 \\ 10 & 14 & 4 \\ 7 & 8 & 1 \end{bmatrix}.$

27 (a) (Row 3 of A) $\cdot$ (column 1 of B) and (Row 3 of A) $\cdot$ (column 2 of B) are both zero.

(b) $\begin{bmatrix} x \\ x \\ 0 \end{bmatrix} \begin{bmatrix} 0 & x & x \end{bmatrix} = \begin{bmatrix} 0 & x & x \\ 0 & x & x \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} x \\ x \\ x \end{bmatrix} \begin{bmatrix} 0 & 0 & x \end{bmatrix} = \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & x \\ 0 & 0 & x \end{bmatrix}$: upper triangular!

28 A times B is $A\begin{bmatrix} | & | & | & | \end{bmatrix}, \quad [\text{——}]B, \quad [\text{——}]\begin{bmatrix} | & | & | & | \end{bmatrix}, \quad \begin{bmatrix} | & | & | \end{bmatrix}\begin{bmatrix} \text{——} \\ \text{——} \end{bmatrix}$

29 $A\boldsymbol{x} = \begin{bmatrix} | & | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1(\text{column } 1) + x_2(\text{column } 2) + x_3(\text{column } 3).$

30 $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}, \quad E = E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$, then $EA = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}.$

31 In Problem 30, $\boldsymbol{c} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix}$, $D - \boldsymbol{c}\boldsymbol{b}/a = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$ in lower corner of EA.

32

$$(A + iB)(\boldsymbol{x} + i\boldsymbol{y}) \rightarrow \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Ax - By \\ Bx + Ay \end{bmatrix} \begin{matrix} \text{real part} \\ \text{imaginary part.} \end{matrix}$$

33 A times $X = [\,\boldsymbol{x}_1 \;\; \boldsymbol{x}_2 \;\; \boldsymbol{x}_3\,]$ will be the identity matrix $I = [\,A\boldsymbol{x}_1 \;\; A\boldsymbol{x}_2 \;\; A\boldsymbol{x}_3\,]$.

34 The solution for $\boldsymbol{b} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}$ is $\boldsymbol{x} = 3\boldsymbol{x}_1 + 5\boldsymbol{x}_2 + 8\boldsymbol{x}_3 = \begin{bmatrix} 3 \\ 8 \\ 16 \end{bmatrix}$; $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ will produce those $\boldsymbol{x}_1 = (1, 1, 1), \boldsymbol{x}_2 = (0, 1, 1), \boldsymbol{x}_3 = (0, 0, 1)$ as columns of its "inverse".

35 The $(2, 2)$ block $S = D - CA^{-1}B$ is the Schur complement: the block version of $d - (cb/a)$.

36 $\begin{bmatrix} a+b & a+b \\ c+d & c+d \end{bmatrix}$ agrees with $\begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix}$ when $b = c$ and $a = d$.

37 $A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$, $A^2 = \begin{bmatrix} 2 & 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 & 1 \\ 1 & 0 & 2 & 0 & 1 \\ 1 & 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 & 2 \end{bmatrix}$, $A^3 = \begin{bmatrix} 0 & 3 & 1 & 1 & 3 \\ 3 & 0 & 3 & 1 & 1 \\ 1 & 3 & 0 & 3 & 1 \\ 1 & 1 & 3 & 0 & 3 \\ 3 & 1 & 1 & 3 & 0 \end{bmatrix}$, $A^3 + A^2$ no zeros so diameter 3

38 $A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$, $A^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$, $A^3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$ $A + A^2 + A^3$ $+A^4$ no zeros diameter 4

39 If A is "northwest" and B is "southeast", AB is upper triangular and BA is lower triangular. Row i of A ends with $i - 1$ zeros. Column j of B starts with $n - j$ zeros. If $i > j$ then (row i of A)$\cdot$(column j of B) $= 0$. So AB is upper triangular. Similarly BA is lower triangular. Problem 2.7.40 asks about inverses and transposes and permutations of a northwest A and a southeast B.

Problem Set 2.5, Page 72

1 $A^{-1} = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{3} & 0 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ -1 & \frac{1}{2} \end{bmatrix}$, $C^{-1} = \begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix}$.

2 $P^{-1} = P$; $P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Always P^{-1} = "*transpose*" of P.

3 $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} .5 \\ -.2 \end{bmatrix}$, $\begin{bmatrix} t \\ z \end{bmatrix} = \begin{bmatrix} -.2 \\ .1 \end{bmatrix}$ so $A^{-1} = \dfrac{1}{10} \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$.

4 $x + 2y = 1,\ \ 3x + 6y = 0$: impossible.

5 $U = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$.

6 (a) Multiply $AB = AC$ by A^{-1} to find $B = C$

(b) $B - C$ can be any matrices $\begin{bmatrix} x & y \\ -x & -y \end{bmatrix}$.

7 (a) In $A\boldsymbol{x} = (1, 0, 0)$, equation 1 + equation 2 − equation 3 is $0 = 1$ (b) The right sides must satisfy $b_1 + b_2 = b_3$ (c) Row 3 becomes a row of zeros—no third pivot.

8 (a) The vector $\boldsymbol{x} = (1, 1, -1)$ solves $A\boldsymbol{x} = \mathbf{0}$ (b) Elimination keeps columns 1+2 = column 3. When columns 1 and 2 end in zeros so does column 3: no third pivot.

9 If you exchange rows 1 and 2 of A, you exchange columns 1 and 2 of A^{-1}.

10 $A^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1/5 \\ 0 & 0 & 1/4 & 0 \\ 0 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} 3 & -2 & 0 & 0 \\ -4 & 3 & 0 & 0 \\ 0 & 0 & 6 & -5 \\ 0 & 0 & -7 & 6 \end{bmatrix}$ (invert each block).

11 (a) $A = I,\ B = -I$ (b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

12 $C = AB$ gives $C^{-1} = B^{-1}A^{-1}$ so $A^{-1} = BC^{-1}$.

13 $M^{-1} = C^{-1}B^{-1}A^{-1}$ so $B^{-1} = CM^{-1}A$.

14 $B^{-1} = A^{-1}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = A^{-1}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$: subtract column 2 of A^{-1} from column 1.

15 If A has a column of zeros, so does BA. So $BA = I$ is impossible. There is no A^{-1}.

16 $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} = (ad - bc)I$. The inverse of one matrix is the other divided by $ad - bc$.

17 $\begin{bmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ & 1 & \\ -1 & & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ -1 & 1 & \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -1 & 1 & \\ 0 & -1 & 1 \end{bmatrix} = E;\ \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 1 & 1 & 1 \end{bmatrix} = L = E^{-1}$ after reversing the order and changing -1 to $+1$.

18 $A^2B = I$ can be written as $A(AB) = I$. Therefore A^{-1} is AB.

19 The $(1, 1)$ entry requires $4a - 3b = 1$; the $(1, 2)$ entry requires $2b - a = 0$. Then $b = \frac{1}{5}$ and $a = \frac{2}{5}$. For the 5 by 5 case $5a - 4b = 1$ and $2b - a = 0$ give $b = \frac{1}{6}$ and $a = \frac{2}{6}$.

20 $A * \mathsf{ones}(4, 1)$ is the zero vector so A cannot be invertible.

21 6 of the 16 are invertible, including all four with three 1's.

22 $\begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} I & A^{-1} \end{bmatrix}$;

$\begin{bmatrix} 1 & 3 & 1 & 0 \\ 3 & 8 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -8 & 3 \\ 0 & 1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} I & A^{-1} \end{bmatrix}$.

23 $\begin{bmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & -1/2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{bmatrix} \to$

$\begin{bmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & -1/2 & 1 & 0 \\ 0 & 0 & 4/3 & 1/3 & -2/3 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 0 & -3/4 & 3/2 & -3/4 \\ 0 & 0 & 4/3 & 1/3 & -2/3 & 1 \end{bmatrix} \to$

$\begin{bmatrix} 2 & 0 & 0 & 3/2 & -1 & 1/2 \\ 0 & 3/2 & 0 & -3/4 & 3/2 & z-3/4 \\ 0 & 0 & 4/3 & 1/3 & -2/3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 3/4 & -1/2 & 1/4 \\ 0 & 1 & 0 & -1/2 & 1 & -1/2 \\ 0 & 0 & 1 & 1/4 & -1/2 & 3/4 \end{bmatrix}.$

24 $\begin{bmatrix} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & a & 0 & 1 & 0 & -b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 & -a & ac-b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$

25 $A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$; $B\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ so B^{-1} does not exist.

26 $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$. Then $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Multiply by $D = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$ to reach I. Here $D^{-1}E_{12}E_{21} = \begin{bmatrix} 3 & -1 \\ -1 & 1/2 \end{bmatrix} = A^{-1}$.

27 $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$ (notice the pattern); $A^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$.

28 $\begin{bmatrix} 2 & 2 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & -1 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/2 & 1/2 \\ 0 & 1 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} I & A^{-1} \end{bmatrix}.$

29 (a) True (AB has a row of zeros) (b) False (matrix of all 1's) (c) True (inverse of A^{-1} is A) (d) True (inverse of A^2 is $(A^{-1})^2$).

30 Not invertible for $c = 7$ (equal columns), $c = 2$ (equal rows), $c = 0$ (zero column).

31 Elimination produces the pivots a and $a-b$ and $a-b$. $A^{-1} = \dfrac{1}{a(a-b)}\begin{bmatrix} a & 0 & -b \\ -a & a & 0 \\ 0 & -a & a \end{bmatrix}$.

32 $A^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. The 5 by 5 A^{-1} also has 1's on the diagonal and superdiagonal.

33 $\boldsymbol{x} = (2, 2, 2, 1)$.

34 $\boldsymbol{x} = (1, 1, \dots, 1)$ has $P\boldsymbol{x} = Q\boldsymbol{x}$ so $(P - Q)\boldsymbol{x} = \mathbf{0}$.

35 $\begin{bmatrix} I & 0 \\ -C & I \end{bmatrix}$ and $\begin{bmatrix} A^{-1} & 0 \\ -D^{-1}CA^{-1} & D^{-1} \end{bmatrix}$ and $\begin{bmatrix} -D & I \\ I & 0 \end{bmatrix}$.

36 If $AC = CA$, multiply left and right by A^{-1} to find $CA^{-1} = A^{-1}C$. If also $BC = CB$, then (using the associative law!!), $(AB)C = A(BC) = A(CB) = (AC)B = (CA)B = C(AB)$.

37 A can be invertible but B is always singular. Each row of B will add to zero, from $0+1+2-3$, so the vector $\boldsymbol{x} = (1,1,1,1)$ will give $B\boldsymbol{x} = \boldsymbol{0}$. I thought A would be invertible as long as you put the 3's on its main diagonal, but that's wrong:

$$A\boldsymbol{x} = \begin{bmatrix} 3 & 0 & 1 & 2 \\ 0 & 3 & 1 & 2 \\ 1 & 2 & 3 & 0 \\ 1 & 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \boldsymbol{0} \quad \text{but} \quad A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 3 & 0 & 1 & 2 \\ 2 & 3 & 0 & 1 \\ 1 & 2 & 3 & 0 \end{bmatrix} \text{ is invertible}$$

38 $AD =$ pascal$(4,1)$ is its own inverse.

39 hilb(6) is not the exact Hilbert matrix because fractions are rounded off.

40 The three Pascal matrices have $S = LU = LL^{\mathrm{T}}$ and then inv$(S) =$ inv(L^{T})inv(L). Note that the triangular L is abs(pascal$(n,1)$) in MATLAB.

41 For $A\boldsymbol{x} = \boldsymbol{b}$ with $A =$ ones$(4,4)$ = singular matrix and $\boldsymbol{b} =$ ones$(4,1)$ in its column space, MATLAB will pick the shortest solution $\boldsymbol{x} = (1,1,1,1)/4$. Any vector in the nullspace of A could be added to this particular solution.

42 If $AC = I$ for square matrices then $C = A^{-1}$ (it is proved in **2I** that $CA = I$ will also be true). The same will be true for C^*. But a square matrix has only one inverse so $C = C^*$.

43
$$\begin{aligned} MM^{-1} &= (I_n - UV)\,(I_n + U(I_m - VU)^{-1}V) \\ &= I_n - UV + U(I_m - VU)^{-1}V - UVU(I_m - VU)^{-1}V \\ &= I_n - UV + U(I_m - VU)(I_m - VU)^{-1}V = I_n \qquad \text{(formulas } \mathbf{1}, \mathbf{2}, \mathbf{4} \text{ are similar)} \end{aligned}$$

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1 $\ell_{21} = 1$; $L = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} \end{bmatrix}$ times $U\boldsymbol{x} = \boldsymbol{c}$ is $A\boldsymbol{x} = \boldsymbol{b}$: $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$.

2 $\ell_{31} = 1$ and $\ell_{32} = 2$ (and $\ell_{33} = 1$): reverse the steps to recover $x + 3y + 6z = 11$ from $U\boldsymbol{x} = \boldsymbol{c}$: 1 times $(x + y + z = 5) + 2$ times $(y + 2z = 2) + 1$ times $(z = 2)$ gives $x + 3y + 6z = 11$.

3 $L\boldsymbol{c} = \boldsymbol{b}$ is $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$; $\boldsymbol{c} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$. $U\boldsymbol{x} = \boldsymbol{c}$ is $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$; $\boldsymbol{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

4 $L\boldsymbol{c} = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} \\ \boldsymbol{c} \\ \\ \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 11 \end{bmatrix}$; $\boldsymbol{c} = \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix}$. $U\boldsymbol{x} = \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 2 \\ & & 1 \end{bmatrix} \begin{bmatrix} \\ \boldsymbol{x} \\ \\ \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix}$; $\boldsymbol{x} = \begin{bmatrix} 5 \\ -2 \\ 2 \end{bmatrix}$.

5 $EA = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix} = U$; $A = LU = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 3 & 0 & 1 \end{bmatrix} U$.

6 $\begin{bmatrix}1&&\\0&1&\\0&-2&1\end{bmatrix}\begin{bmatrix}1&&\\-2&1&\\0&0&1\end{bmatrix}A=\begin{bmatrix}1&1&1\\0&2&3\\0&0&-6\end{bmatrix}=U$. Then $A=\begin{bmatrix}1&0&0\\2&1&0\\0&2&1\end{bmatrix}U=E_{21}^{-1}E_{32}^{-1}U=LU$.

7 $E_{32}E_{31}E_{21}A=\begin{bmatrix}1&&\\&1&\\&-2&1\end{bmatrix}\begin{bmatrix}1&&\\&1&\\-3&&1\end{bmatrix}\begin{bmatrix}1&&\\-2&1&\\&&1\end{bmatrix}\begin{bmatrix}1&0&1\\2&2&2\\3&4&5\end{bmatrix}=\begin{bmatrix}1&0&1\\0&2&0\\0&0&2\end{bmatrix}=U$.

Then put the multipliers $2, 3, 2$ into L and recover $A=LU$: $\begin{bmatrix}1&0&0\\2&1&0\\3&2&1\end{bmatrix}\begin{bmatrix}1&0&1\\0&2&0\\0&0&2\end{bmatrix}=\begin{bmatrix}1&0&1\\2&2&2\\3&4&5\end{bmatrix}$.

8 $E=E_{32}E_{31}E_{21}=\begin{bmatrix}1&&\\&1&\\&-c&1\end{bmatrix}\begin{bmatrix}1&&\\&1&\\-b&&1\end{bmatrix}\begin{bmatrix}1&&\\-a&1&\\&&1\end{bmatrix}=\begin{bmatrix}1&&\\-a&1&\\ac-b&-c&1\end{bmatrix}$. This is $L^{-1}=A^{-1}$. The matrices $E_{21}^{-1},E_{31}^{-1},E_{32}^{-1}$ have entries $+a,+b,+c$ and their product is L.

9 2 by 2: $d=0$ not allowed; $\begin{bmatrix}1&1&0\\1&1&2\\1&2&1\end{bmatrix}=\begin{bmatrix}1&&\\l&1&\\m&n&1\end{bmatrix}\begin{bmatrix}d&e&g\\&f&h\\&&i\end{bmatrix}$ $d=1,\ e=1,$ then $l=1$; $f=0$ is not allowed; **no pivot in row 2**

10 $c=2$ leads to zero in the second pivot position: exchange rows and the matrix will be OK. $c=1$ leads to zero in the third pivot position. In this case the matrix is *singular*.

11 $A=\begin{bmatrix}2&4&8\\0&3&9\\0&0&7\end{bmatrix}$ has $L=I$ and $D=\begin{bmatrix}2&&\\&3&\\&&7\end{bmatrix}$; $A=LU$ has $U=A$ (pivots on the diagonal);

$A=LDU$ has $U=D^{-1}A=\begin{bmatrix}1&2&4\\0&1&3\\0&0&1\end{bmatrix}$ with 1's on the diagonal.

12 $A=\begin{bmatrix}2&4\\4&11\end{bmatrix}=\begin{bmatrix}1&0\\2&1\end{bmatrix}\begin{bmatrix}2&4\\0&3\end{bmatrix}=\begin{bmatrix}1&0\\2&1\end{bmatrix}\begin{bmatrix}2&0\\0&3\end{bmatrix}\begin{bmatrix}1&2\\0&1\end{bmatrix}=LDU$; notice U is L^{T}

$A=\begin{bmatrix}1&&\\4&1&\\0&-1&1\end{bmatrix}\begin{bmatrix}1&4&0\\0&-4&4\\0&0&4\end{bmatrix}=\begin{bmatrix}1&&\\4&1&\\0&-1&1\end{bmatrix}\begin{bmatrix}1&&\\&-4&\\&&4\end{bmatrix}\begin{bmatrix}1&4&0\\0&1&-1\\0&0&1\end{bmatrix}=LDL^{\mathrm{T}}$.

13 $\begin{bmatrix}a&a&a&a\\a&b&b&b\\a&b&c&c\\a&b&c&d\end{bmatrix}=\begin{bmatrix}1&&&\\1&1&&\\1&1&1&\\1&1&1&1\end{bmatrix}\begin{bmatrix}a&a&a&a\\&b-a&b-a&b-a\\&&c-b&c-b\\&&&d-c\end{bmatrix}$. Need $a\neq 0$, $b\neq a$, $c\neq b$, $d\neq c$

14 $\begin{bmatrix}a&r&r&r\\a&b&s&s\\a&b&c&t\\a&b&c&d\end{bmatrix}=\begin{bmatrix}1&&&\\1&1&&\\1&1&1&\\1&1&1&1\end{bmatrix}\begin{bmatrix}a&r&r&r\\&b-r&s-r&s-r\\&&c-s&t-s\\&&&d-t\end{bmatrix}$. Need $a\neq 0$, $b\neq r$, $c\neq s$, $d\neq t$

15 $\begin{bmatrix}1&0\\4&1\end{bmatrix}\boldsymbol{c}=\begin{bmatrix}2\\11\end{bmatrix}$ gives $\boldsymbol{c}=\begin{bmatrix}2\\3\end{bmatrix}$. Then $\begin{bmatrix}2&4\\0&1\end{bmatrix}\boldsymbol{x}=\begin{bmatrix}2\\3\end{bmatrix}$ gives $\boldsymbol{x}=\begin{bmatrix}-5\\3\end{bmatrix}$.

Check that $A=LU=\begin{bmatrix}2&4\\8&17\end{bmatrix}$ times $\boldsymbol{x}$ is $\boldsymbol{b}=\begin{bmatrix}2\\11\end{bmatrix}$.

16 $\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}\boldsymbol{c}=\begin{bmatrix}4\\5\\6\end{bmatrix}$ gives $\boldsymbol{c}=\begin{bmatrix}4\\1\\1\end{bmatrix}$. Then $\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}\boldsymbol{x}=\begin{bmatrix}4\\1\\1\end{bmatrix}$ gives $\boldsymbol{x}=\begin{bmatrix}3\\0\\1\end{bmatrix}$ $A=LU$.

17 (a) L goes to I (b) I goes to L^{-1} (c) LU goes to U.

18 (a) Multiply $LDU=L_1D_1U_1$ by inverses to get $L_1^{-1}LD=D_1U_1U^{-1}$. The left side is lower triangular, the right side is upper triangular $\Rightarrow$ both sides are diagonal.

(b) Since L,U,L_1,U_1 have diagonals of 1's we get $D=D_1$. Then $L_1^{-1}L$ is I and U_1U^{-1} is I.

19 $\begin{bmatrix}1&&\\1&1&\\0&1&1\end{bmatrix}\begin{bmatrix}1&1&0\\&1&1\\&&1\end{bmatrix}=LIU;\ \begin{bmatrix}a&a&0\\a&a+b&b\\0&b&b+c\end{bmatrix}=(\text{same } L)\begin{bmatrix}a&&\\&b&\\&&c\end{bmatrix}(\text{same } U).$

20 A tridiagonal T has 2 nonzeros in the pivot row and only one nonzero below the pivot (so 1 operation to find the multiplier and 1 to find the new pivot!). $T=$ bidiagonal L times U:

$$T=\begin{bmatrix}1&2&0&0\\2&3&1&0\\0&1&2&3\\0&0&3&4\end{bmatrix}\longrightarrow U=\begin{bmatrix}1&2&0&0\\0&-1&1&0\\0&0&3&3\\0&0&0&1\end{bmatrix}.\ \text{Reverse steps by } L=\begin{bmatrix}1&0&0&0\\2&1&0&0\\0&-1&1&0\\0&0&1&1\end{bmatrix}.$$

21 For A, L has the 3 lower zeros but U may not have the upper zero. For B, L has the bottom left zero and U has the upper right zero. One zero in A and two zeros in B are filled in.

22 $\begin{bmatrix}x&x&x\\x&x&x\\x&x&x\end{bmatrix}=\begin{bmatrix}1&0&0\\*&1&0\\*&&1\end{bmatrix}\begin{bmatrix}*&*&*\\0&&\\0&&\end{bmatrix}$ ($*$'s are all known after the first pivot is used).

23 $\begin{bmatrix}5&3&1\\3&3&1\\\mathbf{1}&\mathbf{1}&\mathbf{1}\end{bmatrix}\rightarrow\begin{bmatrix}4&2&0\\\mathbf{2}&\mathbf{2}&\mathbf{0}\\\mathbf{1}&\mathbf{1}&\mathbf{1}\end{bmatrix}\rightarrow\begin{bmatrix}\mathbf{2}&\mathbf{0}&\mathbf{0}\\\mathbf{2}&\mathbf{2}&\mathbf{0}\\\mathbf{1}&\mathbf{1}&\mathbf{1}\end{bmatrix}=L$. Then $A=UL$ with $U=\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}$.

24 $\begin{bmatrix}\mathbf{1}&\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{5}\\2&1&1&0&8\\0&1&3&2&8\\\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{1}&\mathbf{2}\end{bmatrix}\rightarrow\begin{bmatrix}1&1&0&0&5\\0&\mathbf{-1}&\mathbf{1}&0&\mathbf{-2}\\0&\mathbf{1}&\mathbf{1}&0&\mathbf{4}\\0&0&1&1&2\end{bmatrix}$. Solve $\begin{bmatrix}-1&1\\1&1\end{bmatrix}\begin{bmatrix}x_2\\x_3\end{bmatrix}=\begin{bmatrix}-2\\4\end{bmatrix}$ for $x_2=3$ and $x_3=1$ in the middle. Then $x_1=2$ backward and $x_4=1$ forward.

25 The 2 by 2 upper submatrix B has the first two pivots $2,7$. Reason: Elimination on A starts in the upper left corner with elimination on B.

26 The first three pivots for M are still $2,7,6$. To be sure that 9 is the fourth pivot, put zeros in the rest of row 4 and column 4.

27 $$\begin{bmatrix}1&1&1&1&1\\1&2&3&4&5\\1&3&6&10&15\\1&4&10&20&35\\1&5&15&35&70\end{bmatrix}=\begin{bmatrix}1&&&&\\1&1&&&\\1&2&1&&\\1&3&3&1&\\1&4&6&4&1\end{bmatrix}\begin{bmatrix}1&1&1&1&1\\&1&2&3&4\\&&1&3&6\\&&&1&4\\&&&&1\end{bmatrix}.$$ *Pascal's triangle* in L and U. MATLAB's lu code will wreck the pattern. chol does no row exchanges for symmetric matrices with positive pivots.

28 $c = 6$ and also $c = 7$ will make LU impossible ($c = 6$ needs a row exchange).

32 $\mathsf{inv}(A) * \boldsymbol{b}$ should take 3 times as long as $A\backslash\boldsymbol{b}$ (n^3 for A^{-1} *vs* $n^3/3$ multiplications for LU).

34 The upper triangular $\mathsf{triu}(A)$ is theoretically about 6 times faster to invert. Not in reality!

35 Each new *right side* costs only n^2 steps compared to $n^3/3$ for full elimination $A\backslash\boldsymbol{b}$.

36 This L comes from the $-1, 2, -1$ tridiagonal $A = LDL^{\mathrm{T}}$. (Row i of L) $\cdot$ (Column j of L^{-1}) $=$ $\left(\frac{1-i}{i}\right)\left(\frac{j}{i-1}\right)+(1)\left(\frac{j}{i}\right) = 0$ for $i > j$ so $LL^{-1} = I$. Then L^{-1} leads to $A^{-1} = (L^{-1})^{\mathrm{T}}D^{-1}L^{-1}$. ***The*** $-1, 2, -1$ ***matrix has inverse*** $A^{-1}_{ij} = j(n - i + 1)/(n + 1)$ for $i \geq j$ (reverse for $i \leq j$).

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1 $A^{\mathrm{T}} = \begin{bmatrix}1&9\\0&3\end{bmatrix}$, $A^{-1} = \begin{bmatrix}1&0\\-3&1/3\end{bmatrix}$, $(A^{-1})^{\mathrm{T}} = (A^{\mathrm{T}})^{-1} = \begin{bmatrix}1&-3\\0&1/3\end{bmatrix}$; $A^{\mathrm{T}} = A$ and then $A^{-1} = \frac{1}{c^2}\begin{bmatrix}0&c\\c&-1\end{bmatrix} = (A^{-1})^{\mathrm{T}} = (A^{\mathrm{T}})^{-1}$.

2 $(AB)^{\mathrm{T}}$ is not $A^{\mathrm{T}}B^{\mathrm{T}}$ *except when* $AB = BA$. In that case transpose to find: $B^{\mathrm{T}}A^{\mathrm{T}} = A^{\mathrm{T}}B^{\mathrm{T}}$.

3 $\left((AB)^{-1}\right)^{\mathrm{T}} = (B^{-1}A^{-1})^{\mathrm{T}} = (A^{-1})^{\mathrm{T}}(B^{-1})^{\mathrm{T}}$; $(U^{-1})^{\mathrm{T}}$ is *lower* triangular.

4 $A = \begin{bmatrix}0&1\\0&0\end{bmatrix}$ has $A^2 = 0$. But the diagonal entries of $A^{\mathrm{T}}A$ are dot products of columns of A with themselves. If $A^{\mathrm{T}}A = 0$, zero dot products $\Rightarrow$ zero columns $\Rightarrow$ $A =$ zero matrix.

5 (a) $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{y} = a_{22} = 5$ (b) $\boldsymbol{x}^{\mathrm{T}}A = \begin{bmatrix}4&5&6\end{bmatrix}$ (c) $A\boldsymbol{y} = \begin{bmatrix}2\\5\end{bmatrix}$.

6 $M^{\mathrm{T}} = \begin{bmatrix}A^{\mathrm{T}}&C^{\mathrm{T}}\\B^{\mathrm{T}}&D^{\mathrm{T}}\end{bmatrix}$; $M^{\mathrm{T}} = M$ needs $A^{\mathrm{T}} = A, B^{\mathrm{T}} = C, D^{\mathrm{T}} = D$.

7 (a) False (needs $A = A^{\mathrm{T}}$) (b) False (c) True (d) False.

8 The 1 in row 1 has n choices; then the 1 in row 2 has $n - 1$ choices ... ($n!$ choices overall).

9 $P_1P_2 = \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix} \neq P_2P_1$.

10 $(3,1,2,4)$ and $(2,3,1,4)$ keep only 4 in position; 6 more even P's keep 1 or 2 or 3 in position; $(2,1,4,3)$ and $(3,4,1,2)$ exchange 2 pairs. Then $(1,2,3,4)$ and $(4,3,2,1)$ make 12 even P's.

11 $P = \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$; $P_1 = \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$ and $P_2 = \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$ (P_2 gives a column exchange).

12 $(P\boldsymbol{x})^{\mathrm{T}}(P\boldsymbol{y}) = \boldsymbol{x}^{\mathrm{T}}P^{\mathrm{T}}P\boldsymbol{y} = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{y}$ because $P^{\mathrm{T}}P = I$; In general $P\boldsymbol{x}\cdot\boldsymbol{y} = \boldsymbol{x}\cdot P^{\mathrm{T}}\boldsymbol{y} \neq \boldsymbol{x}\cdot P\boldsymbol{y}$:

$$\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}\cdot\begin{bmatrix}1\\1\\2\end{bmatrix} \neq \begin{bmatrix}1\\2\\3\end{bmatrix}\cdot\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\begin{bmatrix}1\\1\\2\end{bmatrix}.$$

13 $P = \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$ or its transpose has $P^3 = I$; $\widehat{P} = \begin{bmatrix}1&0\\0&P\end{bmatrix}$ for the same P has $\widehat{P}^4 = \widehat{P}$.

14 There are $n!$ permutation matrices of order n. Eventually two powers of P *must be the same*: If $P^r = P^s$ then $P^{r-s} = I$. Certainly $r - s \leq n!$

$P = \begin{bmatrix}P_2 & \\ & P_3\end{bmatrix}$ is 5 by 5 with $P_2 = \begin{bmatrix}0&1\\1&0\end{bmatrix}$ and $P_3 = \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$ and $P^6 = I$.

15 (a) $P^{\mathrm{T}}(\text{row } 4) = \text{row } 1$ (b) $P = \begin{bmatrix}E&0\\0&E\end{bmatrix} = P^{\mathrm{T}}$ with $E = \begin{bmatrix}0&1\\1&0\end{bmatrix}$ moves all rows.

16 $A^2 - B^2$ and ABA are symmetric if A and B are symmetric.

17 (a) $A = \begin{bmatrix}1&1\\1&1\end{bmatrix}$ (b) $A = \begin{bmatrix}0&1\\1&1\end{bmatrix}$ (c) $A = \begin{bmatrix}1&1\\1&0\end{bmatrix}$ has $D = \begin{bmatrix}1&0\\0&-1\end{bmatrix}$.

18 (a) $5+4+3+2+1 = 15$ independent entries if $A = A^{\mathrm{T}}$ (b) L has 10 and D has 5: total 15 in LDL^{T} (c) Zero diagonal if $A^{\mathrm{T}} = -A$, leaving $4+3+2+1 = 10$ choices.

19 (a) The transpose of $R^{\mathrm{T}}AR$ is $R^{\mathrm{T}}A^{\mathrm{T}}R^{\mathrm{TT}} = R^{\mathrm{T}}AR = n$ by n

(b) $(R^{\mathrm{T}}R)_{jj} = (\text{column } j \text{ of } R)\cdot(\text{column } j \text{ of } R) = \text{length squared of column } j$.

20 $\begin{bmatrix}1&3\\3&2\end{bmatrix} = \begin{bmatrix}1&0\\3&1\end{bmatrix}\begin{bmatrix}1&0\\0&-7\end{bmatrix}\begin{bmatrix}1&3\\0&1\end{bmatrix}$; $\begin{bmatrix}1&b\\b&c\end{bmatrix} = \begin{bmatrix}1&0\\b&1\end{bmatrix}\begin{bmatrix}1&0\\0&c-b^2\end{bmatrix}\begin{bmatrix}1&b\\0&1\end{bmatrix} = LDL^{\mathrm{T}}$.

$$\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&2\end{bmatrix} = \begin{bmatrix}1&&\\-\frac{1}{2}&1&\\0&-\frac{2}{3}&1\end{bmatrix}\begin{bmatrix}2&&\\&\frac{3}{2}&\\&&\frac{4}{3}\end{bmatrix}\begin{bmatrix}1&-\frac{1}{2}&0\\&1&-\frac{2}{3}\\&&1\end{bmatrix}.$$

21 Lower right 2 by 2 matrix is $\begin{bmatrix}-5&-7\\-7&-32\end{bmatrix}$, $\begin{bmatrix}d-b^2&e-bc\\e-bc&f-c^2\end{bmatrix}$. Still symmetric!

22 $\begin{bmatrix}0&1&\\1&0&\\&&1\end{bmatrix}A = \begin{bmatrix}1&&\\0&1&\\2&3&1\end{bmatrix}\begin{bmatrix}1&0&1\\&1&1\\&&-1\end{bmatrix}$; $\begin{bmatrix}1&&\\&0&1\\&1&0\end{bmatrix}A = \begin{bmatrix}1&&\\1&1&\\2&0&1\end{bmatrix}\begin{bmatrix}1&2&0\\&-1&1\\&&1\end{bmatrix}$

23 $A = \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$ has $L = U = I$; exchange rows 1–2 then rows 2–3 by $P = \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$.

24 $PA = LU$ is $\begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 0 & 3 & 8 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & 1/3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ & 3 & 8 \\ & & -2/3 \end{bmatrix}$. If we wait to exchange and use a_{12} as pivot then $A = L_1P_1U_1 = \begin{bmatrix} 1 & & \\ 3 & 1 & \\ & & 1 \end{bmatrix}\begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$.

25 $abs(A(1,1)) = 0$ and $abs(A(2,1)) > tol$; $A \to \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ and $P \to \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; no more elimination so $L = I$ and $U =$ new A. $abs(A(1,1)) = 0$ and $abs(A(2,1)) > tol$; $A \to \begin{bmatrix} 2 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 5 & 6 \end{bmatrix}$ and $P \to \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $\begin{matrix} abs(A(2,2)) = 0 \\ abs(A(3,2)) > tol \end{matrix}$; $A \to \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 1 \end{bmatrix}$, $L = I$, $P \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

26 $abs(A(1,1)) = 0$ so find $abs(A(2,1)) > tol$; exchange rows to $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 2 & 5 & 4 \end{bmatrix}$ and $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; eliminate to $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 3 & 4 \end{bmatrix}$ and $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$, same P; $abs(A(2,2)) > tol$ so eliminate to $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix} =$ final U and $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$.

27 No solution

28 $L_1 = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 2 & 0 & 1 \end{bmatrix}$ shows the elimination steps as actually done (L is affected by P).

29 One way to decide even vs. odd is to count all pairs that P has in the wrong order. Then P is even or odd when that count is even or odd. Hard step: show that an exchange always reverses that count! Then 3 or 5 exchanges will leave that count odd.

30 $E_{21} = \begin{bmatrix} 1 & & \\ -3 & 1 & \\ & & 1 \end{bmatrix}$ and $E_{21}AE_{21}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 9 \end{bmatrix}$ is still symmetric; $E_{32} = \begin{bmatrix} 1 & & \\ & 1 & \\ & -4 & 1 \end{bmatrix}$ and $E_{32}E_{21}AE_{21}^{\mathrm{T}}E_{32}^{\mathrm{T}} = D$. Elimination from both sides gives the symmetric LDL^{T} directly.

31 Total currents are $A^{\mathrm{T}}\boldsymbol{y} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} y_{BC} \\ y_{CS} \\ y_{BS} \end{bmatrix} = \begin{bmatrix} y_{BC} + y_{BS} \\ -y_{BC} + y_{CS} \\ -y_{CS} - y_{BS} \end{bmatrix}$.

Either way $(A\boldsymbol{x})^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{x}^{\mathrm{T}}(A^{\mathrm{T}}\boldsymbol{y}) = x_By_{BC} + x_By_{BS} - x_Cy_{BC} + x_Cy_{CS} - x_Sy_{CS} - x_Sy_{BS}$.

32 Inputs $\begin{bmatrix} 1 & 50 \\ 40 & 1000 \\ 2 & 50 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\boldsymbol{x}$; $A^{\mathrm{T}}\boldsymbol{y} = \begin{bmatrix} 1 & 40 & 2 \\ 50 & 1000 & 50 \end{bmatrix} \begin{bmatrix} 700 \\ 3 \\ 3000 \end{bmatrix} = \begin{bmatrix} 6820 \\ 188000 \end{bmatrix}$ 1 truck / 1 plane

33 $A\boldsymbol{x} \cdot \boldsymbol{y}$ is the *cost* of inputs while $\boldsymbol{x} \cdot A^{\mathrm{T}}\boldsymbol{y}$ is the *value* of outputs.

34 $P^3 = I$ so three rotations for 360°; P rotates around $(1, 1, 1)$ by 120°.

35 $\begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = EH$

36 $L(U^{\mathrm{T}})^{-1}$ = triangular times triangular. The transpose of $U^{\mathrm{T}}DU$ is $U^{\mathrm{T}}D^{\mathrm{T}}U^{\mathrm{TT}} = U^{\mathrm{T}}DU$ again.

37 These are groups: Lower triangular with diagonal 1's, diagonal invertible D, permutations P, orthogonal matrices with $Q^{\mathrm{T}} = Q^{-1}$.

38 $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{bmatrix}$ (I don't know any rules for constructions like this)

39 Reordering the rows and/or columns of $\left[\begin{smallmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{smallmatrix}\right]$ will move the entry **a**.

40 Certainly B^{T} is northwest. B^2 is a full matrix! B^{-1} is southeast: $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} 0 & 1 \\ 1 & -1 \end{smallmatrix}\right]$. The rows of B are in reverse order from a lower triangular L, so $B = PL$. Then $B^{-1} = L^{-1}P^{-1}$ has the *columns* in reverse order from L^{-1}. So B^{-1} is *southeast*. Northwest times southeast is upper triangular! $B = PL$ and $C = PU$ give $BC = (PLP)U$ = upper times upper.

41 The i, j entry of PAP is the $n - i + 1, n - j + 1$ entry of A. The main diagonal reverses order.

Problem Set 3.1, Page 107

1 $\boldsymbol{x} + \boldsymbol{y} \neq \boldsymbol{y} + \boldsymbol{x}$ and $\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) \neq (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$ and $(c_1 + c_2)\boldsymbol{x} \neq c_1\boldsymbol{x} + c_2\boldsymbol{x}$.

2 The only broken rule is 1 times $\boldsymbol{x}$ equals $\boldsymbol{x}$.

3 (a) $c\boldsymbol{x}$ may not be in our set: not closed under scalar multiplication. Also no $\mathbf{0}$ and no $-\boldsymbol{x}$
(b) $c(\boldsymbol{x} + \boldsymbol{y})$ is the usual $(xy)^c$, while $c\boldsymbol{x} + c\boldsymbol{y}$ is the usual $(x^c)(y^c)$. Those are equal. With $c = 3$, $x = 2$, $y = 1$ they equal 8. This is $3(\mathbf{2} + \mathbf{1})$!! The zero vector is the number 1.

4 The zero vector in **M** is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; $\frac{1}{2}A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ and $-A = \begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix}$. The smallest subspace containing A consists of all matrices cA.

5 (a) One possibility: The matrices cA form a subspace not containing B (b) Yes: the subspace must contain $A - B = I$ (c) All matrices whose main diagonal is all zero.

6 $h(x) = 3f(x) - 4g(x) = 3x^2 - 20x$.

7 Rule 8 is broken: If $c\boldsymbol{f}(x)$ is defined to be the usual $f(cx)$ then $(c_1 + c_2)\boldsymbol{f} = f((c_1 + c_2)x)$ is different from $c_1\boldsymbol{f} + c_2\boldsymbol{f}$ = usual $f(c_1x) + f(c_2x)$.

8 If $(\boldsymbol{f}+\boldsymbol{g})(x)$ is the usual $f(g(x))$ then $(\boldsymbol{g}+\boldsymbol{f})x$ is $g(f(x))$ which is different. In Rule 2 both sides are $f(g(h(x)))$. Rule 4 is broken because there might be no inverse function $f^{-1}(x)$ such that $f(f^{-1}(x)) = x$. If the inverse function exists it will be the vector $-\boldsymbol{f}$.

9 (a) The vectors with integer components allow addition, but not multiplication by $\frac{1}{2}$
(b) Remove the x axis from the xy plane (but leave the origin). Multiplication by any c is allowed but not all vector additions.

10 Only (a) (d) (e) are subspaces.

11 (a) All matrices $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ (b) All matrices $\begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix}$ (c) All diagonal matrices.

12 The sum of $(4,0,0)$ and $(0,4,0)$ is not on the plane.

13 $\mathbf{P}_0$ has the equation $x+y-2z=0$; $(2,0,1)$ and $(0,2,1)$ and their sum $(2,2,2)$ are in $\mathbf{P}_0$.

14 (a) The subspaces of $\mathbf{R}^2$ are $\mathbf{R}^2$ itself, lines through $(0,0)$, and $(0,0)$ itself (b) The subspaces of $\mathbf{R}^4$ are $\mathbf{R}^4$ itself, three-dimensional planes $\boldsymbol{n}\cdot\boldsymbol{v}=0$, two-dimensional subspaces $(\boldsymbol{n}_1\cdot\boldsymbol{v}=0$ and $\boldsymbol{n}_2\cdot\boldsymbol{v}=0)$, one-dimensional lines through $(0,0,0,0)$, and $(0,0,0,0)$ alone.

15 (a) Two planes through $(0,0,0)$ probably intersect in a line through $(0,0,0)$ (b) The plane and line probably intersect in the point $(0,0,0)$ (c) Suppose $\boldsymbol{x}$ is in $\boldsymbol{S}\cap\boldsymbol{T}$ and $\boldsymbol{y}$ is in $\boldsymbol{S}\cap\boldsymbol{T}$. Both vectors are in both subspaces, so $\boldsymbol{x}+\boldsymbol{y}$ and $c\boldsymbol{x}$ are in both subspaces.

16 The smallest subspace containing $\mathbf{P}$ and $\mathbf{L}$ is either $\mathbf{P}$ or $\mathbf{R}^3$.

17 (a) The zero matrix is not invertible (b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is not singular.

18 (a) True (b) True (b) False.

19 The column space of A is the x axis = all vectors $(x,0,0)$. The column space of B is the xy plane = all vectors $(x,y,0)$. The column space of C is the line of vectors $(x,2x,0)$.

20 (a) Solution only if $b_2 = 2b_1$ and $b_3 = -b_1$ (b) Solution only if $b_3 = -b_1$.

21 A combination of the columns of C is also a combination of the columns of A (same column space; B has a different column space).

22 (a) Every $\boldsymbol{b}$ (b) Solvable only if $b_3 = 0$ (c) Solvable only if $b_3 = b_2$.

23 The extra column $\boldsymbol{b}$ enlarges the column space unless $\boldsymbol{b}$ is *already in* the column space of A:
$[A\ \ \boldsymbol{b}] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (larger column space) (no solution to $A\boldsymbol{x}=\boldsymbol{b}$) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ ($\boldsymbol{b}$ already in column space) ($A\boldsymbol{x}=\boldsymbol{b}$ has a solution)

24 The column space of AB is contained in (possibly equal to) the column space of A. If $B=0$ and $A\neq 0$ then $AB=0$ has a smaller column space than A.

25 The solution to $A\boldsymbol{z}=\boldsymbol{b}+\boldsymbol{b}^*$ is $\boldsymbol{z}=\boldsymbol{x}+\boldsymbol{y}$. If $\boldsymbol{b}$ and $\boldsymbol{b}^*$ are in the column space so is $\boldsymbol{b}+\boldsymbol{b}^*$.

26 The column space of any invertible 5 by 5 matrix is $\mathbf{R}^5$. The equation $A\boldsymbol{x}=\boldsymbol{b}$ is always solvable (by $\boldsymbol{x}=A^{-1}\boldsymbol{b}$) so every $\boldsymbol{b}$ is in the column space.

27 (a) False (b) True (c) True (d) False.

28 $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$; $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 3 & 6 & 0 \end{bmatrix}$ (columns on 1 line).

29 Every $\boldsymbol{b}$ is in the column space so that space is $\mathbf{R}^9$.

Problem Set 3.2, Page 118

1 (a) $U = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ Free variables x_2, x_4, x_5 Pivot variables x_1, x_3 (b) $U = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ Free x_3 Pivot x_1, x_2

2 (a) Free variables x_2, x_4, x_5 and solutions $(-2, 1, 0, 0, 0)$, $(0, 0, -2, 1, 0)$, $(0, 0, -3, 0, 1)$

(b) Free variable x_3: solution $(1, -1, 1)$.

3 The complete solutions are $(-2x_2, x_2, -2x_4 - 3x_5, x_4, x_5)$ and $(2x_3, -x_3, x_3)$.

The nullspace contains only $\mathbf{0}$ when there are no free variables.

4 $R = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $R = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, R has the same nullspace as U and A.

5 $\begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & 0 \end{bmatrix}$; $\begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & -3 \end{bmatrix}$.

6 (a) Special solutions $(3, 1, 0)$ and $(5, 0, 1)$ (b) $(3, 1, 0)$. Total count of pivot and free is n.

7 (a) Nullspace of A is the plane $-x + 3y + 5z = 0$; it contains all vectors $(3y + 5z, y, z)$

(b) The *line* through $(3, 1, 0)$ has equations $-x + 3y + 5z = 0$ and $-2x + 6y + 7z = 0$.

8 $R = \begin{bmatrix} 1 & -3 & -5 \\ 0 & 0 & 0 \end{bmatrix}$ with $I = [\,1\,]$; $R = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ with $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

9 (a) False (b) True (c) True (only n columns) (d) True (only m rows).

10 (a) Impossible above diagonal (b) $A =$ invertible $= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ (c) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

(d) $A = 2I, U = 2I, R = I$.

11 $\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

12 $\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

13 If column 4 is all zero then x_4 is a *free* variable. Its special solution is $(0, 0, 0, 1, 0)$.

14 If column 1 = column 5 then x_5 is a free variable. Its special solution is $(-1, 0, 0, 0, 1)$.

15 There are $n-r$ special solutions. The nullspace contains only $\boldsymbol{x} = \mathbf{0}$ when $r = n$. The column space is $\mathbf{R}^m$ when $r = m$.

16 The nullspace contains only $\boldsymbol{x} = \mathbf{0}$ when A has 5 pivots. Also the column space is $\mathbf{R}^5$, because we can solve $A\boldsymbol{x} = \boldsymbol{b}$ and every $\boldsymbol{b}$ is in the column space.

17 $A = [1 \;\; -3 \;\; -1]$; y and z are free; special solutions $(3, 1, 0)$ and $(1, 0, 1)$.

18 Fill in 12 then 3 then 1.

19 If $LU\boldsymbol{x} = \mathbf{0}$, multiply by L^{-1} to find $U\boldsymbol{x} = \mathbf{0}$. Then U and LU have the same nullspace.

20 Column 5 is sure to have no pivot since it is a combination of earlier columns. With 4 pivots in the other columns, the special solution is $\boldsymbol{s} = (1, 0, 1, 0, 1)$. The nullspace contains all multiples of $\boldsymbol{s}$ (a line in $\mathbf{R}^5$).

21 Free variables x_3, x_4: $A = \begin{bmatrix} -1 & 0 & 2 & 3 \\ 0 & -1 & 2 & 1 \end{bmatrix}$.

22 $A = \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$.

23 $A = \begin{bmatrix} 1 & 0 & -1/2 \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{bmatrix}$.

24 This construction is impossible: 2 pivot columns, 2 free variables, only 3 columns.

25 $A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$.

26 $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

27 If nullspace = column space (r pivots) then $n - r = r$. If $n = 3$ then $3 = 2r$ is impossible.

28 If A times every column of B is zero, the column space of B is contained in the *nullspace* of A: $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$.

29 R is most likely to be I; R is most likely to be I with fourth row of zeros.

30 $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ shows that (a)(b)(c) are all false. Notice $\mathsf{rref}(A^{\mathrm{T}}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

31 Three pivots (4 columns and 1 special solution); $R = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ (add any zero rows).

32 Any zero rows come after these rows: $R = [1 \;\; -2 \;\; -3]$, $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $R = I$.

33 (a) $\begin{bmatrix}1&0\\0&1\end{bmatrix}$, $\begin{bmatrix}1&0\\0&0\end{bmatrix}$, $\begin{bmatrix}1&1\\0&0\end{bmatrix}$, $\begin{bmatrix}0&1\\0&0\end{bmatrix}$, $\begin{bmatrix}0&0\\0&0\end{bmatrix}$ (b) All 8 matrices are R's!

34 One reason: A and $-A$ have the same nullspace (and also the same column space).

Problem Set 3.3, page 128

1 (a) and (c) are correct; (d) is false because R might happen to have 1's in nonpivot columns.

2 $R = \begin{bmatrix}1&1&1&1\\0&0&0&0\\0&0&0&0\end{bmatrix}$ $r=1$; $R = \begin{bmatrix}1&0&-1&-2\\0&1&2&3\\0&0&0&0\end{bmatrix}$ $r=2$; $R = \begin{bmatrix}1&-1&1&-1\\0&0&0&0\\0&0&0&0\end{bmatrix}$ $r=1$

3 $R_A = \begin{bmatrix}1&2&0\\0&0&1\\0&0&0\end{bmatrix}$ $R_B = \begin{bmatrix}R_A & R_A\end{bmatrix}$ $R_C \longrightarrow \begin{bmatrix}R_A&0\\0&R_A\end{bmatrix} \longrightarrow$ Zero row in the upper R moves all the way to the bottom.

4 If all pivot variables come last then $R = \begin{bmatrix}0&I\\0&0\end{bmatrix}$. The nullspace matrix is $N = \begin{bmatrix}I\\0\end{bmatrix}$.

5 I think this is true.

6 A and A^{T} have the same rank r. But *pivcol* (the column number) is 2 for A and 1 for A^{T}: $A = \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}$.

7 The special solutions are the columns of $N = \begin{bmatrix}-2&-3\\-4&-5\\1&0\\0&1\end{bmatrix}$ and $N = \begin{bmatrix}1&0\\0&-2\\0&1\end{bmatrix}$.

8 $A = \begin{bmatrix}1&2&4\\2&4&8\\4&8&16\end{bmatrix}$, $B = \begin{bmatrix}2&6&-3\\1&3&-3/2\\2&6&-3\end{bmatrix}$, $M = \begin{bmatrix}a&b\\c&bc/a\end{bmatrix}$.

9 If A has rank 1, the column space is a *line* in $\mathbf{R}^m$. The nullspace is a *plane* in $\mathbf{R}^n$ (given by one equation). The column space of A^{T} is a *line* in $\mathbf{R}^n$.

10 $\boldsymbol{u} = (3,1,4)$, $\boldsymbol{v} = (1,2,2)$; $\boldsymbol{u} = (2,-1)$, $\boldsymbol{v} = (1,1,3,2)$.

11 A rank one matrix has one pivot. The second row of U is zero.

12 $S = \begin{bmatrix}1&3\\1&4\end{bmatrix}$ and $S = \begin{bmatrix}1\end{bmatrix}$ and $S = \begin{bmatrix}1&0\\0&1\end{bmatrix}$.

13 P has rank r (the same as A) because elimination produces the same pivot columns.

14 The rank of R^{T} is also r, and the example matrix A has rank 2:

$$P = \begin{bmatrix}1&3\\2&6\\2&7\end{bmatrix} \qquad P^{\mathrm{T}} = \begin{bmatrix}1&2&2\\3&6&7\end{bmatrix} \qquad S^{\mathrm{T}} = \begin{bmatrix}1&2\\3&7\end{bmatrix} \qquad S = \begin{bmatrix}1&3\\2&7\end{bmatrix}.$$

15 $\text{Rank}(AB) = 1$; $\text{rank}(AM) = 1$ except $AM = 0$ if $c = -1/2$.

16 $(\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}})(\boldsymbol{w}\boldsymbol{z}^{\mathrm{T}}) = \boldsymbol{u}(\boldsymbol{v}^{\mathrm{T}}\boldsymbol{w})\boldsymbol{z}^{\mathrm{T}}$ has rank one unless $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{w} = 0$.

17 (a) By matrix multiplication, each column of AB is A times the corresponding column of B. So a combination of columns of B turns into a combination of columns of AB.

(b) The rank of B is $r = 1$. Multiplying by A cannot increase this rank. The rank stays the same for $A_1 = I$ and it drops to zero for $A_2 = 0$ or $A_2 = [\,1 \;\; 1; \;\; -1 \;\; -1\,]$.

18 If we know that $\text{rank}(B^{\mathrm{T}}A^{\mathrm{T}}) \le \text{rank}(A^{\mathrm{T}})$, then since rank stays the same for transposes, we have $\text{rank}(AB) \le \text{rank}(A)$.

19 We are given $AB = I$ which has rank n. Then $\text{rank}(AB) \le \text{rank}(A)$ forces $\text{rank}(A) = n$.

20 Certainly A and B have at most rank 2. Then their product AB has at most rank 2. Since BA is 3 by 3, it cannot be I even if $AB = I$:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad AB = I \quad \text{and} \quad BA \ne I.$$

21 (a) A and B will both have the same nullspace and row space as R (same R for both matrices).

(b) A equals an *invertible* matrix times B, when they share the same R. A key fact!

22 $$A = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 1 & 3 & 1 & 6 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \end{bmatrix} \text{ (nonzero rows of } R\text{)}.$$

23 $$A = (\text{pivot columns})(\text{nonzero rows of } R) = \begin{bmatrix} 1 & 0 \\ 1 & 4 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 8 \end{bmatrix}.$$

$$B = \begin{bmatrix} 1 & 0 \\ 1 & 4 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 4 \\ 0 & 0 & 8 & 0 & 0 & 8 \end{bmatrix}.$$

24 The m by n matrix Z has r ones at the start of its main diagonal. Otherwise Z is all zeros.

25 $Y = Z$ because the form is decided by the rank which is the same for A and A^{T}.

26 If $c = 1$, $R = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ has x_2, x_3, x_4 free. If $c \ne 1$, $R = \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ has x_3, x_4 free.

Special solutions in $N = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $(c = 1)$ and $N = \begin{bmatrix} -2 & -2 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ $(c \ne 1)$

If $c = 1$, $R = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and x_1 free; if $c = 2$, $R = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$ and x_2 free; $R = I$ if $c \ne 1, 2$

Special solutions in $N = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $(c = 1)$ or $N = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $(c = 2)$ or $N = 2$ by 0 empty matrix.

27 $N = \begin{bmatrix} I \\ -I \end{bmatrix}$; $N = \begin{bmatrix} I \\ -I \end{bmatrix}$; $N = \text{empty}$.

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1 $\begin{bmatrix} 2 & 4 & 6 & 4 & \mathbf{b}_1 \\ 2 & 5 & 7 & 6 & \mathbf{b}_2 \\ 2 & 3 & 5 & 2 & \mathbf{b}_3 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & 6 & 4 & \mathbf{b}_1 \\ 0 & 1 & 1 & 2 & \mathbf{b}_2 - \mathbf{b}_1 \\ 0 & -1 & -1 & -2 & \mathbf{b}_3 - \mathbf{b}_1 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & 6 & 4 & \mathbf{b}_1 \\ 0 & 1 & 1 & 2 & \mathbf{b}_2 - \mathbf{b}_1 \\ 0 & 0 & 0 & 0 & \mathbf{b}_3 + \mathbf{b}_2 - \mathbf{2b}_1 \end{bmatrix}$

$A\boldsymbol{x} = \boldsymbol{b}$ has a solution when $b_3 + b_2 - 2b_1 = 0$; the column space contains all combinations of $(2,2,2)$ and $(4,5,3)$ which is the plane $b_3 + b_2 - 2b_1 = 0$ (!); the nullspace contains all combinations of $\boldsymbol{s}_1 = (-1,-1,1,0)$ and $\boldsymbol{s}_2 = (2,-2,0,1)$; $\boldsymbol{x}_{complete} = \boldsymbol{x}_p + c_1\boldsymbol{s}_1 + c_2\boldsymbol{s}_2$;

$$[\,R \;\; \boldsymbol{d}\,] = \begin{bmatrix} 1 & 0 & 1 & -2 & \mathbf{4} \\ 0 & 1 & 1 & 2 & \mathbf{-1} \\ 0 & 0 & 0 & 0 & \mathbf{0} \end{bmatrix} \text{ gives the particular solution } \boldsymbol{x}_p = (4,-1,0,0).$$

2 $\begin{bmatrix} 2 & 1 & 3 & \mathbf{b}_1 \\ 6 & 3 & 9 & \mathbf{b}_2 \\ 4 & 2 & 6 & \mathbf{b}_3 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 3 & \mathbf{b}_1 \\ 0 & 0 & 0 & \mathbf{b}_2 - \mathbf{3b}_1 \\ 0 & 0 & 0 & \mathbf{b}_3 - \mathbf{2b}_1 \end{bmatrix}$ Then $[\,R \;\; \boldsymbol{d}\,] = \begin{bmatrix} 1 & 1/2 & 3/2 & \mathbf{5} \\ 0 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix}$ $A\boldsymbol{x} = \boldsymbol{b}$ has a solution when $b_2 - 3b_1 = 0$ and $b_3 - 2b_1 = 0$; the column space is the line through $(2,6,4)$ which is the intersection of the planes $b_2 - 3b_1 = 0$ and $b_3 - 2b_1 = 0$; the nullspace contains all combinations of $\boldsymbol{s}_1 = (-1/2,1,0)$ and $\boldsymbol{s}_2 = (-3/2,0,1)$; particular solution $\boldsymbol{x}_p = (5,0,0)$ and complete solution $\boldsymbol{x}_p + c_1\boldsymbol{s}_1 + c_2\boldsymbol{s}_2$.

3 $\boldsymbol{x}_{complete} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$.

4 $\boldsymbol{x}_{complete} = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$.

5 Solvable if $2b_1 + b_2 = b_3$. Then $\boldsymbol{x} = \begin{bmatrix} 5b_1 - 2b_2 \\ b_2 - 2b_1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

6 (a) Solvable if $b_2 = 2b_1$ and $3b_1 - 3b_3 + b_4 = 0$. Then $\boldsymbol{x} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \end{bmatrix}$ (no free variables)

(b) Solvable if $b_2 = 2b_1$ and $3b_1 - 3b_3 + b_4 = 0$. Then $\boldsymbol{x} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

7 $\begin{bmatrix} 1 & 3 & 1 & b_1 \\ 3 & 8 & 2 & b_2 \\ 2 & 4 & 0 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & b_2 \\ 0 & -1 & -1 & b_2 - 3b_1 \\ 0 & -2 & -2 & b_3 - 2b_1 \end{bmatrix} \to$ row 3 − 2 (row 2) + 4 (row 1) is the zero row $[\,0 \;\; 0 \;\; 0 \;\; b_3 - 2b_2 + 4b_1\,]$

8 (a) Every $\boldsymbol{b}$ is in the column space: *independent rows.* (b) Need $b_3 = 2b_2$. Row $3-2$ row $2 = 0$.

9 $L[\,U \quad \boldsymbol{c}\,] = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 + b_2 - 5b_1 \end{bmatrix} = [\,A \quad \boldsymbol{b}\,];$

$\boldsymbol{x}_p = (-9, 0, 3, 0)$ so $-9(1,2,3) + 3(3,8,7) = (0,6,-6)$ is exactly $A\boldsymbol{x}_p = \boldsymbol{b}$.

10 $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$

11 A 1 by 3 system has at least two free variables.

12 (a) $\boldsymbol{x}_1 - \boldsymbol{x}_2$ and $\boldsymbol{0}$ solve $A\boldsymbol{x} = \boldsymbol{0}$ (b) $2\boldsymbol{x}_1 - 2\boldsymbol{x}_2$ solves $A\boldsymbol{x} = \boldsymbol{0}$; $2\boldsymbol{x}_1 - \boldsymbol{x}_2$ solves $A\boldsymbol{x} = \boldsymbol{b}$.

13 (a) The particular solution $\boldsymbol{x}_p$ is always multiplied by 1 (b) Any solution can be the particular solution (c) $\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$. Then $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is shorter (length $\sqrt{2}$) than $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$
(d) The "homogeneous" solution in the nullspace is $\boldsymbol{x}_n = \boldsymbol{0}$ when A is invertible.

14 If column 5 has no pivot, x_5 is a free variable. The zero vector *is not* the only solution to $A\boldsymbol{x} = \boldsymbol{0}$. If $A\boldsymbol{x} = \boldsymbol{b}$ has a solution, it has *infinitely many* solutions.

15 If row 3 of U has no pivot, that is a *zero row.* $U\boldsymbol{x} = \boldsymbol{c}$ is solvable only if $c_3 = 0$. $A\boldsymbol{x} = \boldsymbol{b}$ *might not* be solvable, because U may have other zero rows.

16 The largest rank is 3. Then there is a pivot in every *row.* The solution *always exists.* The column space is $\mathbf{R}^3$. An example is $A = [\,I \quad F\,]$ for any 3 by 2 matrix F.

17 The largest rank is 4. There is a pivot in every *column.* The solution is *unique.* The nullspace contains only the *zero vector.* An example is $A = [\,I\,;\ G\,]$ for any 4 by 2 matrix G.

18 Rank = 3; rank = 3 unless $q = 2$ (then rank = 2).

19 All ranks = 2.

20 $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix}; \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & -2 & 3 \\ 0 & 0 & 11 & -5 \end{bmatrix}.$

21 (a) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$

22 If $A\boldsymbol{x}_1 = \boldsymbol{b}$ and $A\boldsymbol{x}_2 = \boldsymbol{b}$ then we can add $\boldsymbol{x}_1 - \boldsymbol{x}_2$ to any solution of $A\boldsymbol{x} = \boldsymbol{B}$. But there will be *no* solution to $A\boldsymbol{x} = \boldsymbol{B}$ if $\boldsymbol{B}$ is not in the column space.

23 For A, $q = 3$ gives rank 1, every other q gives rank 2. For B, $q = 6$ gives rank 1, every other q gives rank 2.

24 (a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (b) $[1 \;\; 1]$ (c) $[\,0\,]$ or any $r < m, r < n$ (d) Invertible.

25 (a) $r < m$, always $r \le n$ (b) $r = m$, $r < n$ (c) $r < m$, $r = n$ (d) $r = m = n$.

26 $R = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad R = I.$

27 R has n pivots equal to 1. Zeros above and below pivots make $R = I$.

28 $\begin{bmatrix} 1\,2\,3\,\mathbf{0} \\ 0\,0\,4\,\mathbf{0} \end{bmatrix} \rightarrow \begin{bmatrix} 1\,2\,0\,\mathbf{0} \\ 0\,0\,1\,\mathbf{0} \end{bmatrix}$; $\boldsymbol{x}_n = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$; $\begin{bmatrix} 1\,2\,3\,\mathbf{5} \\ 0\,0\,4\,\mathbf{8} \end{bmatrix} \rightarrow \begin{bmatrix} 1\,2\,0\,\mathbf{-1} \\ 0\,0\,1\quad\mathbf{2} \end{bmatrix}$ $\boldsymbol{x}_p = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$.

The pivot columns contain I so -1 and 2 go into $\boldsymbol{x}_p$.

29 $R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\boldsymbol{x}_n = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$; $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 5 \end{bmatrix}$: no solution because of row 3.

30 $\begin{bmatrix} 1 & 0 & 2 & 3 & \mathbf{2} \\ 1 & 3 & 2 & 0 & \mathbf{5} \\ 2 & 0 & 4 & 9 & \mathbf{10} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 3 & \mathbf{2} \\ 0 & 3 & 0 & -3 & \mathbf{3} \\ 0 & 0 & 0 & 3 & \mathbf{6} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 & \mathbf{-4} \\ 0 & 1 & 0 & 0 & \mathbf{3} \\ 0 & 0 & 0 & 1 & \mathbf{2} \end{bmatrix}$; $\boldsymbol{x}_p = \begin{bmatrix} 4 \\ -3 \\ 0 \\ -2 \end{bmatrix}$ and $\boldsymbol{x}_n = x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

31 $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 0 & 3 \end{bmatrix}$; B cannot exist since 2 equations in 3 unknowns cannot have a unique solution.

32 $A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\boldsymbol{x} = \begin{bmatrix} 7 \\ -2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -7 \\ 2 \\ 1 \end{bmatrix}$ and then no solution.

33 $A = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$.

34 The matrix A has rank $4 - 1 = 3$; the complete solution is $\boldsymbol{x} = c\boldsymbol{s}$ for any c.

$$R = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{with } -2, -3 \text{ in the free column.}$$

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1 $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0$ gives $c_3 = c_2 = c_1 = 0$. But $\boldsymbol{v}_1 + \boldsymbol{v}_2 - 4\boldsymbol{v}_3 + \boldsymbol{v}_4 = \mathbf{0}$ (dependent).

2 $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are independent. All six vectors are on the plane $(1, 1, 1, 1) \cdot \boldsymbol{v} = 0$ so no four of these six vectors can be independent.

3 If $a = 0$ then column 1 $= \mathbf{0}$; if $d = 0$ then b(column 1) $- a$(column 2) $= \mathbf{0}$; if $f = 0$ then all columns end in zero (all are perpendicular to $(0, 0, 1)$, all in the xy plane, must be dependent).

4 $U\boldsymbol{x} = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ gives $z = 0$ then $y = 0$ then $x = 0$.

5 (a) $\begin{bmatrix}1&2&3\\3&1&2\\2&3&1\end{bmatrix} \to \begin{bmatrix}1&2&3\\0&-5&-7\\0&-1&-5\end{bmatrix} \to \begin{bmatrix}1&2&3\\0&-5&-7\\0&0&-18/5\end{bmatrix}$: invertible $\Rightarrow$ independent columns

(b) $\begin{bmatrix}1&2&-3\\-3&1&2\\2&-3&1\end{bmatrix} \to \begin{bmatrix}1&2&-3\\0&7&-7\\0&-7&7\end{bmatrix} \to \begin{bmatrix}1&2&-3\\0&7&-7\\0&0&0\end{bmatrix}$; $A\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$, columns add to $\mathbf{0}$.

6 Columns 1, 2, 4 are independent. Also 1, 3, 4 and 2, 3, 4 and others (but not 1, 2, 3). Same column numbers (not same columns!) for A.

7 The sum $\boldsymbol{v}_1 - \boldsymbol{v}_2 + \boldsymbol{v}_3 = \mathbf{0}$ because $(\boldsymbol{w}_2 - \boldsymbol{w}_3) - (\boldsymbol{w}_1 - \boldsymbol{w}_3) + (\boldsymbol{w}_1 - \boldsymbol{w}_2) = \mathbf{0}$.

8 If $c_1(\boldsymbol{w}_2+\boldsymbol{w}_3)+c_2(\boldsymbol{w}_1+\boldsymbol{w}_3)+c_3(\boldsymbol{w}_1+\boldsymbol{w}_2) = \mathbf{0}$ then $(c_2+c_3)\boldsymbol{w}_1+(c_1+c_3)\boldsymbol{w}_2+(c_1+c_2)\boldsymbol{w}_3 = \mathbf{0}$. Since the $\boldsymbol{w}$'s are independent this requires $c_2 + c_3 = 0, c_1 + c_3 = 0, c_1 + c_2 = 0$. The only solution is $c_1 = c_2 = c_3 = 0$. Only this combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ gives zero.

9 (a) The four vectors are the columns of a 3 by 4 matrix A. There is a nonzero solution to $A\boldsymbol{x} = \mathbf{0}$ because there is at least one free variable (b) dependent if $[\,\boldsymbol{v}_1 \quad \boldsymbol{v}_2\,]$ has rank 0 or 1 (c) $0\boldsymbol{v}_1 + 3(0,0,0) = \mathbf{0}$.

10 The plane is the nullspace of $A = [\,1 \quad 2 \quad -3 \quad -1\,]$. Three free variables give three solutions $(x,y,z,t) = (2,-1,0,0)$ and $(3,0,1,0)$ and $(1,0,0,1)$.

11 (a) Line in $\mathbf{R}^3$ (b) Plane in $\mathbf{R}^3$ (c) Plane in $\mathbf{R}^3$ (d) All of $\mathbf{R}^3$.

12 $\boldsymbol{b}$ is in the column space when there is a solution to $A\boldsymbol{x} = \boldsymbol{b}$; $\boldsymbol{c}$ is in the row space when there is a solution to $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{c}$. *False.* The zero vector is always in the row space.

13 All dimensions are 2. The row spaces of A and U are the same.

14 The dimension of $\mathbf{S}$ is (a) zero when $\boldsymbol{x} = \mathbf{0}$ (b) one when $\boldsymbol{x} = (1,1,1,1)$ (c) three when $\boldsymbol{x} = (1,1,-1,-1)$ because all rearrangements of this $\boldsymbol{x}$ are perpendicular to $(1,1,1,1)$ (d) four when the $\boldsymbol{x}$'s are not equal and don't add to zero. **No $\boldsymbol{x}$ gives** $\dim \mathbf{S} = 2$.

15 $\boldsymbol{v} = \frac{1}{2}(\boldsymbol{v}+\boldsymbol{w}) + \frac{1}{2}(\boldsymbol{v}-\boldsymbol{w})$ and $\boldsymbol{w} = \frac{1}{2}(\boldsymbol{v}+\boldsymbol{w}) - \frac{1}{2}(\boldsymbol{v}-\boldsymbol{w})$. The two pairs *span* the same space. They are a basis when $\boldsymbol{v}$ and $\boldsymbol{w}$ are *independent.*

16 The n independent vectors span a space of dimension n. They are a *basis* for that space. If they are the columns of A then m is *not less* than n $(m \geq n)$.

17 These bases are not unique! (a) $(1,1,1,1)$ (b) $(1,-1,0,0), (1,0,-1,0), (1,0,0,-1)$ (c) $(1,-1,-1,0)$, $(1,-1,0,-1)$ (d) $(1,0)(0,1)$; $(-1,0,1,0,0), (0,-1,0,1,0), (-1,0,0,0,1)$.

18 Any bases for $\mathbf{R}^2$; (row 1 and row 2) or (row 1 and row 1 + row 2).

19 (a) The 6 vectors *might not* span $\mathbf{R}^4$ (b) The 6 vectors *are not* independent (c) Any four *might be* a basis.

20 Independent columns $\Rightarrow$ rank n. Columns span $\mathbf{R}^m \Rightarrow$ rank m. Columns are basis for $\mathbf{R}^m \Rightarrow$ rank $= m = n$.

21 One basis is $(2,1,0)$, $(-3,0,1)$. The vector $(2,1,0)$ is a basis for the intersection with the xy plane. The normal vector $(1,-2,3)$ is a basis for the line perpendicular to the plane.

22 (a) The only solution is $\boldsymbol{x} = \mathbf{0}$ because *the columns are independent* (b) $A\boldsymbol{x} = \boldsymbol{b}$ is solvable because *the columns span* $\mathbf{R}^5$.

23 (a) True (b) False because the basis vectors may not be in $\mathbf{S}$.

24 Columns 1 and 2 are bases for the (different) column spaces; rows 1 and 2 are bases for the (equal) row spaces; $(1, -1, 1)$ is a basis for the (equal) nullspaces.

25 (a) False for $[1 \;\; 1]$ (b) False (c) True: Both dimensions $= 2$ if A is invertible, dimensions $= 0$ if $A = 0$, otherwise dimensions $= 1$ (d) False, columns may be dependent.

26 Rank 2 if $c = 0$ and $d = 2$; rank 2 except when $c = d$ or $c = -d$.

27 (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) Add $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$ are a basis for all $A = -A^{\mathrm{T}}$.

28 I, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$; echelon matrices do *not* form a subspace; they *span* the upper triangular matrices (not every U is echelon).

29 $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}; \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}$.

30 $-\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} + \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} - \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix} + \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} + \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} - \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix} = 0$

31 (a) All 3 by 3 matrices (b) Upper triangular matrices (c) All multiples cI.

32 $\begin{bmatrix} -1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ -1 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 2 \end{bmatrix}$.

33 (a) $y(x) =$ constant C (b) $y(x) = 3x$ (c) $y(x) = 3x + C = \boldsymbol{y}_p + \boldsymbol{y}_n$.

34 $y(0) = 0$ requires $A + B + C = 0$. One basis is $\cos x - \cos 2x$ and $\cos x - \cos 3x$.

35 (a) $y(x) = e^{2x}$ (b) $y = x$ (one basis vector in each case).

36 $y_1(x), y_2(x), y_3(x)$ can be $x, 2x, 3x$ (dim 1) or $x, 2x, x^2$ (dim 2) or x, x^2, x^3 (dim 3).

37 Basis $1, x, x^2, x^3$; basis $x - 1, x^2 - 1, x^3 - 1$.

38 Basis for $\mathbf{S}$: $(1, 0, -1, 0)$, $(0, 1, 0, 0)$, $(1, 0, 0, -1)$; basis for $\mathbf{T}$: $(1, -1, 0, 0)$ and $(0, 0, 2, 1)$; $\mathbf{S} \cap \mathbf{T}$ has dimension 1.

39 See Solution 30 for $I =$ combination of five other P's. Check the $(1,1)$ entry, then $(3,2)$, then $(3,3)$, then $(1,2)$ to show that those five P's are independent.

Four conditions on the 9 entries make all row sums and column sums equal: row sum 1 = row sum 2 = row sum 3 = column sum 1 = column sum 2 (= column sum 3 is automatic).

40 The subspace of matrices that have $AS = SA$ has dimension *three*.

41 (a) No, don't span (b) No, dependent (c) Yes, a basis (d) No, dependent

42 If the 5 by 5 matrix $[\,A \quad \boldsymbol{b}\,]$ is invertible, $\boldsymbol{b}$ is not a combination of the columns of A. If $[\,A \quad \boldsymbol{b}\,]$ is singular, and the 4 columns of A are independent, $\boldsymbol{b}$ *is* a combination of those columns.

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1 (a) Row and column space dimensions $= 5$, nullspace dimension $= 4$, left nullspace dimension $= 2$ sum $= 16 = m + n$ (b) Column space is $\mathbf{R}^3$; left nullspace contains only $\mathbf{0}$.

2 A: Row space $(1, 2, 4)$; nullspace $(-2, 1, 0)$ and $(-4, 0, 1)$; column space $(1, 2)$; left nullspace $(-2, 1)$. B: Row space $(1, 2, 4)$ and $(2, 5, 8)$; column space $(1, 2)$ and $(2, 5)$; nullspace $(-4, 0, 1)$; left nullspace basis is empty.

3 Row space $(0, 1, 2, 3, 4)$ and $(0, 0, 0, 1, 2)$; column space $(1, 1, 0)$ and $(3, 4, 1)$; nullspace basis $(1, 0, 0, 0, 0), (0, 2, -1, 0, 0), (0, 2, 0, -2, 1)$; left nullspace $(1, -1, 1)$.

4 (a) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) Impossible: $r + (n - r)$ must be 3 (c) $\begin{bmatrix} 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} -9 & -3 \\ 3 & 1 \end{bmatrix}$

(e) Impossible: Row space = column space requires $m = n$. Then $m - r = n - r$.

5 $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}$.

6 A: Row space $(0, 3, 3, 3)$ and $(0, 1, 0, 1)$; column space $(3, 0, 1)$ and $(3, 0, 0)$; nullspace $(1, 0, 0, 0)$ and $(0, -1, 0, 1)$; left nullspace $(0, 1, 0)$. B: Row space (1), column space $(1, 4, 5)$, nullspace: empty basis, left nullspace $(-4, 1, 0)$ and $(-5, 0, 1)$.

7 Invertible A: row space basis = column space basis = $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$; nullspace basis and left nullspace basis are empty. Matrix B: row space basis $(1, 0, 0, 1, 0, 0)$, $(0, 1, 0, 0, 1, 0)$ and $(0, 0, 1, 0, 0, 1)$; column space basis $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$; nullspace basis $(-1, 0, 0, 1, 0, 0)$ and $(0, -1, 0, 0, 1, 0)$ and $(0, 0, -1, 0, 0, 1)$; left nullspace basis is empty.

8 Row space dimensions $3, 3, 0$; column space dimensions $3, 3, 0$; nullspace dimensions $2, 3, 2$; left nullspace dimensions $0, 2, 3$.

9 (a) Same row space and nullspace. Therefore rank (dimension of row space) is the same
(b) Same column space and left nullspace. Same rank (dimension of column space).

10 Most likely rank $= 3$, nullspace and left nullspace contain only $(0, 0, 0)$. When the matrix is 3 by 5: Most likely rank $= 3$ and dimension of nullspace is 2.

11 (a) No solution means that $r < m$. Always $r \leq n$. Can't compare m and n
(b) If $m - r > 0$, the left nullspace contains a nonzero vector.

12 $\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 1 \end{bmatrix}$; $r + (n - r) = n = 3$ but $2 + 2$ is 4.

13 (a) False (b) True (c) False (choose A and B same size and invertible).

14 Row space basis $(1,2,3,4)$, $(0,1,2,3)$, $(0,0,1,2)$; nullspace basis $(0,1,-2,1)$; column space basis $(1,0,0)$, $(0,1,0)$, $(0,0,1)$; left nullspace has empty basis.

15 Row space and nullspace stay the same; $(2,1,3,4)$ is in the new column space.

16 If $A\boldsymbol{v}=\mathbf{0}$ and $\boldsymbol{v}$ is a row of A then $\boldsymbol{v}\cdot\boldsymbol{v}=0$.

17 Row space $=yz$ plane; column space $=xy$ plane; nullspace $=x$ axis; left nullspace $=z$ axis. For $I+A$: Row space $=$ column space $=\mathbf{R}^3$, nullspaces contain only zero vector.

18 Row $3-2$ row $2+$ row $1=$ zero row so the vectors $c(1,-2,1)$ are in the left nullspace. The same vectors happen to be in the nullspace.

19 Elimination leads to $0=b_3-b_2-b_1$ so $(-1,-1,1)$ is in the left nullspace. Elimination leads to $b_3-2b_1=0$ and $b_4+b_2-4b_1=0$, so $(-2,0,1,0)$ and $(-4,1,0,1)$ are in the left nullspace.

20 (a) All combinations of $(-1,2,0,0)$ and $(-\frac{1}{4},0,-3,1)$ (b) One (c) $(1,2,3)$, $(0,1,4)$.

21 (a) $\boldsymbol{u}$ and $\boldsymbol{w}$ (b) $\boldsymbol{v}$ and $\boldsymbol{z}$ (c) rank <2 if $\boldsymbol{u}$ and $\boldsymbol{w}$ are dependent or $\boldsymbol{v}$ and $\boldsymbol{z}$ are dependent (d) The rank of $\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}+\boldsymbol{w}\boldsymbol{z}^{\mathrm{T}}$ is 2.

22 $\begin{bmatrix}1&2\\2&2\\4&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&1\end{bmatrix}=\begin{bmatrix}1&2&2\\2&2&2\\4&1&1\end{bmatrix}$.

23 Row space basis $(3,0,3)$, $(1,1,2)$; column space basis $(1,4,2)$, $(2,5,7)$; rank is only 2.

24 $A^{\mathrm{T}}\boldsymbol{y}=\boldsymbol{d}$ puts $\boldsymbol{d}$ in the *row space* of A; unique solution if the *left nullspace* (nullspace of A^{T}) contains only $\boldsymbol{y}=\mathbf{0}$.

25 (a) True (same rank) (b) False $A=[1\ \ 0]$ (c) False (A can be invertible and also unsymmetric) (d) True.

26 The rows of $AB=C$ are combinations of the rows of B. So rank $C\le$ rank B. Also rank $C\le$ rank A. (The columns of C are combinations of the columns of A).

27 Choose $d=bc/a$. Then the row space has basis (a,b) and the nullspace has basis $(-b,a)$.

28 Both ranks are 2; if $p\ne 0$, rows 1 and 2 are a basis for the row space. $\boldsymbol{N}(B^{\mathrm{T}})$ has six vectors with 1 and -1 separated by a zero; $\boldsymbol{N}(C^{\mathrm{T}})$ has $(-1,0,0,0,0,0,0,1)$ and $(0,-1,0,0,0,0,1,0)$ and columns $3,4,5,6$ of I; $\boldsymbol{N}(C)$ is a challenge.

29 $a_{11}=1, a_{12}=0, a_{13}=1, a_{22}=0, a_{32}=1, a_{31}=0, a_{23}=1, a_{33}=0, a_{21}=1$ (not unique).

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1 Both nullspace vectors are orthogonal to the row space vector in $\mathbf{R}^3$. Column space is perpendicular to the nullspace of A^{T} in $\mathbf{R}^2$.

2 The nullspace is $\mathbf{Z}$ (only zero vector) so $\boldsymbol{x}_n=\mathbf{0}$. and row space $=\mathbf{R}^2$. Plane $\perp$ line in $\mathbf{R}^3$.

3 (a) $\begin{bmatrix} 1 & 2 & -3 \\ 2 & -3 & 1 \\ -3 & 5 & -2 \end{bmatrix}$ (b) Impossible, $\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$ not orthogonal to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in $\boldsymbol{C}(A)$ and $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ in $\boldsymbol{N}(A^{\mathrm{T}})$ is impossible: not perpendicular (d) This asks for $A^2 = 0$; take $A = \left[\begin{smallmatrix} 1 & -1 \\ 1 & -1 \end{smallmatrix}\right]$ (e) $(1, 1, 1)$ will be in the nullspace and row space; no such matrix.

4 If $AB = 0$, the columns of B are in the *nullspace* of A. The rows of A are in the *left nullspace* of B. If rank $= 2$, all four subspaces would have dimension 2 which is impossible for 3 by 3.

5 (a) If $A\boldsymbol{x} = \boldsymbol{b}$ has a solution and $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$, then $\boldsymbol{y}$ is perpendicular to $\boldsymbol{b}$. $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} = (A\boldsymbol{x})^{\mathrm{T}}\boldsymbol{y} = 0$. (b) $\boldsymbol{b}$ is not in the column space; so not $\perp$ to all $\boldsymbol{y}$ in the left nullspace (see **7**).

6 Multiply the equations by $y_1 = 1$, $y_2 = 1$, $y_3 = -1$. They add to $0 = 1$ so no solution: $\boldsymbol{y} = (1, 1, -1)$ is in the left nullspace. Can't have $0 = (\boldsymbol{y}^{\mathrm{T}}A)\boldsymbol{x} = \boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} = 1$.

7 Multiply by $\boldsymbol{y} = (1, 1, -1)$, then $x_1 - x_2 = 1$ plus $x_2 - x_3 = 1$ minus $x_1 - x_3 = 1$ is $0 = 1$.

8 $\boldsymbol{x} = \boldsymbol{x}_r + \boldsymbol{x}_n$, where $\boldsymbol{x}_r$ is in the row space and $\boldsymbol{x}_n$ is in the nullspace. Then $A\boldsymbol{x}_n = \mathbf{0}$ and $A\boldsymbol{x} = A\boldsymbol{x}_r + A\boldsymbol{x}_n = A\boldsymbol{x}_r$. All vectors $A\boldsymbol{x}$ are combinations of the columns of A.

9 $A\boldsymbol{x}$ is always in the *column space* of A. If $A^{\mathrm{T}}A\boldsymbol{x} = \mathbf{0}$ then $A\boldsymbol{x}$ is also in the nullspace of A^{T}. Perpendicular to itself, so $A\boldsymbol{x} = \mathbf{0}$.

10 (a) For a symmetric matrix the column space and row space are the same (b) $\boldsymbol{x}$ is in the nullspace and $\boldsymbol{z}$ is in the column space = row space: so these "eigenvectors" have $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{z} = 0$.

11 The nullspace of A is spanned by $(-2, 1)$, the row space is spanned by $(1, 2)$. The nullspace of B is spanned by $(0, 1)$, the row space is spanned by $(1, 0)$.

12 $\boldsymbol{x}$ splits into $\boldsymbol{x}_r + \boldsymbol{x}_n = (1, -1) + (1, 1) = (2, 0)$.

13 $V^{\mathrm{T}}W$ = zero matrix makes each basis vector for $\boldsymbol{V}$ orthogonal to each basis vector for $\boldsymbol{W}$. Then every $\boldsymbol{v}$ in $\boldsymbol{V}$ is orthogonal to every $\boldsymbol{w}$ in $\boldsymbol{W}$ (they are combinations of the basis vectors).

14 $A\boldsymbol{x} = B\widehat{\boldsymbol{x}}$ means that $[\,A \quad B\,]\left[\begin{smallmatrix} \mathbf{x} \\ -\widehat{\mathbf{x}} \end{smallmatrix}\right] = \mathbf{0}$. Three homogeneous equations in four unknowns always have a nonzero solution. Here $\boldsymbol{x} = (3, 1)$ and $\widehat{\boldsymbol{x}} = (1, 0)$ and $A\boldsymbol{x} = B\widehat{\boldsymbol{x}} = (5, 6, 5)$ is in both column spaces. Two planes in $\mathbf{R}^3$ must intersect in a line at least!

15 A p-dimensional and a q-dimensional subspace of $\mathbf{R}^n$ share at least a line if $p + q > n$.

16 $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0} \Rightarrow (A\boldsymbol{x})^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}\boldsymbol{y} = 0$. Then $\boldsymbol{y} \perp A\boldsymbol{x}$ and $\boldsymbol{N}(A^{\mathrm{T}}) \perp \boldsymbol{C}(A)$.

17 If $\boldsymbol{S}$ is the subspace of $\mathbf{R}^3$ containing only the zero vector, then $\boldsymbol{S}^{\perp}$ is $\mathbf{R}^3$. If $\boldsymbol{S}$ is spanned by $(1, 1, 1)$, then $\boldsymbol{S}^{\perp}$ is spanned by $(1, -1, 0)$ and $(1, 0, -1)$. If $\boldsymbol{S}$ is spanned by $(2, 0, 0)$ and $(0, 0, 3)$, then $\boldsymbol{S}^{\perp}$ is spanned by $(0, 1, 0)$.

18 $\boldsymbol{S}^{\perp}$ is the nullspace of $A = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 2 & 2 \end{bmatrix}$. Therefore $\boldsymbol{S}^{\perp}$ is a *subspace* even if $\boldsymbol{S}$ is not.

19 $\boldsymbol{L}^{\perp}$ is the 2-*dimensional subspace* (*a plane*) in $\mathbf{R}^3$ perpendicular to $\boldsymbol{L}$. Then $(\boldsymbol{L}^{\perp})^{\perp}$ is a 1-*dimensional subspace* (*a line*) perpendicular to $\boldsymbol{L}^{\perp}$. In fact $(\boldsymbol{L}^{\perp})^{\perp}$ is $\boldsymbol{L}$.

20 If $\boldsymbol{V}$ is the whole space $\mathbf{R}^4$, then $\boldsymbol{V}^{\perp}$ contains only the *zero vector*. Then $(\boldsymbol{V}^{\perp})^{\perp} = \mathbf{R}^4 = \boldsymbol{V}$.

21 For example $(-5, 0, 1, 1)$ and $(0, 1, -1, 0)$ span $\boldsymbol{S}^{\perp}$ = nullspace of $A = [1 \;\; 2 \;\; 2 \;\; 3;\;\; 1 \;\; 3 \;\; 3 \;\; 2]$.

22 $(1,1,1,1)$ is a basis for $\boldsymbol{P}^{\perp}$. $A = [1 \;\; 1 \;\; 1 \;\; 1]$ has the plane $\boldsymbol{P}$ as its nullspace.

23 $\boldsymbol{x}$ in $\boldsymbol{V}^{\perp}$ is perpendicular to any vector in $\boldsymbol{V}$. Since $\boldsymbol{V}$ contains all the vectors in $\boldsymbol{S}$, $\boldsymbol{x}$ is also perpendicular to any vector in $\boldsymbol{S}$. So every $\boldsymbol{x}$ in $\boldsymbol{V}^{\perp}$ is also in $\boldsymbol{S}^{\perp}$.

24 Column 1 of A^{-1} is orthogonal to the space spanned by the 2nd, 3rd, . . ., nth rows of A.

25 If the columns of A are unit vectors, all mutually perpendicular, then $A^{\mathrm{T}}A = I$.

26 $A = \begin{bmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{bmatrix}$, $A^{\mathrm{T}}A = 9I$ is *diagonal*: $(A^{\mathrm{T}}A)_{ij} =$ (column i of A)$\cdot$(column j of A).

27 The lines $3x + y = b_1$ and $6x + 2y = b_2$ are parallel. They are the same line if $b_2 = 2b_1$. In that case (b_1, b_2) is perpendicular to $(-2,1)$. The nullspace is the line $3x + y = 0$. One particular vector in the nullspace is $(-1,3)$.

28 (a) $(1,-1,0)$ is in both planes. Normal vectors are perpendicular, but planes still intersect!
(b) Need *three* orthogonal vectors to span the whole orthogonal complement.
(c) Lines can meet without being orthogonal.

29 $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 0 \\ 3 & 0 & -1 \end{bmatrix}$; $\boldsymbol{v}$ can *not* be in the nullspace and row space, or in the left nullspace and column space. These spaces are orthogonal and $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{v} \neq 0$.

30 When $AB = 0$, the column space of B is contained in the nullspace of A. Therefore the dimension of $\boldsymbol{C}(B) \leq$ dimension of $\boldsymbol{N}(A)$. This means $\operatorname{rank}(B) \leq 4 - \operatorname{rank}(A)$.

31 null(N') produces a basis for the *row space* of A (perpendicular to $\boldsymbol{N}(A)$).

Problem Set 4.2, page 181

1 (a) $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} = 5/3$; $\boldsymbol{p} = (5/3, 5/3, 5/3)$; $\boldsymbol{e} = (-2/3, 1/3, 1/3)$
(b) $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} = -1$; $\boldsymbol{p} = (1,3,1)$; $\boldsymbol{e} = (0,0,0)$.

2 (a) $\boldsymbol{p} = (\cos\theta, 0)$ (b) $\boldsymbol{p} = (0,0)$ since $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b} = 0$.

3 $P_1 = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and $P_1\boldsymbol{b} = \frac{1}{3}\begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$ and $P_1^2 = P_1$. $P_2 = \frac{1}{11}\begin{bmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{bmatrix}$ and $P_2\boldsymbol{b} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$.

4 $P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $P_2 = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. $P_1P_2 \neq 0$ and $P_1 + P_2$ is not a projection matrix.

5 $P_1 = \frac{1}{9}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}$, $P_2 = \frac{1}{9}\begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix}$. $P_1P_2 =$ zero matrix because $\boldsymbol{a}_1 \perp \boldsymbol{a}_2$.

6 $\boldsymbol{p}_1 = (\frac{1}{9}, -\frac{2}{9}, -\frac{2}{9})$ and $\boldsymbol{p}_2 = (\frac{4}{9}, \frac{4}{9}, -\frac{2}{9})$ and $\boldsymbol{p}_3 = (\frac{4}{9}, -\frac{2}{9}, \frac{4}{9})$. Then $\boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3 = (1,0,0) = \boldsymbol{b}$.

7 $P_1 + P_2 + P_3 = \frac{1}{9}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix} + \frac{1}{9}\begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix} + \frac{1}{9}\begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix} = I.$

8 $\boldsymbol{p}_1 = (1,0)$ and $\boldsymbol{p}_2 = (0.6, 1.2)$. Then $\boldsymbol{p}_1 + \boldsymbol{p}_2 \neq \boldsymbol{b}$.

9 Since A is invertible, $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}} = AA^{-1}(A^{\mathrm{T}})^{-1}A^{\mathrm{T}} = I$: project onto all of $\mathbf{R}^2$.

10 $P_2 = \begin{bmatrix} 0.2 & 0.4 \\ 0.4 & 0.8 \end{bmatrix}$, $P_2\boldsymbol{a}_1 = \begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix}$, $P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $P_1P_2\boldsymbol{a}_1 = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}$. *No*, $P_1P_2 \neq (P_1P_2)^2$.

11 (a) $\boldsymbol{p} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b} = (2,3,0)$ and $\boldsymbol{e} = (0,0,4)$ (b) $\boldsymbol{p} = (4,4,6)$ and $\boldsymbol{e} = (0,0,0)$.

12 $P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ = projection on xy plane. $P_2 = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

13 $\boldsymbol{p} = (1,2,3,0)$. P = square matrix = $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

14 The projection of this $\boldsymbol{b}$ onto the column space of A is $\boldsymbol{b}$ itself, but P is not necessarily I.

$$P = \frac{1}{21}\begin{bmatrix} 5 & 8 & -4 \\ 8 & 17 & 2 \\ -4 & 2 & 20 \end{bmatrix} \text{ and } \boldsymbol{p} = (0,2,4).$$

15 The column space of $2A$ is the same as the column space of A. $\widehat{\boldsymbol{x}}$ for $2A$ is *half* of $\widehat{\boldsymbol{x}}$ for A.

16 $\frac{1}{2}(1,2,-1) + \frac{3}{2}(1,0,1) = (2,1,1)$. Therefore $\boldsymbol{b}$ is in the plane. Projection shows $P\boldsymbol{b} = \boldsymbol{b}$.

17 $P^2 = P$ and therefore $(I-P)^2 = (I-P)(I-P) = I - PI - IP + P^2 = I - P$. When P projects onto the column space of A then $I - P$ projects onto the *left nullspace* of A.

18 (a) $I - P$ is the projection matrix onto $(1,-1)$ in the perpendicular direction to $(1,1)$

(b) $I - P$ is the projection matrix onto the plane $x + y + z = 0$ perpendicular to $(1,1,1)$.

19 For any choice, say $(1,1,0)$ and $(2,0,1)$, the matrix P is $\begin{bmatrix} 5/6 & 1/6 & 1/3 \\ 1/6 & 5/6 & -1/3 \\ 1/3 & -1/3 & 1/3 \end{bmatrix}$.

20 $\boldsymbol{e} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$, $Q = \boldsymbol{e}\boldsymbol{e}^{\mathrm{T}}/\boldsymbol{e}^{\mathrm{T}}\boldsymbol{e} = \begin{bmatrix} 1/6 & -1/6 & -1/3 \\ -1/6 & 1/6 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$, $P = I - Q = \begin{bmatrix} 5/6 & 1/6 & 1/3 \\ 1/6 & 5/6 & -1/3 \\ 1/3 & -1/3 & 1/3 \end{bmatrix}$.

21 $\left(A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\right)^2 = A(A^{\mathrm{T}}A)^{-1}(A^{\mathrm{T}}A)(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$. Therefore $P^2 = P$. $P\boldsymbol{b}$ is always in the column space (where P projects). Therefore its projection $P(P\boldsymbol{b})$ is $P\boldsymbol{b}$.

22 $P^{\mathrm{T}} = \left(A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\right)^{\mathrm{T}} = A\left((A^{\mathrm{T}}A)^{-1}\right)^{\mathrm{T}}A^{\mathrm{T}} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}} = P$. ($A^{\mathrm{T}}A$ is symmetric.)

23 If A is invertible then its column space is all of $\mathbf{R}^n$. So $P = I$ and $\boldsymbol{e} = \mathbf{0}$.

24 The nullspace of A^{T} is *orthogonal* to the column space $\boldsymbol{C}(A)$. So if $A^{\mathrm{T}}\boldsymbol{b} = \mathbf{0}$, the projection of $\boldsymbol{b}$ onto $\boldsymbol{C}(A)$ should be $\boldsymbol{p} = \mathbf{0}$. Check $P\boldsymbol{b} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b} = A(A^{\mathrm{T}}A)^{-1}\mathbf{0} = \mathbf{0}$.

25 The column space of P will be $\boldsymbol{S}$ (n-dimensional). Then r = dimension of column space $= n$.

26 A^{-1} exists since the rank is $r = m$. Multiply $A^2 = A$ by A^{-1} to get $A = I$.

27 $A\boldsymbol{x}$ is in the nullspace of A^{T}. But $A\boldsymbol{x}$ is always in the column space of A. To be in both of those perpendicular spaces, $A\boldsymbol{x}$ must be zero. So A and $A^{\mathrm{T}}A$ have the *same nullspace.*

28 $P^2 = P = P^{\mathrm{T}}$ give $P^{\mathrm{T}}P = P$. Then the $(2,2)$ entry of P equals the $(2,2)$ entry of $P^{\mathrm{T}}P$ which is the length squared of column 2.

29 Set $A = B^{\mathrm{T}}$. Then A has independent columns. By **4*G***, $A^{\mathrm{T}}A = BB^{\mathrm{T}}$ is invertible.

30 (a) The column space is the line through $\boldsymbol{a} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ so $P_C = \dfrac{\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}} = \dfrac{1}{25}\begin{bmatrix} 9 & 12 \\ 12 & 25 \end{bmatrix}$. We can't use $(A^{\mathrm{T}}A)^{-1}$ because A has dependent columns. (b) The row space is the line through $\boldsymbol{v} = (1,2,2)$ and $P_R = \boldsymbol{v}\boldsymbol{v}^{\mathrm{T}}/\boldsymbol{v}^{\mathrm{T}}\boldsymbol{v}$. Always $P_C A = A$ and $AP_R = A$ and then $P_C A P_R = A$!

Problem Set 4.3, page 192

1 $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$ give $A^{\mathrm{T}}A = \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix}$ and $A^{\mathrm{T}}\boldsymbol{b} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$.

$A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$ gives $\widehat{\boldsymbol{x}} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\boldsymbol{p} = A\widehat{\boldsymbol{x}} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix}$ and $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = \begin{bmatrix} -1 \\ 3 \\ -5 \\ 3 \end{bmatrix}$. $E = \|\boldsymbol{e}\|^2 = 44$.

2 $\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$. Change the right side to $\boldsymbol{p} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix}$; $\widehat{\boldsymbol{x}} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ exactly solves $A\widehat{\boldsymbol{x}} = \boldsymbol{b}$.

3 $\boldsymbol{p} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b} = (1,5,13,17)$. $\boldsymbol{e} = (-1,3,-5,3)$. $\boldsymbol{e}$ is indeed perpendicular to both columns of A. The shortest distance $\|\boldsymbol{e}\|$ is $\sqrt{44}$.

4 $E = (C + \mathbf{0}D)^2 + (C + \mathbf{1}D - 8)^2 + (C + \mathbf{3}D - 8)^2 + (C + \mathbf{4}D - 20)^2$. Then $\partial E/\partial C = 2C + 2(C{+}D{-}8){+}2(C{+}3D{-}8){+}2(C{+}4D{-}20) = 0$ and $\partial E/\partial D = 1{\cdot}2(C{+}D{-}8){+}3{\cdot}2(C{+}3D{-}8){+}4 \cdot 2(C + 4D - 20) = 0$. These normal equations are again $\begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$.

5 $E = (C-0)^2 + (C-8)^2 + (C-8)^2 + (C-20)^2$. $A^{\mathrm{T}} = [\,1 \ \ 1 \ \ 1 \ \ 1\,]$, $A^{\mathrm{T}}A = [\,4\,]$ and $A^{\mathrm{T}}\boldsymbol{b} = [\,36\,]$ and $(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b} = 9 =$ best height C. Errors $\boldsymbol{e} = (-9,-1,-1,11)$.

6 $\widehat{\boldsymbol{x}} = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} = 9$ and projection $\boldsymbol{p} = (9,9,9,9)$; $\boldsymbol{e}^{\mathrm{T}}\boldsymbol{a} = (-9,-1,-1,11)^{\mathrm{T}}(1,1,1,1) = 0$ and $\|\boldsymbol{e}\| = \sqrt{204}$.

7 $A = [\,0 \ \ 1 \ \ 3 \ \ 4\,]^{\mathrm{T}}$, $A^{\mathrm{T}}A = [\,26\,]$ and $A^{\mathrm{T}}\boldsymbol{b} = [\,112\,]$. Best $D = 112/26 = 56/13$.

8 $\widehat{\boldsymbol{x}} = 56/13$, $\boldsymbol{p} = (56/13)(0,1,3,4)$. $C = 9$, $D = 56/13$ don't match $(C,D) = (1,4)$; the columns of A were not perpendicular so we can't project separately to find $C = 1$ and $D = 4$.

9 Closest parabola: $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$. $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = \begin{bmatrix} 4 & 8 & 26 \\ 8 & 26 & 92 \\ 26 & 92 & 338 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \\ 400 \end{bmatrix}$.

10 $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$. Then $\begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 0 \\ 47 \\ -28 \\ 5 \end{bmatrix}$. Exact cubic so $\boldsymbol{p} = \boldsymbol{b}$, $\boldsymbol{e} = \boldsymbol{0}$.

11 (a) The best line is $x = 1 + 4t$, which goes through the center point $(\widehat{t}, \widehat{b}) = (2, 9)$

(b) From the first equation: $C \cdot m + D \cdot \sum_{i=1}^m t_i = \sum_{i=1}^m b_i$. Divide by m to get $C + D\widehat{t} = \widehat{b}$.

12 (a) $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} = m$, $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b} = b_1 + \cdots + b_m$. Therefore $\widehat{\boldsymbol{x}}$ is the mean of the b's (b) $\boldsymbol{e} = \boldsymbol{b} - \widehat{\boldsymbol{x}}\boldsymbol{a}$.

$\|\boldsymbol{e}\|^2 = \sum_{i=1}^m (b_i - \widehat{\boldsymbol{x}})^2$ (c) $\boldsymbol{p} = (3, 3, 3)$, $\boldsymbol{e} = (-2, -1, 3)$, $\boldsymbol{p}^{\mathrm{T}}\boldsymbol{e} = 0$. $P = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

13 $(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}(\boldsymbol{b} - A\boldsymbol{x}) = \widehat{\boldsymbol{x}} - \boldsymbol{x}$. Errors $\boldsymbol{b} - A\boldsymbol{x} = (\pm 1, \pm 1, \pm 1)$ add to $\boldsymbol{0}$, so the $\widehat{\boldsymbol{x}} - \boldsymbol{x}$ add to $\boldsymbol{0}$.

14 $(\widehat{\boldsymbol{x}} - \boldsymbol{x})(\widehat{\boldsymbol{x}} - \boldsymbol{x})^{\mathrm{T}} = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}(\boldsymbol{b} - A\boldsymbol{x})(\boldsymbol{b} - A\boldsymbol{x})^{\mathrm{T}}A(A^{\mathrm{T}}A)^{-1}$. Average $(\boldsymbol{b} - A\boldsymbol{x})(\boldsymbol{b} - A\boldsymbol{x})^{\mathrm{T}} = \sigma^2 I$ gives the *covariance matrix* $(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\sigma^2 A(A^{\mathrm{T}}A)^{-1}$ which simplifies to $\sigma^2(A^{\mathrm{T}}A)^{-1}$.

15 Problem 14 gives the expected error $(\widehat{\boldsymbol{x}} - x)^2$ as $\sigma^2(A^{\mathrm{T}}A)^{-1} = \sigma^2/m$. By taking m measurements, the variance drops from σ^2 to σ^2/m.

16 $\frac{1}{10}b_{10} + \frac{\mathbf{9}}{\mathbf{10}}\widehat{\boldsymbol{x}}_9 = \frac{1}{10}(b_1 + \cdots + b_{10})$.

17 $\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix}$. The solution $\widehat{\boldsymbol{x}} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$ comes from $\begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 35 \\ 42 \end{bmatrix}$.

18 $\boldsymbol{p} = A\widehat{\boldsymbol{x}} = (5, 13, 17)$ gives the heights of the closest line. The error is $\boldsymbol{b} - \boldsymbol{p} = (2, -6, 4)$.

19 If $\boldsymbol{b} = \boldsymbol{e}$ then $\boldsymbol{b}$ is perpendicular to the column space of A. Projection $\boldsymbol{p} = \boldsymbol{0}$.

20 If $\boldsymbol{b} = A\widehat{\boldsymbol{x}} = (5, 13, 17)$ then error $\boldsymbol{e} = \boldsymbol{0}$ since $\boldsymbol{b}$ is *in the column space of* A.

21 $\boldsymbol{e}$ is in $\boldsymbol{N}(A^{\mathrm{T}})$; $\boldsymbol{p}$ is in $\boldsymbol{C}(A)$; $\widehat{\boldsymbol{x}}$ is in $\boldsymbol{C}(A^{\mathrm{T}})$; $\boldsymbol{N}(A) = \{\boldsymbol{0}\}$ = zero vector.

22 The least squares equation is $\begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$. Solution: $C = 1$, $D = -1$.

23 The square of the distance between points on two lines is $E = (y - x)^2 + (3y - x)^2 + (1 + x)^2$. Set $\frac{1}{2}\partial E/\partial x = -(y - x) - (3y - x) + (x + 1) = 0$ and $\frac{1}{2}\partial E/\partial y = (y - x) + 3(3y - x) = 0$. The solution is $x = -5/7, y = -2/7; E = 2/7$, and the minimal distance is $\sqrt{2/7}$.

24 $\boldsymbol{e}$ is orthogonal to $\boldsymbol{p}$; $\|\boldsymbol{e}\|^2 = \boldsymbol{e}^{\mathrm{T}}(\boldsymbol{b} - \boldsymbol{p}) = \boldsymbol{e}^{\mathrm{T}}\boldsymbol{b} = \boldsymbol{b}^{\mathrm{T}}\boldsymbol{b} - \boldsymbol{b}^{\mathrm{T}}\boldsymbol{p}$.

25 The derivatives of $\|A\boldsymbol{x} - \boldsymbol{b}\|^2$ are zero when $\boldsymbol{x} = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b}$.

26 Direct approach to 3 points on a line: *Equal slopes* $(b_2 - b_1)/(t_2 - t_1) = (b_3 - b_2)/(t_3 - t_2)$. Linear algebra approach: If $\boldsymbol{y}$ is orthogonal to the columns $(1, 1, 1)$ and (t_1, t_2, t_3) and $\boldsymbol{b}$ is in the column space then $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} = 0$. This $\boldsymbol{y} = (t_2 - t_3, t_3 - t_1, t_1 - t_2)$ is in the left nullspace. Then $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} = 0$ is the same equal slopes condition written as $(b_2 - b_1)(t_3 - t_2) = (b_3 - b_2)(t_2 - t_1)$.

27 $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix}$ has $A^{\mathrm{T}}A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $A^{\mathrm{T}}\boldsymbol{b} = \begin{bmatrix} 8 \\ -2 \\ -3 \end{bmatrix}$, $\begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -3/2 \end{bmatrix}$. At $x, y = 0, 0$ the best plane $2 - x - \frac{3}{2}y$ has height $C = 2$ which is the average of $0, 1, 3, 4$.

Problem Set 4.4, page 203

1 (a) *Independent* (b) *Independent* and *orthogonal* (c) *Independent* and *orthonormal.* For orthonormal, (a) becomes $(1, 0), (0, 1)$ and (b) is $(.6, .8), (.8, -.6)$.

2 $\boldsymbol{q}_1 = (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$. $\boldsymbol{q}_2 = (-\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$. $Q^{\mathrm{T}}Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ but $QQ^{\mathrm{T}} = \begin{bmatrix} 5/9 & 2/9 & -4/9 \\ 2/9 & 8/9 & 2/9 \\ -4/9 & 2/9 & 5/9 \end{bmatrix}$.

3 (a) $A^{\mathrm{T}}A = 16I$ (b) $A^{\mathrm{T}}A$ is diagonal with entries 1, 4, 9.

4 (a) $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $QQ^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (b) $(1, 0)$ and $(0, 0)$ are *orthogonal*, not *independent*
(c) $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$, $(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$, $(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$.

5 *Orthogonal* vectors are $(1, -1, 0)$ and $(1, 1, -1)$. *Orthonormal* are $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0), (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$.

6 If Q_1 and Q_2 are orthogonal matrices then $(Q_1Q_2)^{\mathrm{T}}Q_1Q_2 = Q_2^{\mathrm{T}}Q_1^{\mathrm{T}}Q_1Q_2 = Q_2^{\mathrm{T}}Q_2 = I$ which means that Q_1Q_2 is orthogonal also.

7 The least squares solution to $Q^{\mathrm{T}}Q\widehat{\boldsymbol{x}} = Q^{\mathrm{T}}\boldsymbol{b}$ is $\widehat{\boldsymbol{x}} = Q^{\mathrm{T}}\boldsymbol{b}$. This is $\mathbf{0}$ if $Q = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

8 If $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ are *orthonormal* vectors in $\mathbf{R}^5$ then $(\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{b})\boldsymbol{q}_1 + (\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{b})\boldsymbol{q}_2$ is closest to $\boldsymbol{b}$.

9 (a) $P = QQ^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (b) $(QQ^{\mathrm{T}})(QQ^{\mathrm{T}}) = Q(Q^{\mathrm{T}}Q)Q^{\mathrm{T}} = QQ^{\mathrm{T}}$.

10 (a) If $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\boldsymbol{q}_3$ are *orthonormal* then the dot product of $\boldsymbol{q}_1$ with $c_1\boldsymbol{q}_1 + c_2\boldsymbol{q}_2 + c_3\boldsymbol{q}_3 = \mathbf{0}$ gives $c_1 = 0$. Similarly $c_2 = c_3 = 0$ *independent* (b) $Q\boldsymbol{x} = \mathbf{0} \Rightarrow Q^{\mathrm{T}}Q\boldsymbol{x} = \mathbf{0} \Rightarrow \boldsymbol{x} = \mathbf{0}$.

11 (a) Two *orthonormal* vectors are $\frac{1}{10}(1, 3, 4, 5, 7)$ and $\frac{1}{10}(7, -3, -4, 5, -1)$ (b) The closest vector in the plane is the *projection* $QQ^{\mathrm{T}}(1, 0, 0, 0, 0) = (0.5, -0.18, -0.24, 0.4, 0)$.

12 (a) $\boldsymbol{a}_1^{\mathrm{T}}\boldsymbol{b} = \boldsymbol{a}_1^{\mathrm{T}}(x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + x_3\boldsymbol{a}_3) = x_1(\boldsymbol{a}_1^{\mathrm{T}}\boldsymbol{a}_1) = x_1$
(b) $\boldsymbol{a}_1^{\mathrm{T}}\boldsymbol{b} = \boldsymbol{a}_1^{\mathrm{T}}(x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + x_3\boldsymbol{a}_3) = x_1(\boldsymbol{a}_1^{\mathrm{T}}\boldsymbol{a}_1)$. Therefore $x_1 = \boldsymbol{a}_1^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}_1^{\mathrm{T}}\boldsymbol{a}_1$
(c) x_1 is the first component of A^{-1} times $\boldsymbol{b}$.

13 The multiple to subtract is $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$. Then $\boldsymbol{B} = \boldsymbol{b} - \frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}\boldsymbol{a} = (4, 0) - 2 \cdot (1, 1) = (2, -2)$.

14 $\begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{q}_1 & \boldsymbol{q}_2 \end{bmatrix} \begin{bmatrix} \|\boldsymbol{a}\| & \boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{b} \\ 0 & \|\boldsymbol{B}\| \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 2\sqrt{2} \\ 0 & 2\sqrt{2} \end{bmatrix} = QR.$

15 (a) $\boldsymbol{q}_1 = \frac{1}{3}(1,2,-2)$, $\boldsymbol{q}_2 = \frac{1}{3}(2,1,2)$, $\boldsymbol{q}_3 = \frac{1}{3}(2,-2,-1)$ (b) The nullspace of A^{T} contains $\boldsymbol{q}_3$ (c) $\widehat{\boldsymbol{x}} = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}(1,2,7) = (1,2)$.

16 The projection $\boldsymbol{p} = (\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a})\boldsymbol{a} = 14\boldsymbol{a}/49 = 2\boldsymbol{a}/7$ is closest to $\boldsymbol{b}$; $\boldsymbol{q}_1 = \boldsymbol{a}/\|\boldsymbol{a}\| = \boldsymbol{a}/7$ is $(4,5,2,2)/7$. $\boldsymbol{B} = \boldsymbol{b} - \boldsymbol{p} = (-1,4,-4,-4)/7$ has $\|B\| = 1$ so $\boldsymbol{q}_2 = \boldsymbol{B}$.

17 $\boldsymbol{p} = (\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a})\boldsymbol{a} = (3,3,3)$ and $\boldsymbol{e} = (-2,0,2)$. $\boldsymbol{q}_1 = (1,1,1)/\sqrt{3}$ and $\boldsymbol{q}_2 = (-1,0,1)/\sqrt{2}$.

18 $\boldsymbol{A} = \boldsymbol{a} = (1,-1,0,0)$; $\boldsymbol{B} = \boldsymbol{b} - \boldsymbol{p} = (\frac{1}{2},\frac{1}{2},-1,0)$; $\boldsymbol{C} = \boldsymbol{c} - \boldsymbol{p}_A - \boldsymbol{p}_B = (\frac{1}{3},\frac{1}{3},\frac{1}{3},-1)$. Notice the pattern in those orthogonal vectors $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$.

19 If $A = QR$ then $A^{\mathrm{T}}A = R^{\mathrm{T}}R =$ *lower* times *upper* triangular. Pivots of $A^{\mathrm{T}}A$ are 3 and 8.

20 (a) True (b) True. $Q\boldsymbol{x} = x_1\boldsymbol{q}_1 + x_2\boldsymbol{q}_2$. $\|Q\boldsymbol{x}\|^2 = x_1^2 + x_2^2$ because $\boldsymbol{q}_1 \cdot \boldsymbol{q}_2 = 0$.

21 The orthonormal vectors are $\boldsymbol{q}_1 = (1,1,1,1)/2$ and $\boldsymbol{q}_2 = (-5,-1,1,5)/\sqrt{52}$. Then $\boldsymbol{b} = (-4,-3,3,0)$ projects to $\boldsymbol{p} = (-7,-3,-1,3)/2$. Check that $\boldsymbol{b} - \boldsymbol{p} = (-1,-3,7,-3)/2$ is orthogonal to both $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$.

22 $A = (1,1,2)$, $B = (1,-1,0)$, $C = (-1,-1,1)$. Not yet orthonormal.

23 $\boldsymbol{q}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\boldsymbol{q}_2 = \begin{bmatrix}0\\0\\1\end{bmatrix}$, $\boldsymbol{q}_3 = \begin{bmatrix}0\\1\\0\end{bmatrix}$. $A = \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}1&2&4\\0&3&6\\0&0&5\end{bmatrix}$.

24 (a) One basis for this subspace is $\boldsymbol{v}_1 = (1,-1,0,0)$, $\boldsymbol{v}_2 = (1,0,-1,0)$, $\boldsymbol{v}_3 = (1,0,0,1)$ (b) $(1,1,1,-1)$ (c) $\boldsymbol{b}_2 = (\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2})$ and $\boldsymbol{b}_1 = (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2})$.

25 $\begin{bmatrix}2&1\\1&1\end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix}2&-1\\1&2\end{bmatrix} \cdot \frac{1}{\sqrt{5}}\begin{bmatrix}5&3\\0&1\end{bmatrix}$. Singular $\begin{bmatrix}1&1\\1&1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1&-1\\1&1\end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix}2&2\\0&0\end{bmatrix}$.
The Gram-Schmidt process breaks down when A is singular and $ad - bc = 0$.

26 $(\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{C}^*)\boldsymbol{q}_2 = \dfrac{\boldsymbol{B}^{\mathrm{T}}\boldsymbol{c}}{\boldsymbol{B}^{\mathrm{T}}\boldsymbol{B}}\boldsymbol{B}$ because $\boldsymbol{q}_2 = \dfrac{\boldsymbol{B}}{\|\boldsymbol{B}\|}$ and the extra $\boldsymbol{q}_1$ in $\boldsymbol{C}^*$ is orthogonal to $\boldsymbol{q}_2$.

27 When $\boldsymbol{a}$ and $\boldsymbol{b}$ are not orthogonal, the projections onto these lines do not add to the projection onto their plane.

28 $\boldsymbol{q}_1 = \frac{1}{3}(2,2,-1)$, $\boldsymbol{q}_2 = \frac{1}{3}(2,-1,2)$, $\boldsymbol{q}_3 = \frac{1}{3}(1,-2,-2)$.

29 There are mn multiplications in (11) and $\frac{1}{2}m^2n$ multiplications in each part of (12).

30 The columns of the wavelet matrix W are orthonormal. Then $W^{-1} = W^{\mathrm{T}}$. See Section 7.3 for more about wavelets.

31 (a) $c = \frac{1}{2}$ (b) Change all signs in rows 2, 3, 4; then in columns 2, 3, 4.

32 $\boldsymbol{p}_1 = \frac{1}{2}(-1,1,1,1)$ and $\boldsymbol{p}_2 = (0,0,1,1)$.

33 $Q_1 = \begin{bmatrix}1&0\\0&-1\end{bmatrix}$ reflects across x axis, $Q_2 = \begin{bmatrix}1&0&0\\0&0&-1\\0&-1&0\end{bmatrix}$ across plane $y + z = 0$.

34 (a) $Q\boldsymbol{u} = (I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})\boldsymbol{u} = \boldsymbol{u} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u}$. This is $-\boldsymbol{u}$, provided that $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u}$ equals 1
(b) $Q\boldsymbol{v} = (I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})\boldsymbol{v} = \boldsymbol{u} - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{v} = \boldsymbol{u}$, provided that $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{v} = 0$.

35 No solution

36 Orthogonal and lower triangular $\Rightarrow \pm 1$ on the main diagonal, 0 elsewhere.

Problem Set 5.1, page 213

1 $\det(2A) = 8$ and $\det(-A) = (-1)^4 \det A = \frac{1}{2}$ and $\det(A^2) = \frac{1}{4}$ and $\det(A^{-1}) = 2$.

2 $\det(\frac{1}{2}A) = (\frac{1}{2})^3 \det A = -\frac{1}{8}$ and $\det(-A) = (-1)^3 \det A = 1$; $\det(A^2) = 1$; $\det(A^{-1}) = -1$.

3 (a) False: 2 by 2 I (b) True (c) False: 2 by 2 I (d) False (but trace = 0).

4 Exchange rows 1 and 3. Exchange rows 1 and 4, then 2 and 3.

5 $|J_5| = 1$, $|J_6| = -1$, $|J_7| = -1$. The determinants are $1, 1, -1, -1$ repeating, so $|J_{101}| = 1$.

6 Multiply the zero row by t. The determinant is multiplied by t but the matrix is the same $\Rightarrow \det = 0$.

7 $\det(Q) = 1$ for rotation, $\det(Q) = -1$ for reflection $(1 - 2\sin^2\theta - 2\cos^2\theta = -1)$.

8 $Q^{\mathrm{T}}Q = I \Rightarrow |Q|^2 = 1 \Rightarrow |Q| = \pm 1$; Q^n stays orthogonal so can't blow up. Same for Q^{-1}.

9 $\det A = 1, \det B = 2, \det C = 0$.

10 If the entries in every row add to zero, then $(1, 1, \ldots, 1)$ is in the nullspace: singular A has $\det = 0$. (The columns add to the zero column so they are linearly dependent.) If every row adds to one, then rows of $A - I$ add to zero (not necessarily $\det A = 1$).

11 $CD = -DC \Rightarrow |CD| = (-1)^n |DC|$ and *not* $-|DC|$. If n is even we can have $|CD| \neq 0$.

12 $\det(A^{-1}) = \det \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \dfrac{ad-bc}{(ad-bc)^2} = \dfrac{1}{ad-bc}.$

13 Pivots $1, 1, 1$ give $\det = 1$; pivots $1, -2, -3/2$ give $\det = 3$.

14 $\det(A) = 24$ and $\det(A) = 5$.

15 $\det = 0$ and $\det = 1 - 2t^2 + t^4 = (1 - t^2)^2$.

16 A singular rank one matrix has $\det = 0$; Also $\det K = 0$.

17 Any 3 by 3 skew-symmetric K has $\det(K^{\mathrm{T}}) = \det(-K) = (-1)^3 \det(K)$. This is $-\det(K)$. But also $\det(K^{\mathrm{T}}) = \det(K)$, so we must have $\det(K) = 0$.

18 $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = (b-a)(c-a) \begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix} = (b-a)(c-a)(c-b).$

19 $\det(U) = 6$, $\det(U^{-1}) = \frac{1}{6}$, $\det(U^2) = 36$, $\det(U) = ad$, $\det(U^2) = a^2d^2$. If $ad \neq 0$ then $\det(U^{-1}) = 1/ad$.

20 $\det \begin{bmatrix} a - Lc & b - Ld \\ c - la & d - lb \end{bmatrix} = (ad - bc)(1 - Ll).$

21 Rules 5 and 3 give Rule 2. (Since Rules 4 and 3 give 5, they also give Rule 2.)

22 $\det(A) = 3$, $\det(A^{-1}) = \frac{1}{3}$, $\det(A - \lambda I) = \lambda^2 - 4\lambda + 3$. Then $\lambda = 1$ and $\lambda = 3$ give $\det(A - \lambda I) = 0$. *Note to instructor*: If you discuss this exercise, you can explain that this is the reason determinants come before eigenvalues. Identify 1 and 3 as the eigenvalues.

23 $\det(A) = 10$, $A^2 = \begin{bmatrix} 18 & 7 \\ 14 & 11 \end{bmatrix}$, $\det(A^2) = 100$, $A^{-1} = \frac{1}{10}\begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix}$, $\det(A^{-1}) = \frac{1}{10}$. $\det(A - \lambda I) = \lambda^2 - 7\lambda + 10 = 0$ when $\lambda = 2$ or $\lambda = 5$.

24 $\det(L) = 1$, $\det(U) = -6$, $\det(A) = -6$, $\det(U^{-1}L^{-1}) = -\frac{1}{6}$, and $\det(U^{-1}L^{-1}A) = 1$.

25 Row 2 = 2 times row 1 so $\det A = 0$.

26 Row 3 − row 2 = row 2 − row 1 so A is singular.

27 $\det A = abc$, $\det B = -abcd$, $\det C = a(b-a)(c-b)$.

28 (a) *True*: $\det(AB) = \det(A)\det(B) = 0$ (b) *False*: may exchange rows
(c) *False*: $A = 2I$ and $B = I$ (d) *True*: $\det(AB) = \det(A)\det(B) = \det(BA)$.

29 A is rectangular so $\det(A^{\mathrm{T}}A) \neq (\det A^{\mathrm{T}})(\det A)$: these are not defined.

30 $\begin{bmatrix} \partial f/\partial a & \partial f/\partial c \\ \partial f/\partial b & \partial f/\partial d \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = A^{-1}$.

31 The Hilbert determinants are $1, .08, 4.6\times10^{-4}, 1.6\times10^{-7}, 3.7\times10^{-12}, 5.4\times10^{-18}, 4.8\times10^{-25}$, $2.7\times10^{-33}, 9.7\times10^{-43}, 2.2\times10^{-53}$. Pivots are ratios of determinants, 10th pivot is near 10^{-10}.

32 Typical determinants of rand(n) are $10^6, 10^{25}, 10^{79}, 10^{218}$ for $n = 50, 100, 200, 400$. Using randn(n) with normal bell-shaped probabilities these are $10^{31}, 10^{78}, 10^{186}$, Inf means $\geq 2^{1024}$. MATLAB computes $1.999999999999999 \times 2^{1023} \approx 1.8 \times 10^{308}$ but one more 9 gives Inf!

33
```
n=5; p=(n-1)^2; A0=ones(n); maxdet=0;     for k=0:2^p-1
B=rem(floor(k.*2.^(-p+1:0)),2); A=A0; A(2:n,2:n)=1-2*reshape(B,n-1,n-1);
if abs(det(A))>maxdet, maxdet=abs(det(A)); maxA=A;   end   end
```

Output: maxA =

$$\begin{matrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 \end{matrix}$$

maxdet = 48. The maxdet for $n = 1$ to 8 is $1, 2, 4, 16, 48, 160, 576, 4096$ (symmetry assumed for $n = 6, 7$); $n = 4, 8$ are orthogonal Hadamard.

Note that row 1 and column 1 are normalized to +1's. Subtracting row 1 from all other rows and dividing by −2 gives an equivalent problem for 0–1 matrices (cofactor of size $n-1$).

34 Reduce B to [row 3 : row 2; row 1]. Then $\det B = -6$.

Problem Set 5.2, page 225

1 $\det A = 1 + 18 + 12 - 9 - 4 - 6 = 12$, rows are independent; $\det B = 0$, rows are dependent; $\det C = -1$, independent rows.

2 $\det A = -2$, independent; $\det B = 0$, dependent; $\det C = (-2)(0)$, dependent.

3 Each of the 6 terms in $\det A$ is zero; the rank is at most 2; column 2 has no pivot.

4 (a) The last three rows must be dependent (b) In each of the 120 terms: Choices from the last 3 rows must use 3 columns; at least one choice will be zero.

5 $a_{11}a_{23}a_{32}a_{44}$ gives −1, $a_{14}a_{23}a_{32}a_{41}$ gives +1 so $\det A = 0$; $\det B = 2\cdot4\cdot4\cdot2 - 1\cdot4\cdot4\cdot1 = 48$.

6 Four zeros in a row guarantee det = 0; $A = I$ has 12 zeros.

7 (a) If $a_{11} = a_{22} = a_{33} = 0$ then 4 terms are sure zeros (b) 15 terms are certainly zero.

8 $5!/2 = 60$ permutation matrices have det = +1. Put row 5 of I at the top (4 exchanges).

9 Some term $a_{1\alpha}a_{2\beta}\cdots a_{n\omega}$ is not zero! Move rows 1, 2, . . ., n into rows α, β, . . ., ω. Then these nonzero a's will be on the main diagonal.

10 To get $+1$ for the even permutations the matrix needs an *even* number of -1's. For the odd P's the matrix needs an *odd* number of -1's. So six 1's and det $= 6$ are impossible: max(det) $= 4$.

11 $\det(I + P\text{even}) = 16$ or 4 or 0 (16 comes from $I + I$).

12 $C = \begin{bmatrix} 6 & -3 \\ -1 & 2 \end{bmatrix}$. $C = \begin{bmatrix} 0 & 42 & -35 \\ 0 & -21 & 14 \\ -3 & 6 & -3 \end{bmatrix}$. $\det B = 1(0) + 2(42) + 3(-35) = -21$.

13 $C = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ and $AC^{\mathrm{T}} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. Therefore $A^{-1} = \frac{1}{4}C^{\mathrm{T}}$.

14 $|B_4| = 2\det\begin{bmatrix} 1 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{bmatrix} + \det\begin{bmatrix} 1 & -1 & \\ -1 & 2 & \\ & -1 & -1 \end{bmatrix} = 2|B_3| - \det\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = 2|B_3| - |B_2|$.

15 (a) $C_1 = 0$, $C_2 = -1$, $C_3 = 0$, $C_4 = 1$ (b) $C_n = -C_{n-2}$ by cofactors of row 1 then cofactors of column 1. Therefore $C_{10} = -C_8 = C_6 = -C_4 = -1$.

16 Must choose 1's from column 2 then column 1, column 4 then column 3, and so on. Therefore n must be even to have $\det A_n \neq 0$. The number of row exchanges is $\frac{1}{2}n$ so $C_n = (-1)^{n/2}$.

17 The 1,1 cofactor is E_{n-1}. The 1,2 cofactor has a single 1 in its first column, with cofactor E_{n-2}. Signs give $E_n = E_{n-1} - E_{n-2}$. Then 1, 0, -1, -1, 0, 1 repeats by sixes; $E_{100} = -1$.

18 The 1,1 cofactor is F_{n-1}. The 1,2 cofactor has a 1 in column 1, with cofactor F_{n-2}. Multiply by $(-1)^{1+2}$ and also (-1) from the 1,2 entry to find $F_n = F_{n-1} + F_{n-2}$ (so Fibonacci).

19 $|B_n| = |A_n| - |A_{n-1}| = (n + 1) - n = 1$.

20 Since x, x^2, x^3 are all in the same row, they are never multiplied in $\det V_4$. The determinant is zero at $x = a$ or b or c, so $\det V$ has factors $(x - a)(x - b)(x - c)$. Multiply by the cofactor V_3. Any Vandermonde matrix $V_{ij} = (c_i)^{j-1}$ has $\det V =$ product of all $(c_l - c_k)$ for $l > k$.

21 $G_2 = -1$, $G_3 = 2$, $G_4 = -3$, and $G_n = (-1)^{n-1}(n - 1) =$ (product of the n eigenvalues!)

22 $S_1 = 3$, $S_2 = 8$, $S_3 = 21$. The rule looks like every second number in Fibonacci's sequence $\ldots 3, 5, 8, 13, 21, 34, 55, \ldots$ so the guess is $S_4 = 55$. Following the solution to Problem 32 with 3's instead of 2's confirms $S_4 = 81 + 1 - 9 - 9 - 9 = 55$.

23 The problem asks us to show that $F_{2n+2} = 3F_{2n} - F_{2n-2}$. Keep using the Fibonacci rule:

$$F_{2n+2} = F_{2n+1} + F_{2n} = F_{2n} + F_{2n-1} + F_{2n} = F_{2n} + (F_{2n} - F_{2n-2}) + F_{2n} = 3F_{2n} - F_{2n-2}.$$

24 Changing 3 to 2 in the corner reduces the determinant F_{2n+2} by 1 times the cofactor of that corner entry. This cofactor is the determinant of S_{n-1} (one size smaller) which is F_{2n}. Therefore changing 3 to 2 changes the determinant to $F_{2n+2} - F_{2n}$ which is F_{2n+1}.

25 (a) If we choose an entry from B we must choose an entry from the zero block; result zero. This leaves a pair of entries from A times a pair from D leading to $(\det A)(\det D)$

(b) and (c) Take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

26 (a) All L's have det $= 1$; $\det U_k = \det A_k = 2, 6, -6$ for $k = 1, 2, 3$ (b) Pivots $2, \frac{3}{2}, -\frac{1}{3}$.

27 Problem 25 gives $\det \begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} = 1$ and $\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = |A|$ times $|D - CA^{-1}B|$ which is $|AD - ACA^{-1}B|$. If $AC = CA$ this is $|AD - CAA^{-1}B| = \det(AD - CB)$.

28 If A is a row and B is a column then $\det M = \det AB =$ dot product of A and B. If A is a column and B is a row then AB has rank 1 and $\det M = \det AB = 0$ (unless $m = n = 1$).

29 (a) $\det A = a_{11}C_{11} + \cdots + a_{1n}C_{1n}$. The derivative with respect to a_{11} is the cofactor C_{11}.

30 Row 1 $-$ 2 row 2 $+$ row 3 $= 0$ so the matrix is singular.

31 There are five nonzero products, all 1's with a plus or minus sign. Here are the (row, column) numbers and the signs: $+(1,1)(2,2)(3,3)(4,4) + (1,2)(2,1)(3,4)(4,3) - (1,2)(2,1)(3,3)(4,4) - (1,1)(2,2)(3,4)(4,3) - (1,1)(2,3)(3,2)(4,4)$. Total $1 + 1 - 1 - 1 - 1 = -1$.

32 The 5 products in solution 31 change to $16 + 1 - 4 - 4 - 4$ since A has 2's and -1's:

$$(2)(2)(2)(2) + (-1)(-1)(-1)(-1) - (-1)(-1)(2)(2) - (2)(2)(-1)(-1) - (2)(-1)(-1)(2).$$

33 $\det P = -1$ because the cofactor of $P_{14} = 1$ in row one has sign $(-1)^{1+4}$. The big formula for $\det P$ has only one term $(1 \cdot 1 \cdot 1 \cdot 1)$ with minus sign because three exchanges take $4, 1, 2, 3$ into $1, 2, 3, 4$; $\det(P^2) = (\det P)(\det P) = +1$ so $\det \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is *not right.*

34 With $a_{11} = 1$, the $-1, 2, -1$ matrix has det $= 1$ and inverse $(A^{-1})_{ij} = n + 1 - \max(i, j)$.

35 With $a_{11} = 2$, the $-1, 2, -1$ matrix has det $= n+1$ and $(n+1)(A^{-1})_{ij} = i(n - j + 1)$ for $i \le j$ and symmetrically $(n+1)(A^{-1})_{ij} = j(n - i + 1)$ for $i \ge j$.

36 Subtracting 1 from the n, n entry subtracts its cofactor C_{nn} from the determinant. That cofactor is $C_{nn} = 1$ (smaller Pascal matrix). Subtracting 1 from 1 leaves 0.

Problem Set 5.3, page 240

1 (a) $\det A = 3$, $\det B_1 = -6$, $\det B_2 = 3$ so $x_1 = -6/3 = -2$ and $x_2 = 3/3 = 1$ (b) $|A| = 4$, $|B_1| = 3$, $|B_2| = -2$, $|B_3| = 1$. Therefore $x_1 = \frac{3}{4}$ and $x_2 = -\frac{1}{2}$ and $x_3 = \frac{1}{4}$.

2 (a) $y = -c/(ad - bc)$ (b) $y = (fg - id)/D$.

3 (a) $x_1 = 3/0$ and $x_2 = -2/0$: no solution (b) $x_1 = 0/0$ and $x_2 = 0/0$: *undetermined.*

4 (a) $x_1 = \det([\,\boldsymbol{b} \;\; \boldsymbol{a}_2 \;\; \boldsymbol{a}_3\,])/\det A$, if $\det A \neq 0$ (b) The determinant is linear in its first column so $x_1|\boldsymbol{a}_1 \; \boldsymbol{a}_2 \; \boldsymbol{a}_3| + x_2|\boldsymbol{a}_2 \; \boldsymbol{a}_2 \; \boldsymbol{a}_3| + x_3|\boldsymbol{a}_3 \; \boldsymbol{a}_2 \; \boldsymbol{a}_3|$. The last two determinants are zero.

5 If the first column in A is also the right side $\boldsymbol{b}$ then $\det A = \det B_1$. Both B_2 and B_3 are singular since a column is repeated. Therefore $x_1 = |B_1|/|A| = 1$ and $x_2 = x_3 = 0$.

6 (a) $\begin{bmatrix} 1 & -\frac{2}{3} & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & -\frac{4}{3} & 1 \end{bmatrix}$ (b) $\frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$. The inverse of a symmetric matrix is symmetric.

7 If all cofactors $= 0$ (even in 1 row or column) then $\det A = 0$ (no inverse). $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ has no zero cofactors but it is not invertible.

8 $C = \begin{bmatrix} 6 & -3 & 0 \\ 3 & 1 & -1 \\ -6 & 2 & 1 \end{bmatrix}$ and $AC^{\mathrm{T}} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Therefore $\det A = 3$. Cofactor of 100 is 0.

9 If we know the cofactors and $\det A = 1$ then $C^{\mathrm{T}} = A^{-1}$ and $\det A^{-1} = 1$. Now A is the inverse of A^{-1}, so A is the cofactor matrix for C.

10 Take the determinant of both sides. The left side gives $\det AC^{\mathrm{T}} = (\det A)(\det C)$ while the right side gives $(\det A)^n$. Divide by $\det A$ to reach $\det C = (\det A)^{n-1}$.

11 We find $\det A = (\det C)^{\frac{1}{n-1}}$ with $n = 4$. Then $\det A^{-1}$ is $1/\det A$. Construct A^{-1} using the cofactors. Invert to find A.

12 The cofactors of A are integers. Division by $\det A = \pm 1$ gives integer entries in A^{-1}.

13 Both $\det A$ and $\det A^{-1}$ are integers since the matrices contain only integers. But $\det A^{-1} = 1/\det A$ so $\det A = 1$ or -1.

14 $A = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$ has cofactor matrix $C = \begin{bmatrix} -1 & 2 & 1 \\ 3 & -6 & 2 \\ 1 & 3 & -1 \end{bmatrix}$ and $A^{-1} = \frac{1}{5}C^{\mathrm{T}}$.

15 (a) Cofactors $C_{21} = C_{31} = C_{32} = 0$ (b) $C_{12} = C_{21}, C_{31} = C_{13}, C_{32} = C_{23}$ make S^{-1} symmetric.

16 For $n = 5$ the matrix C contains *25* cofactors and each 4 by 4 cofactor contains *24* terms and each term needs *3* multiplications: total 1800 multiplications vs. 125 for Gauss-Jordan.

17 (a) Area $\begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} = 10$ (b) 5 (c) 5.

18 Volume $= \begin{vmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{vmatrix} = 20$. Area of faces = length of cross product $\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & 1 & 1 \\ 1 & 3 & 1 \end{vmatrix} = -2\boldsymbol{i} - 2\boldsymbol{j} + 8\boldsymbol{k} = 6\sqrt{2}$.

19 (a) Area $\frac{1}{2}\begin{vmatrix} 2 & 1 & 1 \\ 3 & 4 & 1 \\ 0 & 5 & 1 \end{vmatrix} = 5$ (b) 5 + new triangle area $\frac{1}{2}\begin{vmatrix} 2 & 1 & 1 \\ 0 & 5 & 1 \\ -1 & 0 & 1 \end{vmatrix} = 5 + 7 = 12$.

20 $\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} = 4 = \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix}$ because the transpose has the same determinant. See #23.

21 The edges of the hypercube have length $\sqrt{1+1+1+1} = 2$. The volume $\det H$ is $2^4 = 16$. ($H/2$ has orthonormal columns. Then $\det(H/2) = 1$ leads again to $\det H = 16$.)

22 The maximum volume is $L_1L_2L_3L_4$ reached when the four edges are orthogonal in $\mathbf{R}^4$. With entries 1 and -1 all lengths are $\sqrt{1+1+1+1} = 2$. The maximum determinant is $2^4 = 16$, achieved by Hadamard above. For a 3 by 3 matrix, $\det A = (\sqrt{3})^3$ can't be achieved.

23 A student (Dave Nelson) suggested a way to move in 3 steps from the parallelogram P with sides (a,b) and (c,d) to its "transpose" P' with sides (a,c) and (b,d). Each step slides one edge of a parallelogram along itself, with no change in area: a triangle is added at one end and lost at the other end. The origin stays fixed.

First slide the side from (c,d) to $(a,b)+(c,d)$ along to the y axis. The new corners will be $(0,e)$ and $(a,b)+(0,e)$. Then slide the vertical side that goes from (a,b) to $(a,b)+(0,e)$ until it goes from (a,c) to $(a,c)+(0,e)$. Finally slide the new side that goes from $(0,e)$ to $(0,e)+(a,c)$ along itself until it goes from (b,d) to $(b,d)+(a,c)$. This is now the transposed parallelogram P'.

Check an example with $(a,b)=(3,2)$ and $(c,d)=(1,4)$ and area 10. Then $e=10/3$ because with vertical sides we must have area $=e$ times a. The line from $(0,e)$ to $(a,c)+(0,e)$ in step 3 has the equation $y=e+cx/a$. Step 3 works because (b,d) is on that line!—$d=e+cb/a$ is true since $ae=$ area $=ad-bc$.

24 $A^{\mathrm{T}}A=\begin{bmatrix}\boldsymbol{a}^{\mathrm{T}}\\ \boldsymbol{b}^{\mathrm{T}}\\ \boldsymbol{c}^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c}\end{bmatrix}=\begin{bmatrix}\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} & 0 & 0\\ 0 & \boldsymbol{b}^{\mathrm{T}}\boldsymbol{b} & 0\\ 0 & 0 & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{c}\end{bmatrix}$ has $\begin{array}{lcl}\det A^{\mathrm{T}}A & = & (\|a\|\|b\|\|c\|)^2\\ \det A & = & \pm\|a\|\|b\|\|c\|\end{array}$.

25 The box has height 4. The volume is $4=\det\begin{bmatrix}1&0&0\\0&1&0\\2&3&4\end{bmatrix}$; $\boldsymbol{i}\times\boldsymbol{j}=\boldsymbol{k}$ and $(\boldsymbol{k}\cdot\boldsymbol{w})=4$.

26 The n-dimensional cube has 2^n corners, $n2^{n-1}$ edges and $2n$ $(n-1)$-dimensional faces. Coefficients from $(2+x)^n$ in Worked Example **2.4 A**. The cube whose edges are the rows of $2I$ has volume 2^n.

27 The pyramid has volume $\frac{1}{6}$. The 4-dimensional pyramid has volume $\frac{1}{24}$.

28 $J=r$. The columns are orthogonal and their lengths are 1 and r.

29 $J=\begin{vmatrix}\sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta\\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta\\ \cos\varphi & -\rho\sin\varphi & 0\end{vmatrix}=\rho^2\sin\varphi$, needed for triple integrals inside spheres.

30 $\begin{vmatrix}\partial r/\partial x & \partial r/\partial y\\ \partial\theta/\partial x & \partial\theta/\partial y\end{vmatrix}=\begin{vmatrix}\cos\theta & \sin\theta\\ -\frac{1}{r}\sin\theta & \frac{1}{r}\cos\theta\end{vmatrix}=\frac{1}{r}$.

31 The triangle with corners $(0,0)$, $(6,0)$, $(1,4)$ has area 24. Rotated by $\theta=60^0$ the area is *unchanged*. The determinant of the rotation matrix is $J=\begin{vmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{vmatrix}=\begin{vmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{vmatrix}=1$.

32 Base area 10, height 2, volume 20.

33 $V=\det\begin{bmatrix}2&4&0\\-1&3&0\\1&2&2\end{bmatrix}=20$.

34 $\begin{bmatrix}u_1&u_2&u_3\\v_1&v_2&v_3\\w_1&w_2&w_3\end{bmatrix}=u_1\begin{bmatrix}v_2&v_3\\w_2&w_3\end{bmatrix}-u_2\begin{bmatrix}v_1&v_3\\w_1&w_3\end{bmatrix}+u_3\begin{bmatrix}v_1&v_2\\w_1&w_2\end{bmatrix}=\boldsymbol{u}\cdot(\boldsymbol{v}\times\boldsymbol{w})$.

35 $(\boldsymbol{w}\times\boldsymbol{u})\cdot\boldsymbol{v}=(\boldsymbol{v}\times\boldsymbol{w})\cdot\boldsymbol{u}=(\boldsymbol{u}\times\boldsymbol{v})\cdot\boldsymbol{w}$: *Cyclic* = *even* permutation of $(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})$.

36 $S=(2,1,-1)$. The area is $\|PQ\times PS\|=\|(-2,-2,-1)\|=3$. The other four corners could be $(0,0,0)$, $(0,0,2)$, $(1,2,2)$, $(1,1,0)$. The volume of the tilted box is $|det|=1$.

37 If $(1,1,0)$, $(1,2,1)$, (x,y,z) are in a plane the volume is $\det\begin{bmatrix} x & y & z \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} = x - y + z = 0$.

38 $\det\begin{bmatrix} x & y & z \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} = 0 = 7x - 5y + z$; plane contains the two vectors.

39 (a) Doubling each row multiplies the volume by 2^n (b) From (a) it follows that $2\det A = \det(2A)$ only if $n = 1$.

Problem Set 6.1, page 253

1 A and A^2 and A^∞ all have the same eigenvectors. The eigenvalues are 1 and 0.5 for A, 1 and 0.25 for A^2, 1 and 0 for A^∞. Therefore A^2 is halfway between A and A^∞.

Exchanging the rows of A changes the eigenvalues to 1 and -0.5 (it is still a Markov matrix with eigenvalue 1, and the trace is now $0.2 + 0.3$—so the other eigenvalue is -0.5).

Singular matrices stay singular during elimination, so $\lambda = 0$ does not change.

2 $\lambda_1 = -1$ and $\lambda_2 = 5$ with eigenvectors $\boldsymbol{x}_1 = (-2,1)$ and $\boldsymbol{x}_2 = (1,1)$. The matrix $A + I$ has the same eigenvectors, with eigenvalues increased by 1 to 0 and 6.

3 A has $\lambda_1 = 4$ and $\lambda_2 = -1$ (check trace and determinant) with $\boldsymbol{x}_1 = (1,2)$ and $\boldsymbol{x}_2 = (2,-1)$. A^{-1} has the same eigenvectors as A, with eigenvalues $1/\lambda_1 = 1/4$ and $1/\lambda_2 = -1$.

4 A has $\lambda_1 = -3$ and $\lambda_2 = 2$ (check trace and determinant) with $\boldsymbol{x}_1 = (3,-2)$ and $\boldsymbol{x}_2 = (1,1)$. A^2 has the same eigenvectors as A, with eigenvalues $\lambda_1^2 = 9$ and $\lambda_2^2 = 4$.

5 A and B have $\lambda_1 = 1$ and $\lambda_2 = 1$. $A + B$ has $\lambda_1 = 1$, $\lambda_2 = 3$. Eigenvalues of $A + B$ *are not equal* to eigenvalues of A plus eigenvalues of B.

6 A and B have $\lambda_1 = 1$ and $\lambda_2 = 1$. AB and BA have $\lambda = \frac{1}{2}(3 \pm \sqrt{5})$. Eigenvalues of AB *are not equal* to eigenvalues of A times eigenvalues of B. Eigenvalues of AB and BA ***are equal.***

7 The eigenvalues of U are the *pivots*. The eigenvalues of L are all 1's. The eigenvalues of A are not the same as the pivots.

8 (a) Multiply $A\boldsymbol{x}$ to see $\lambda\boldsymbol{x}$ which reveals λ (b) Solve $(A - \lambda I)\boldsymbol{x} = \boldsymbol{0}$ to find $\boldsymbol{x}$.

9 (a) Multiply by A: $A(A\boldsymbol{x}) = A(\lambda\boldsymbol{x}) = \lambda A\boldsymbol{x}$ gives $A^2\boldsymbol{x} = \lambda^2\boldsymbol{x}$ (b) Multiply by A^{-1}: $A^{-1}A\boldsymbol{x} = A^{-1}\lambda\boldsymbol{x} = \lambda A^{-1}\boldsymbol{x}$ gives $A^{-1}\boldsymbol{x} = \frac{1}{\lambda}\boldsymbol{x}$ (c) Add $I\boldsymbol{x} = \boldsymbol{x}$: $(A + I)\boldsymbol{x} = (\lambda + 1)\boldsymbol{x}$.

10 A has $\lambda_1 = 1$ and $\lambda_2 = .4$ with $\boldsymbol{x}_1 = (1,2)$ and $\boldsymbol{x}_2 = (1,-1)$. A^∞ has $\lambda_1 = 1$ and $\lambda_2 = 0$ (same eigenvectors). A^{100} has $\lambda_1 = 1$ and $\lambda_2 = (.4)^{100}$ which is near zero. So A^{100} is very near A^∞.

11 $M = (A - \lambda_2 I)(A - \lambda_1 I) =$ zero matrix so the columns of $A - \lambda_1 I$ are in the nullspace of $A - \lambda_2 I$. This "Cayley-Hamilton Theorem" $M = 0$ in Problem 6.2.35 has a short proof: by Problem 9, M has eigenvalues $(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_1) = 0$ and $(\lambda_2 - \lambda_2)(\lambda_2 - \lambda_1) = 0$. Same $\boldsymbol{x}_1, \boldsymbol{x}_2$.

12 P has $\lambda = 1,\ 0,\ 1$ with eigenvectors $(1,2,0)$, $(2,-1,0)$, $(0,0,1)$. Add the first and last vectors: $(1,2,1)$ also has $\lambda = 1$. $P^{100} = P$ so P^{100} gives the same answers.

13 (a) $P\boldsymbol{u} = (\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u}) = \boldsymbol{u}$ so $\lambda = 1$ (b) $P\boldsymbol{v} = (\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})\boldsymbol{v} = \boldsymbol{u}(\boldsymbol{u}^{\mathrm{T}}\boldsymbol{v}) = \mathbf{0}$ so $\lambda = 0$
(c) $\boldsymbol{x}_1 = (-1,1,0,0)$, $\boldsymbol{x}_2 = (-3,0,1,0)$, $\boldsymbol{x}_3 = (-5,0,0,1)$ are eigenvectors with $\lambda = 0$.

14 The eigenvectors are $\boldsymbol{x}_1 = (1,i)$ and $\boldsymbol{x}_2 = (1,-i)$.

15 $\lambda = \frac{1}{2}(-1 \pm i\sqrt{3})$; the three eigenvalues are $1, 1, -1$.

16 Set $\lambda = 0$ to find $\det A = (\lambda_1)(\lambda_2)\cdots(\lambda_n)$.

17 If A has $\lambda_1 = 3$ and $\lambda_2 = 4$ then $\det(A - \lambda I) = (\lambda - 3)(\lambda - 4) = \lambda^2 - 7\lambda + 12$. Always $\lambda_1 = \frac{1}{2}(a + d + \sqrt{(a-d)^2 + 4bc})$ and $\lambda_2 = \frac{1}{2}(a + d - \sqrt{})$. Their sum is $a + d$.

18 $\begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}$, $\begin{bmatrix} 3 & 2 \\ -1 & 6 \end{bmatrix}$, $\begin{bmatrix} 2 & 2 \\ -3 & 7 \end{bmatrix}$.

19 (a) rank $= 2$ (b) $\det(B^{\mathrm{T}}B) = 0$ (d) eigenvalues of $(B+I)^{-1}$ are $1, \frac{1}{2}, \frac{1}{3}$.

20 $A = \begin{bmatrix} 0 & 1 \\ -28 & 11 \end{bmatrix}$ has trace 11 and determinant 28.

21 $a = 0$, $b = 9$, $c = 0$ multiply $1, \lambda, \lambda^2$ in $\det(A - \lambda I) = 9\lambda - \lambda^3$: $A =$ *companion matrix.*

22 $(A - \lambda I)$ has the same determinant as $(A - \lambda I)^{\mathrm{T}}$. $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$: different eigenvectors.

23 $\lambda = 1$ (for Markov), 0 (for singular), $-\frac{1}{2}$ (so sum of eigenvalues $=$ trace $= \frac{1}{2}$).

24 $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$. Always $A^2 =$ zero matrix if $\lambda = 0, 0$ (Cayley-Hamilton 6.2.35).

25 $\lambda = 0,\ 0,\ 6$ with $\boldsymbol{x}_1 = (0,-2,1)$, $\boldsymbol{x}_2 = (1,-2,0)$, $\boldsymbol{x}_3 = (1,2,1)$.

26 $A\boldsymbol{x} = c_1\lambda_1\boldsymbol{x}_1 + \cdots + c_n\lambda_n\boldsymbol{x}_n$ equals $B\boldsymbol{x} = c_1\lambda_1\boldsymbol{x}_1 + \cdots + c_n\lambda_n\boldsymbol{x}_n$ for all $\boldsymbol{x}$. So $A = B$.

27 $\lambda = 1,\ 2,\ 5,\ 7$.

28 $\operatorname{rank}(A) = 1$ with $\lambda = 0,\ 0,\ 0,\ 4$; $\operatorname{rank}(C) = 2$ with $\lambda = 0,\ 0,\ 2,\ 2$.

29 B has $\lambda = -1,\ -1,\ -1,\ 3$ so $\det B = -3$. The 5 by 5 matrix A has $\lambda = 0,\ 0,\ 0,\ 0,\ 5$ and $B = A - I$ has $\lambda = -1,\ -1,\ -1,\ -1,\ 4$.

30 $\lambda(A) = 1,\ 4,\ 6$; $\lambda(B) = 2,\ \sqrt{3},\ -\sqrt{3}$; $\lambda(C) = 0,\ 0,\ 6$.

31 $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+b \\ c+d \end{bmatrix} = (a+b)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $\lambda_2 = d - b$ to produce trace $= a + d$.

32 Eigenvector $(1,3,4)$ for A with $\lambda = 11$ and eigenvector $(3,1,4)$ for PAP.

33 (a) $\boldsymbol{u}$ is a basis for the nullspace, $\boldsymbol{v}$ and $\boldsymbol{w}$ give a basis for the column space
(b) $\boldsymbol{x} = (0, \frac{1}{3}, \frac{1}{5})$ is a particular solution. Add any $c\boldsymbol{u}$ from the nullspace
(c) If $A\boldsymbol{x} = \boldsymbol{u}$ had a solution, $\boldsymbol{u}$ would be in the column space, giving dimension 3.

34 With $\lambda_1 = e^{2\pi i/3}$ and $\lambda_2 = e^{-2\pi i/3}$, the determinant is $\lambda_1\lambda_2 = 1$ and the trace is $\lambda_1 + \lambda_2 = -1$:

$$e^{2\pi i/3} + e^{-2\pi i/3} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} + \cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3} = -1. \text{ Also } \lambda_1^3 = \lambda_2^3 = 1.$$

$A = \left[\begin{smallmatrix} -1 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ has this trace -1 and determinant 1. Then $A^3 = I$ and every $(M^{-1}AM)^3 = I$. Choosing $\lambda_1 = \lambda_2 = 1$ leads to I or else to a matrix like $A = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ that has $A^3 \neq I$.

35 $\det(P-\lambda I)=0$ gives the equation $\lambda^3=1$. This reflects the fact that $P^3=I$. The solutions of $\lambda^3=1$ are $\lambda=1$ (real) and $\lambda=e^{2\pi i/3}, \lambda=e^{-2\pi i/3}$ (complex conjugates). The real eigenvector $\boldsymbol{x}_1=(1,1,1)$ is not changed by the permutation P. The complex eigenvectors are $\boldsymbol{x}_2=(1,e^{-2\pi i/3},e^{-4\pi i/3})$ and $\boldsymbol{x}_3=(1,e^{2\pi i/3},e^{4\pi i/3})=\overline{\boldsymbol{x}}_2$.

36 For 3 by 3 permutations: determinant $=1$ or -1, all pivots $=1$, trace $=0,1$ or 3, eigenvalues $=1$ or -1 or $e^{2\pi i/3}$ or $e^{4\pi i/3}$ (from the previous problem).

Problem Set 6.2, page 266

1 $\begin{bmatrix}1&2\\0&3\end{bmatrix}=\begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}1&0\\0&3\end{bmatrix}\begin{bmatrix}1&-1\\0&1\end{bmatrix};\quad \begin{bmatrix}1&1\\2&2\end{bmatrix}=\begin{bmatrix}1&1\\-1&2\end{bmatrix}\begin{bmatrix}0&0\\0&3\end{bmatrix}\begin{bmatrix}\frac{2}{3}&-\frac{1}{3}\\\frac{1}{3}&\frac{1}{3}\end{bmatrix}.$

2 If $A=S\Lambda S^{-1}$ then $A^3=S\Lambda^3S^{-1}$ and $A^{-1}=S\Lambda^{-1}S^{-1}$.

3 $A=\begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}2&0\\0&5\end{bmatrix}\begin{bmatrix}1&-1\\0&1\end{bmatrix}=\begin{bmatrix}2&3\\0&5\end{bmatrix}.$

4 If $A=S\Lambda S^{-1}$ then the eigenvalue matrix for $A+2I$ is $\Lambda+2I$ and the eigenvector matrix is still S. $A+2I=S(\Lambda+2I)S^{-1}=S\Lambda S^{-1}+S(2I)S^{-1}=A+2I$.

5 (a) False: don't know λ's (b) True (c) True (d) False: need eigenvectors of S!.

6 A is a diagonal matrix. If S is triangular, then S^{-1} is triangular, so $S\Lambda S^{-1}$ is also triangular.

7 The columns of S are nonzero multiples of $(2,1)$ and $(0,1)$ in either order. Same for A^{-1}.

8 $\begin{bmatrix}a&b\\b&a\end{bmatrix}$ for any a and b.

9 $A^2=\begin{bmatrix}2&1\\1&1\end{bmatrix},\ A^3=\begin{bmatrix}3&2\\2&1\end{bmatrix},\ A^4=\begin{bmatrix}5&3\\3&2\end{bmatrix};\ F_{20}=6765.$

10 (a) $A=\begin{bmatrix}.5&.5\\1&0\end{bmatrix}$ has $\lambda_1=1,\ \lambda_2=-\frac{1}{2}$ with $\boldsymbol{x}_1=(1,1),\ \boldsymbol{x}_2=(1,-2)$

(b) $A^n=\begin{bmatrix}1&1\\1&-2\end{bmatrix}\begin{bmatrix}1^n&0\\0&(-.5)^n\end{bmatrix}\begin{bmatrix}\frac{2}{3}&\frac{1}{3}\\\frac{1}{3}&-\frac{1}{3}\end{bmatrix}\to A^\infty=\begin{bmatrix}\frac{2}{3}&\frac{1}{3}\\\frac{2}{3}&\frac{1}{3}\end{bmatrix}$

(c) $\begin{bmatrix}G_{k+1}\\G_k\end{bmatrix}=A^k\begin{bmatrix}G_1\\G_0\end{bmatrix}\to\begin{bmatrix}\frac{2}{3}&\frac{1}{3}\\\frac{2}{3}&\frac{1}{3}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}\frac{2}{3}\\\frac{2}{3}\end{bmatrix}.$

11 $A=S\Lambda S^{-1}=\begin{bmatrix}1&1\\1&0\end{bmatrix}=\frac{1}{\lambda_1-\lambda_2}\begin{bmatrix}\lambda_1&\lambda_2\\1&1\end{bmatrix}\begin{bmatrix}\lambda_1&0\\0&\lambda_2\end{bmatrix}\begin{bmatrix}1&-\lambda_2\\-1&\lambda_1\end{bmatrix}.$

$S\Lambda^kS^{-1}=\frac{1}{\lambda_1-\lambda_2}\begin{bmatrix}\lambda_1&\lambda_2\\1&1\end{bmatrix}\begin{bmatrix}\lambda_1^k&0\\0&\lambda_2^k\end{bmatrix}\begin{bmatrix}1&-\lambda_2\\-1&\lambda_1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}-\\(\lambda_1^k-\lambda_2^k)/(\lambda_1-\lambda_2)\end{bmatrix}.$

12 The equation for the λ's is $\lambda^2-\lambda-1=0$ or $\lambda^2=\lambda+1$. Multiply by λ^k to get $\lambda^{k+2}=\lambda^{k+1}+\lambda^k$.

13 Direct computation gives $L_0,\ldots,L_{10}$ as $2,1,3,4,7,11,18,29,47,76,123$. My calculator gives $\lambda_1^{10}=(1.618\ldots)^{10}=122.991\ldots$.

14 The rule $F_{k+2} = F_{k+1} + F_k$ produces the pattern: even, odd, odd, even, odd, odd, . . .

15 (a) True (b) False (c) False (might have 2 or 3 independent eigenvectors).

16 (a) False: don't know λ (b) True: missing an eigenvector (c) True.

17 $A = \begin{bmatrix} 8 & 3 \\ -3 & 2 \end{bmatrix}$ (or other), $A = \begin{bmatrix} 9 & 4 \\ -4 & 1 \end{bmatrix}$, $A = \begin{bmatrix} 10 & 5 \\ -5 & 0 \end{bmatrix}$; only eigenvectors are $(c, -c)$.

18 The rank of $A - 3I$ is one. Changing any entry except $a_{12} = 1$ makes A diagonalizable.

19 $S\Lambda^k S^{-1}$ approaches zero if and only if every $|\lambda| < 1$; $B^k \to 0$.

20 $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & .2 \end{bmatrix}$ and $S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$; $\Lambda^k \to \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $S\Lambda^k S^{-1} \to \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$: *steady state.*

21 $\Lambda = \begin{bmatrix} .9 & 0 \\ 0 & .3 \end{bmatrix}$, $S = \begin{bmatrix} 3 & -3 \\ 1 & 1 \end{bmatrix}$; $B^{10} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = (.9)^{10} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $B^{10} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = (.3)^{10} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, $B^{10} \begin{bmatrix} 6 \\ 0 \end{bmatrix} =$ sum of those two.

22 $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ and $A^k = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3^k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

23 $B^k = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}^k \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 3^k & 3^k - 2^k \\ 0 & 2^k \end{bmatrix}$.

24 $\det A = (\det S)(\det \Lambda)(\det S^{-1}) = \det \Lambda = \lambda_1 \cdots \lambda_n$. This works when A is *diagonalizable.*

25 $\operatorname{trace} AB = (aq + bs) + (cr + dt) = (qa + rc) + (sb + td) = \operatorname{trace} BA$. Proof for diagonalizable case: the trace of $S\Lambda S^{-1}$ is the trace of $(\Lambda S^{-1})S = \Lambda$ which is *the sum of* the λ's.

26 $AB - BA = I$: impossible since $\operatorname{trace} AB - \operatorname{trace} BA = \textit{zero} \neq \operatorname{trace} I$. $E = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

27 If $A = S\Lambda S^{-1}$ then $B = \begin{bmatrix} A & 0 \\ 0 & 2A \end{bmatrix} = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & 2\Lambda \end{bmatrix} \begin{bmatrix} S^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix}$.

28 The A's form a subspace since cA and $A_1 + A_2$ have the same S. When $S = I$ the A's give the subspace of diagonal matrices. Dimension 4.

29 If A has columns $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ then $A^2 = A$ means every $A\boldsymbol{x}_i = \boldsymbol{x}_i$. All vectors in the column space are eigenvectors with $\lambda = 1$. Always the nullspace has $\lambda = 0$. Dimensions of those spaces add to n by the Fundamental Theorem so A is diagonalizable (n independent eigenvectors).

30 Two problems: The nullspace and column space can overlap, so $\boldsymbol{x}$ could be in both. There may not be r independent eigenvectors in the column space.

31 $R = S\sqrt{\Lambda}S^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ has $R^2 = A$. $\sqrt{B}$ would have $\lambda = \sqrt{9}$ and $\lambda = \sqrt{-1}$ so its trace is not real. Note $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ can have $\sqrt{-1} = i$ and $-i$, and real square root $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

32 $A^{\mathrm{T}} = A$ gives $\boldsymbol{x}^{\mathrm{T}} AB\boldsymbol{x} = (A\boldsymbol{x})^{\mathrm{T}}(B\boldsymbol{x}) \leq \|A\boldsymbol{x}\|\|B\boldsymbol{x}\|$ by the Schwarz inequality. $B^{\mathrm{T}} = -B$ gives $-\boldsymbol{x}^{\mathrm{T}} BA\boldsymbol{x} = (B\boldsymbol{x})^{\mathrm{T}} A\boldsymbol{x} \leq \|A\boldsymbol{x}\|\|B\boldsymbol{x}\|$. Add these to get Heisenberg when $AB - BA = I$.

33 The factorizations of A and B into $S\Lambda S^{-1}$ are the same. So $A = B$.

34 $A = S\Lambda_1 S^{-1}$ and $B = S\Lambda_2 S^{-1}$. Diagonal matrices always give $\Lambda_1\Lambda_2 = \Lambda_2\Lambda_1$. Then $AB = BA$ from $S\Lambda_1 S^{-1} S\Lambda_2 S^{-1} = S\Lambda_1\Lambda_2 S^{-1} = S\Lambda_2\Lambda_1 S^{-1} = S\Lambda_2 S^{-1} S\Lambda_1 S^{-1} = BA$.

35 If $A = S\Lambda S^{-1}$ then the product $(A - \lambda_1 I)\cdots(A - \lambda_n I)$ equals $S(\Lambda - \lambda_1 I)\cdots(\Lambda - \lambda_n I)S^{-1}$. The factor $\Lambda - \lambda_j I$ is zero in row j. *The product is zero in all rows* = zero matrix.

36 $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ has $A^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $A^2 - A - I$ = zero matrix confirms Cayley-Hamilton.

37 $(A - aI)(A - dI) = \begin{bmatrix} 0 & b \\ 0 & d-a \end{bmatrix}\begin{bmatrix} a-d & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

38 (a) The eigenvectors for $\lambda = 0$ always span the nullspace (b) The eigenvectors for $\lambda \neq 0$ span the column space if there are r independent eigenvectors: then algebraic multiplicity = geometric multiplicity for each nonzero λ.

39 The eigenvalues $2, -1, 0$ and their eigenvectors are in Λ and S. Then $A^k = S\Lambda^k S^{-1}$ is

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 2^k & & \\ & (-1)^k & \\ & & 0^k \end{bmatrix}\frac{1}{6}\begin{bmatrix} 4 & 1 & 1 \\ 2 & -2 & -2 \\ 0 & 3 & -3 \end{bmatrix} = \frac{2^k}{6}\begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} + \frac{(-1)^k}{3}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

Check $k = 1$! The $(2, 2)$ entry of A^4 is $2^4/6 + (-1)^4/3 = 18/6 = 3$. The 4-step paths that begin and end at node 2 are 2 to 1 to 1 to 1 to 2, 2 to 1 to 2 to 1 to 2, and 2 to 1 to 3 to 1 to 2. Harder to find the eleven 4-step paths that start and end at node 1.

Notice the column times row multiplication above. Since $A = A^{\mathrm{T}}$ the eigenvectors in the columns of S are orthogonal. They are in the rows of S^{-1} divided by their length squared.

40 B has the same eigenvectors $(1, 0)$ and $(0, 1)$ as A, so B is also diagonal. The 4 equations $AB - BA = \begin{bmatrix} a & b \\ 2c & 2d \end{bmatrix} - \begin{bmatrix} a & 2b \\ c & 2d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ have coefficient matrix with rank 2.

41 $AB = BA$ always has the solution $B = A$. (In case $A = 0$ every B is a solution.)

42 B has $\lambda = i$ and $-i$, so B^4 has $\lambda^4 = 1$ and 1; C has $\lambda = (1 \pm \sqrt{3}i)/2 = \exp(\pm\pi i/3)$ so $\lambda^3 = -1$ and -1. Then $C^3 = -I$ and $C^{1024} = -C$.

Problem Set 6.3, page 279

1 $\boldsymbol{u}_1 = e^{4t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\boldsymbol{u}_2 = e^t\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. If $\boldsymbol{u}(0) = (5, -2)$, then $\boldsymbol{u}(t) = 3e^{4t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2e^t\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

2 $z(t) = -2e^t$; then $dy/dt = 4y - 6e^t$ with $y(0) = 5$ gives $y(t) = 3e^{4t} + 2e^t$ as in Problem 1.

3 $\begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix}$. Then $\lambda = \frac{1}{2}(5 \pm \sqrt{41})$.

4 $\begin{bmatrix} 6 & -2 \\ 2 & 1 \end{bmatrix}$ has $\lambda_1 = 5$, $\boldsymbol{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\lambda_2 = 2$, $\boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$; rabbits $r(t) = 20e^{5t} + 10e^{2t}$, $w(t) = 10e^{5t} + 20e^{2t}$. The ratio of rabbits to wolves approaches 20/10; e^{5t} dominates.

5 $d(v+w)/dt = dv/dt + dw/dt = (w-v)+(v-w) = 0$, so the total $v+w$ is constant. $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ has $\lambda_1 = 0$ and $\lambda_2 = -2$ with $\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$; $\begin{matrix} v(1) = 20 + 10e^{-2} \\ w(1) = 20 - 10e^{-2} \end{matrix}$.

6 $\lambda_1 = 0$ and $\lambda_2 = 2$. Now $v(t) = 20 + 10e^{2t} \to \infty$ as $t \to \infty$.

7 $e^{At} = I + t\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \text{zeros} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$.

8 $A = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix}$ has trace 6, det 9, $\lambda = 3$ and 3 with only one independent eigenvector $(1,3)$.

9 $my'' + by' + ky = 0$ is $\begin{bmatrix} m & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y' \\ y \end{bmatrix}' = \begin{bmatrix} -b & -k \\ 1 & 0 \end{bmatrix}\begin{bmatrix} y' \\ y \end{bmatrix}$.

10 When A is skew-symmetric, $\|\boldsymbol{u}(t)\| = \|e^{At}\boldsymbol{u}(0)\| = \|\boldsymbol{u}(0)\|$. So e^{At} is an *orthogonal* matrix.

11 (a) $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ i \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ -i \end{bmatrix}$. Then $\boldsymbol{u}(t) = \frac{1}{2}e^{it}\begin{bmatrix} 1 \\ i \end{bmatrix} + \frac{1}{2}e^{-it}\begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$.

12 $y(t) = \cos t$ starts at $y(0) = 1$ and $y'(0) = 0$.

13 $\boldsymbol{u}_p = A^{-1}\boldsymbol{b} = 4$ and $u(t) = ce^{2t} + 4$; $\boldsymbol{u}_p = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and $\boldsymbol{u}(t) = c_1e^{2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2e^{3t}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

14 Substituting $\boldsymbol{u} = e^{ct}\boldsymbol{v}$ gives $ce^{ct}\boldsymbol{v} = Ae^{ct}\boldsymbol{v} - e^{ct}\boldsymbol{b}$ or $(A - cI)\boldsymbol{v} = \boldsymbol{b}$ or $\boldsymbol{v} = (A - cI)^{-1}\boldsymbol{b} =$ particular solution. If c is an eigenvalue then $A - cI$ is not invertible.

15 $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. In each case e^{At} blows up.

16 $d/dt(e^{At}) = A + A^2t + \frac{1}{2}A^3t^2 + \frac{1}{6}A^4t^3 + \cdots = A(I + At + \frac{1}{2}A^2t^2 + \frac{1}{6}A^3t^3 + \cdots) = Ae^{At}$.

17 $e^{Bt} = I + Bt = \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}$. Derivative $= \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = B$.

18 The solution at time $t + T$ is also $e^{A(t+T)}\boldsymbol{u}(0)$. Thus e^{At} times e^{AT} equals $e^{A(t+T)}$.

19 $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$; $e^{At} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} e^t & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} e^t & e^t - 1 \\ 0 & 1 \end{bmatrix}$.

20 If $A^2 = A$ then $e^{At} = I + At + \frac{1}{2}At^2 + \frac{1}{6}At^3 + \cdots = I + (e^t - 1)A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} e^t - 1 & e^t - 1 \\ 0 & 0 \end{bmatrix}$.

21 $e^A = \begin{bmatrix} e & e-1 \\ 0 & 1 \end{bmatrix}$, $e^B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, $e^Ae^B \neq e^Be^A = \begin{bmatrix} e & e-2 \\ 0 & 1 \end{bmatrix} \neq e^{A+B} = \begin{bmatrix} e & 0 \\ 0 & 1 \end{bmatrix}$.

22 $A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}$, then $e^{At} = \begin{bmatrix} e^t & \frac{1}{2}(e^{3t} - e^t) \\ 0 & e^{3t} \end{bmatrix}$.

23 $A^2 = A$ so $A^3 = A$ and by Problem 20 $e^{At} = I + (e^t - 1)A = \begin{bmatrix} e^t & 3(e^t - 1) \\ 0 & 1 \end{bmatrix}$.

24 (a) The inverse of e^{At} is e^{-At} (b) If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ then $e^{At}\boldsymbol{x} = e^{\lambda t}\boldsymbol{x}$ and $e^{\lambda t} \neq 0$.

25 $x(t) = e^{4t}$ and $y(t) = -e^{4t}$ is a growing solution. The correct matrix for the exchanged unknown $\boldsymbol{u} = (y, x)$ is $\begin{bmatrix} 2 & -2 \\ -4 & 0 \end{bmatrix}$ and it *does* have the same eigenvalues as the original matrix.

Problem Set 6.4, page 290

1 $A = \begin{bmatrix} 1 & 3 & 6 \\ 3 & 3 & 3 \\ 6 & 3 & 5 \end{bmatrix} + \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix} = \frac{1}{2}(A+A^{\mathrm{T}}) + \frac{1}{2}(A-A^{\mathrm{T}})$ = symmetric + skew-symmetric.

2 $(A^{\mathrm{T}}CA)^{\mathrm{T}} = A^{\mathrm{T}}C^{\mathrm{T}}(A^{\mathrm{T}})^{\mathrm{T}} = A^{\mathrm{T}}CA$. When A is 6 by 3, C is 6 by 6 and $A^{\mathrm{T}}CA$ is 3 by 3.

3 $\lambda = 0, 2, -1$ with unit eigenvectors $\pm(0,1,-1)/\sqrt{2}$ and $\pm(2,1,1)/\sqrt{6}$ and $\pm(1,-1,-1)/\sqrt{3}$.

4 $Q = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

5 $Q = \frac{1}{3} \begin{bmatrix} 2 & 1 & 2 \\ 2 & -2 & -1 \\ -1 & -2 & 2 \end{bmatrix}$.

6 $Q = \begin{bmatrix} .8 & .6 \\ -.6 & .8 \end{bmatrix}$ or $\begin{bmatrix} -.8 & .6 \\ .6 & .8 \end{bmatrix}$ or exchange columns.

7 (a) $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ has $\lambda = -1$ and 3 (b) The pivots have the same signs as the λ's (c) trace $= \lambda_1 + \lambda_2 = 2$, so A can't have two negative eigenvalues.

8 If $A^3 = 0$ then all $\lambda^3 = 0$ so all $\lambda = 0$ as in $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. If A is symmetric then $A^3 = Q\Lambda^3 Q^{\mathrm{T}} = 0$ gives $\Lambda = 0$ and the only symmetric possibility is $A = Q\,0\,Q^{\mathrm{T}}$ = zero matrix.

9 If λ is complex then $\overline{\lambda}$ is also an eigenvalue ($A\overline{\boldsymbol{x}} = \overline{\lambda}\overline{\boldsymbol{x}}$). Always $\lambda + \overline{\lambda}$ is real. The trace is real so the third eigenvalue must be real.

10 If $\boldsymbol{x}$ is not real then $\lambda = \boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x}/\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}$ is *not* necessarily real. Can't assume real eigenvectors!

11 $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = 2\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} + 4\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$; $\begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} = 0\begin{bmatrix} .64 & -.48 \\ -.48 & .36 \end{bmatrix} + 25\begin{bmatrix} .36 & .48 \\ .48 & .64 \end{bmatrix}$

12 $[\,\boldsymbol{x}_1 \;\; \boldsymbol{x}_2\,]$ is an orthogonal matrix so $P_1 + P_2 = \boldsymbol{x}_1\boldsymbol{x}_1^{\mathrm{T}} + \boldsymbol{x}_2\boldsymbol{x}_2^{\mathrm{T}} = [\,\boldsymbol{x}_1 \;\; \boldsymbol{x}_2\,]\begin{bmatrix} \boldsymbol{x}_1^{\mathrm{T}} \\ \boldsymbol{x}_2^{\mathrm{T}} \end{bmatrix} = I$; $P_1P_2 = \boldsymbol{x}_1(\boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{x}_2)\boldsymbol{x}_2^{\mathrm{T}} = 0$. Second proof: $P_1P_2 = P_1(I - P_1) = P_1 - P_1 = 0$ since $P_1^2 = P_1$.

13 $\lambda = ib$ and $-ib$; $A = \begin{bmatrix} 0 & 3 & 0 \\ -3 & 0 & 4 \\ 0 & -4 & 0 \end{bmatrix}$ has $\det(A - \lambda I) = -\lambda^3 - 25\lambda = 0$ and $\lambda = 0,\ 5i,\ -5i$.

14 Skew-symmetric and orthogonal; $\lambda = i,\ i,\ -i,\ -i$ to have trace zero.

15 A has $\lambda = 0, 0$ and only one independent eigenvector $\boldsymbol{x} = (i, 1)$.

16 (a) If $A\boldsymbol{z} = \lambda\boldsymbol{y}$ and $A^{\mathrm{T}}\boldsymbol{y} = \lambda\boldsymbol{z}$ then $B[\,\boldsymbol{y};\ -\boldsymbol{z}\,] = [\,-A\boldsymbol{z};\ A^{\mathrm{T}}\boldsymbol{y}\,] = -\lambda[\,\boldsymbol{y};\ -\boldsymbol{z}\,]$. So $-\lambda$ is also an eigenvalue of B. (b) $A^{\mathrm{T}}A\boldsymbol{z} = A^{\mathrm{T}}(\lambda\boldsymbol{y}) = \lambda^2\boldsymbol{z}$. The eigenvalues of $A^{\mathrm{T}}A$ are ≥ 0 (c) $\lambda = -1, -1, 1, 1$; $\boldsymbol{x}_1 = (1,0,-1,0)$, $\boldsymbol{x}_2 = (0,1,0,-1)$, $\boldsymbol{x}_3 = (1,0,1,0)$, $\boldsymbol{x}_4 = (0,1,0,1)$.

17 The eigenvalues of B are 0, $\sqrt{2}$, $-\sqrt{2}$ with $\boldsymbol{x}_1 = (1,-1,0)$, $\boldsymbol{x}_2 = (1,1,\sqrt{2})$, $\boldsymbol{x}_3 = (1,1,-\sqrt{2})$.

18 $\boldsymbol{y}$ is in the nullspace of A and $\boldsymbol{x}$ is in the column space. $A = A^{\mathrm{T}}$ has column space $=$ row space, and this is perpendicular to the nullspace. Then $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{x} = 0$. If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ and $A\boldsymbol{y} = \beta\boldsymbol{y}$ then shift by β: $(A - \beta I)\boldsymbol{x} = (\lambda - \beta)\boldsymbol{x}$ and $(A - \beta I)\boldsymbol{y} = \boldsymbol{0}$ and again $\boldsymbol{x} \perp \boldsymbol{y}$.

19 B has eigenvectors in $S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1+d \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$; independent but not perpendicular.

20 $\lambda = -5$ and 5 have the same signs as the pivots -3 and $25/3$.

21 (a) False. $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ (b) True (c) True. $A^{-1} = Q\Lambda^{-1}Q^{\mathrm{T}}$ is also symmetric (d) False.

22 If $A^{\mathrm{T}} = -A$ then $A^{\mathrm{T}}A = AA^{\mathrm{T}} = -A^2$. If A is orthogonal then $A^{\mathrm{T}}A = AA^{\mathrm{T}} = I$. $A = \begin{bmatrix} a & 1 \\ -1 & d \end{bmatrix}$ is normal only if $a = d$. Then $\boldsymbol{x} = \begin{bmatrix} 1 \\ i \end{bmatrix}$ is perpendicular to $\begin{bmatrix} 1 \\ -i \end{bmatrix}$.

23 A and A^{T} have the same λ's but the *order* of the $\boldsymbol{x}$'s can change. $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has $\lambda_1 = i$ and $\lambda_2 = -i$ with $\boldsymbol{x}_1 = (1, i)$ for A but $\boldsymbol{x}_1 = (1, -i)$ for A^{T}.

24 A is invertible, orthogonal, permutation, diagonalizable, Markov; B is projection, diagonalizable, Markov. $QR, S\Lambda S^{-1}, Q\Lambda Q^{\mathrm{T}}$ possible for A; $S\Lambda S^{-1}$ and $Q\Lambda Q^{\mathrm{T}}$ possible for B.

25 Symmetry gives $Q\Lambda Q^{\mathrm{T}}$ when $b = 1$; repeated λ and no S when $b = -1$; singular if $b = 0$.

26 Orthogonal and symmetric requires $|\lambda| = 1$ and λ real, so every $\lambda = \pm 1$. Then $A = \pm I$ or $A = Q\Lambda Q^{\mathrm{T}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} =$ reflection.

27 Eigenvectors $(1, 0)$ and $(1, 1)$ give a 45° angle even with A^{T} very close to A.

28 The roots of $\lambda^2 + b\lambda + c = 0$ differ by $\sqrt{b^2 - 4c}$. For $\det(A + tB - \lambda I)$ we have $b = -3 - 8t$ and $c = 2 + 16t - t^2$. The minimum of $b^2 - 4c$ is $1/17$ at $t = 2/17$. Then $\lambda_2 - \lambda_1 = 1/\sqrt{17}$.

29 We get good eigenvectors for the "symmetric part" $\frac{1}{2}(P + P^{\mathrm{T}})$ which MATLAB would recognize as symmetric. But the projection matrix $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}} =$ product of 3 matrices is not recognized as exactly symmetric.

Problem Set 6.5, page 302

1 A_4 has two positive eigenvalues because $a = 1$ and $ac - b^2 = 1$; $\boldsymbol{x}^{\mathrm{T}}A_1\boldsymbol{x}$ is zero for $\boldsymbol{x} = (1, -1)$ and $\boldsymbol{x}^{\mathrm{T}}A_1\boldsymbol{x} < 0$ for $\boldsymbol{x} = (6, -5)$.

2 Positive definite for $-3 < b < 3$: $\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 9 - b^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 9 - b^2 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = LDL^{\mathrm{T}}$;

Positive definite for $c > 8$: $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & c - 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & c - 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = LDL^{\mathrm{T}}$.

3 $f(x, y) = x^2 + 4xy + 9y^2 = (x + 2y)^2 + 5y^2$; $f(x, y) = x^2 + 6xy + 9y^2 = (x + 3y)^2$.

4 $x^2 + 4xy + 3y^2 = (x + 2y)^2 - y^2$ is negative at $x = 2$, $y = -1$.

5 $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ produces $f(x,y) = [\,x \;\; y\,] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2xy$. A has $\lambda = 1$ and -1.

6 $\boldsymbol{x}^{\mathrm{T}} A^{\mathrm{T}} A \boldsymbol{x} = (A\boldsymbol{x})^{\mathrm{T}}(A\boldsymbol{x}) = 0$ only if $A\boldsymbol{x} = \mathbf{0}$. Since A has independent columns this only happens when $\boldsymbol{x} = \mathbf{0}$.

7 $A^{\mathrm{T}}A = \begin{bmatrix} 1 & 2 \\ 2 & 13 \end{bmatrix}$ and $A^{\mathrm{T}}A = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix}$ are positive definite; $A^{\mathrm{T}}A = \begin{bmatrix} 2 & 3 & 3 \\ 3 & 5 & 4 \\ 3 & 4 & 5 \end{bmatrix}$ is singular.

8 $A = \begin{bmatrix} 3 & 6 \\ 6 & 16 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. Pivots outside squares, and L inside.

9 $A = \begin{bmatrix} 4 & -4 & 8 \\ -4 & 4 & -8 \\ 8 & -8 & 16 \end{bmatrix}$ has only one pivot $= 4$, rank $A = 1$, eigenvalues are 24, 0, 0, $\det A = 0$.

10 $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ has pivots $2, \frac{3}{2}, \frac{4}{3}$; $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ is singular; $A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

11 $|A_1| = 2$, $|A_2| = 6$, $|A_3| = 30$. The pivots are $2/1$, $6/2$, $30/6$.

12 A is positive definite for $c > 1$; determinants $c, c^2 - 1, c^3 + 2 - 3c > 0$. B is never positive definite (determinants $d - 4$ and $-4d + 12$ are never both positive).

13 $A = \begin{bmatrix} 1 & 5 \\ 5 & 10 \end{bmatrix}$ has $a + c > 2b$ but $ac < b^2$, so not positive definite.

14 The eigenvalues of A^{-1} are positive because they are $1/\lambda(A)$. And the entries of A^{-1} pass the determinant tests. And $\boldsymbol{x}^{\mathrm{T}} A^{-1} \boldsymbol{x} = (A^{-1}\boldsymbol{x})^{\mathrm{T}} A (A^{-1}\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \neq \mathbf{0}$.

15 Since $\boldsymbol{x}^{\mathrm{T}} A \boldsymbol{x} > 0$ and $\boldsymbol{x}^{\mathrm{T}} B \boldsymbol{x} > 0$ we have $\boldsymbol{x}^{\mathrm{T}}(A + B)\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}} A \boldsymbol{x} + \boldsymbol{x}^{\mathrm{T}} B \boldsymbol{x} > 0$ for all $\boldsymbol{x} \neq \mathbf{0}$. Then $A + B$ is a positive definite matrix.

16 $\boldsymbol{x}^{\mathrm{T}} A \boldsymbol{x}$ is not positive when $(x_1, x_2, x_3) = (0, 1, 0)$ because of the zero on the diagonal.

17 If a_{jj} were smaller than all the eigenvalues, $A - a_{jj}I$ would have *positive* eigenvalues (so positive definite). But $A - a_{jj}I$ has a *zero* in the (j, j) position; impossible by Problem 16.

18 If $A\boldsymbol{x} = \lambda \boldsymbol{x}$ then $\boldsymbol{x}^{\mathrm{T}} A \boldsymbol{x} = \lambda \boldsymbol{x}^{\mathrm{T}} \boldsymbol{x}$. If A is positive definite this leads to $\lambda = \boldsymbol{x}^{\mathrm{T}} A \boldsymbol{x} / \boldsymbol{x}^{\mathrm{T}} \boldsymbol{x} > 0$ (ratio of positive numbers).

19 All cross terms are $\boldsymbol{x}_i^{\mathrm{T}} \boldsymbol{x}_j = 0$ because symmetric matrices have orthogonal eigenvectors.

20 (a) The determinant is positive, all $\lambda > 0$ (b) All projection matrices except I are singular (c) The diagonal entries of D are its eigenvalues (d) $-I$ has det $= 1$ when n is even.

21 A is positive definite when $s > 8$; B is positive definite when $t > 5$ (check determinants).

22 $R = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{9} & \\ & \sqrt{1} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$; $R = Q \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} Q^{\mathrm{T}} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

23 $\lambda_1 = 1/a^2$ and $\lambda_2 = 1/b^2$ so $a = 1/\sqrt{\lambda_1}$ and $b = 1/\sqrt{\lambda_2}$. The ellipse $9x^2 + 16y^2 = 1$ has axes with half-lengths $a = \frac{1}{3}$ and $b = \frac{1}{4}$.

24 The ellipse $x^2 + xy + y^2 = 1$ has axes with half-lengths $a = 1/\sqrt{\lambda_1} = \sqrt{2}$ and $b = \sqrt{2/3}$.

25 $A = \begin{bmatrix} 9 & 3 \\ 3 & 5 \end{bmatrix}$; $C = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$.

26 $C = L\sqrt{D} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & \sqrt{5} \end{bmatrix}$ have *square roots* of the pivots from D.

27 $ax^2 + 2bxy + cy^2 = a(x + \frac{b}{a}y)^2 + \frac{ac-b^2}{a}y^2$; $2x^2 + 8xy + 10y^2 = 2(x + 2y)^2 + 2y^2$.

28 $\det A = 10$; $\lambda = 2$ and 5; $\boldsymbol{x}_1 = (\cos\theta, \sin\theta)$, $\boldsymbol{x}_2 = (-\sin\theta, \cos\theta)$; the λ's are positive.

29 $A_1 = \begin{bmatrix} 6x^2 & 2x \\ 2x & 2 \end{bmatrix}$ is positive definite if $x \neq 0$; $f_1 = (\frac{1}{2}x^2 + y)^2 = 0$ on the curve $\frac{1}{2}x^2 + y = 0$;

$A_2 = \begin{bmatrix} 6x & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 1 & 0 \end{bmatrix}$ is indefinite and $(0, 1)$ is a saddle point.

30 $ax^2 + 2bxy + cy^2$ has a saddle point if $ac < b^2$. The matrix is *indefinite* ($\lambda < 0$ and $\lambda > 0$).

31 If $c > 9$ the graph of z is a bowl, if $c < 9$ the graph has a saddle point. When $c = 9$ the graph of $z = (2x + 3y)^2$ is a trough staying at zero on the line $2x + 3y = 0$.

32 Orthogonal matrices, exponentials e^{At}, matrices with $\det = 1$ are groups. Examples of subgroups are orthogonal matrices with $\det = 1$, exponentials e^{An} for integer n.

Problem Set 6.6, page 310

1 $C = (MN)^{-1}A(MN)$ so if B is similar to A and C is similar to B, then A *is similar to* C.

2 $B = (FG^{-1})^{-1}A(FG^{-1})$. If C is similar to A and also to B then A *is similar to* B.

3 $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; $M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$; $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ gives $B = M^{-1}AM$.

4 A has no repeated λ so it can be diagonalized: $S^{-1}AS = \Lambda$ makes A similar to Λ.

5 $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ are similar; $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ by itself and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ by itself.

6 Eight families of similar matrices: 6 matrices have $\lambda = 0, 1$; 3 matrices have $\lambda = 1, 1$ and 3 have $\lambda = 0, 0$ (two families each!); one has $\lambda = 1, -1$; one has $\lambda = 2, 0$; two have $\lambda = \frac{1}{2}(1 \pm \sqrt{5})$.

7 (a) $(M^{-1}AM)(M^{-1}\boldsymbol{x}) = M^{-1}(A\boldsymbol{x}) = M^{-1}\boldsymbol{0} = \boldsymbol{0}$ (b) The nullspaces of A and of $M^{-1}AM$ have the same *dimension*. Different vectors and different bases.

8 $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ have the same line of eigenvectors and the same eigenvalues 0, 0.

9 $A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $A^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$, every $A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$. $A^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

10 $J^2 = \begin{bmatrix} c^2 & 2c \\ 0 & c^2 \end{bmatrix}$, $J^3 = \begin{bmatrix} c^3 & 3c^2 \\ 0 & c^3 \end{bmatrix}$, $J^k = \begin{bmatrix} c^k & kc^{k-1} \\ 0 & c^k \end{bmatrix}$; $J^0 = I$, $J^{-1} = \begin{bmatrix} c^{-1} & -c^{-2} \\ 0 & c^{-1} \end{bmatrix}$.

11 $w(t) = \big(w(0) + tx(0) + \frac{1}{2}t^2y(0) + \frac{1}{6}t^3z(0)\big)e^{5t}$.

12 If $M^{-1}JM = K$ then $JM = \begin{bmatrix} m_{21} & m_{22} & m_{23} & m_{24} \\ 0 & 0 & 0 & 0 \\ m_{41} & m_{42} & m_{43} & m_{44} \\ 0 & 0 & 0 & 0 \end{bmatrix} = MK = \begin{bmatrix} 0 & m_{12} & m_{13} & 0 \\ 0 & m_{22} & m_{23} & 0 \\ 0 & m_{32} & m_{33} & 0 \\ 0 & m_{42} & m_{43} & 0 \end{bmatrix}$.
That means $m_{21} = m_{22} = m_{23} = m_{24} = 0$ and M is not invertible.

13 (1) Choose M_i = reverse diagonal matrix to get $M_i^{-1}J_iM_i = M_i^{\mathrm{T}}$ in each block (2) M_0 has those blocks M_i on its block diagonal to get $M_0^{-1}JM_0 = J^{\mathrm{T}}$. (3) $A^{\mathrm{T}} = (M^{-1})^{\mathrm{T}}J^{\mathrm{T}}M^{\mathrm{T}}$ is $(M^{-1})^{\mathrm{T}}M_0^{-1}JM_0M^{\mathrm{T}} = (MM_0M^{\mathrm{T}})^{-1}A(MM_0M^{\mathrm{T}})$, and A^{T} is similar to A.

14 Every matrix MJM^{-1} will be similar to J.

15 $\det(M^{-1}AM - \lambda I) = \det(M^{-1}AM - M^{-1}\lambda IM) = \det(M^{-1}(A - \lambda I)M) = \det(A - \lambda I)$.

16 $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is similar to $\begin{bmatrix} d & c \\ b & a \end{bmatrix}$; $\begin{bmatrix} b & a \\ d & c \end{bmatrix}$ is similar to $\begin{bmatrix} c & d \\ a & b \end{bmatrix}$. I is not similar to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

17 (a) True: One has $\lambda = 0$, the other doesn't (b) False. Diagonalize a nonsymmetric matrix and Λ is symmetric (c) False: $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ are similar (d) True: All eigenvalues of $A + I$ are increased by 1, so different from the eigenvalues of A.

18 $AB = B^{-1}(BA)B$ so AB is similar to BA. *Also* $AB\boldsymbol{x} = \lambda\boldsymbol{x}$ leads to $BA(B\boldsymbol{x}) = \lambda(B\boldsymbol{x})$.

19 Diagonals 6 by 6 and 4 by 4; AB has all the same eigenvalues as BA plus $6 - 4$ zeros.

20 (a) $A = M^{-1}BM \Rightarrow A^2 = (M^{-1}BM)(M^{-1}BM) = M^{-1}B^2M$ (b) A may not be similar to $B = -A$ (but it could be!) (c) $\begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}$ is diagonalizable to $\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ because $\lambda_1 \neq \lambda_2$

(d) $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ has only one eigenvector, so not diagonalizable (e) PAP^{T} is similar to A.

21 J^2 has three 1's down the *second* superdiagonal, and two independent eigenvectors for $\lambda = 0$. Its 5 by 5 Jordan form is $\begin{bmatrix} J_3 & \\ & J_2 \end{bmatrix}$ with $J_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $J_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Note to professors: You could list all 3 by 3 and 4 by 4 Jordan J's (any a, b, c, d !):
$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}, \begin{bmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix}, \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix}$ with $3, 2, 1$ eigenvectors; 4 by 4 $\mathsf{diag}(a, b, c, d)$ and

$$\begin{bmatrix} a & 1 & & \\ & a & & \\ & & b & \\ & & & c \end{bmatrix}, \begin{bmatrix} a & 1 & & \\ & a & & \\ & & b & 1 \\ & & & b \end{bmatrix}, \begin{bmatrix} a & 1 & & \\ & a & 1 & \\ & & a & \\ & & & b \end{bmatrix}, \begin{bmatrix} a & 1 & & \\ & a & 1 & \\ & & a & 1 \\ & & & a \end{bmatrix}$$ 4, 3, 2, 2, 1 eigenvectors.

Problem Set 6.7, page 318

1 $A^{\mathrm{T}}A = \begin{bmatrix} 5 & 20 \\ 20 & 80 \end{bmatrix}$ has $\sigma_1^2 = 85$, $\boldsymbol{v}_1 = \begin{bmatrix} 1/\sqrt{17} \\ 4/\sqrt{17} \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} 4/\sqrt{17} \\ -1/\sqrt{17} \end{bmatrix}$.

2 (a) $AA^{\mathrm{T}} = \begin{bmatrix} 17 & 34 \\ 34 & 68 \end{bmatrix}$ has $\sigma_1^2 = 85$, $\boldsymbol{u}_1 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$, $\boldsymbol{u}_2 = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$.

(b) $A\boldsymbol{v}_1 = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} 1/\sqrt{17} \\ 4/\sqrt{17} \end{bmatrix} = \begin{bmatrix} \sqrt{17} \\ 2\sqrt{17} \end{bmatrix} = \sqrt{85} \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \sigma_1 \boldsymbol{u}_1$.

3 $\boldsymbol{u}_1 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$ for the column space, $\boldsymbol{v}_1 = \begin{bmatrix} 1/\sqrt{17} \\ 4/\sqrt{17} \end{bmatrix}$ for the row space, $\boldsymbol{u}_2 = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$ for the nullspace, $\boldsymbol{v}_2 = \begin{bmatrix} 4/\sqrt{17} \\ -1/\sqrt{17} \end{bmatrix}$ for the left nullspace.

4 $A^{\mathrm{T}}A = AA^{\mathrm{T}} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ has eigenvalues $\sigma_1^2 = \dfrac{3+\sqrt{5}}{2}$ and $\sigma_2^2 = \dfrac{3-\sqrt{5}}{2}$.

Since $A = A^{\mathrm{T}}$ the eigenvectors of $A^{\mathrm{T}}A$ are the same as for A. Since $\lambda_2 = \frac{1-\sqrt{5}}{2}$ is *negative*, $\sigma_1 = \lambda_1$ but $\sigma_2 = -\lambda_2$. The eigenvectors are the same as in Section 6.2 for A, except for the effect of this minus sign: $\boldsymbol{u}_1 = \boldsymbol{v}_1 = \begin{bmatrix} \lambda_1/\sqrt{1+\lambda_1^2} \\ 1/\sqrt{1+\lambda_1^2} \end{bmatrix}$ and $\boldsymbol{u}_2 = -\boldsymbol{v}_2 = \begin{bmatrix} \lambda_2/\sqrt{1+\lambda_2^2} \\ 1/\sqrt{1+\lambda_2^2} \end{bmatrix}$.

6 A proof that ***eigshow*** finds the SVD for 2 by 2 matrices. Starting at the orthogonal pair $\boldsymbol{V}_1 = (1,0)$, $\boldsymbol{V}_2 = (0,1)$ the demo finds $A\boldsymbol{V}_1$ and $A\boldsymbol{V}_2$ at angle θ. After a 90° turn by the mouse to $\boldsymbol{V}_2, -\boldsymbol{V}_1$ the demo finds $A\boldsymbol{V}_2$ and $-A\boldsymbol{V}_1$ at angle $\pi - \theta$. Somewhere between, the constantly orthogonal $\boldsymbol{v}_1, \boldsymbol{v}_2$ must have produced $A\boldsymbol{v}_1$ and $A\boldsymbol{v}_2$ at angle $\theta = \pi/2$. Those are the orthogonal directions for $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$.

7 $AA^{\mathrm{T}} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ has $\sigma_1^2 = 3$ with $\boldsymbol{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ and $\sigma_2^2 = 1$ with $\boldsymbol{u}_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$. $A^{\mathrm{T}}A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ has $\sigma_1^2 = 3$ with $\boldsymbol{v}_1 = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$, $\sigma_2^2 = 1$ with $\boldsymbol{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$; and $\boldsymbol{v}_3 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$.

Then $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{bmatrix}^{\mathrm{T}}$.

8 $A = UV^{\mathrm{T}}$ since all $\sigma_j = 1$.

9 $A = 12\,UV^{\mathrm{T}}$.

10 $A = W\Sigma W^{\mathrm{T}}$ is the same as $A = U\Sigma V^{\mathrm{T}}$.

11 Multiply $U\Sigma V^{\mathrm{T}}$ using columns (of U) times rows (of ΣV^{T}).

12 Since $A^{\mathrm{T}} = A$ we have $\sigma_1^2 = \lambda_1^2$ and $\sigma_2^2 = \lambda_2^2$. But λ_2 is negative, so $\sigma_1 = 3$ and $\sigma_2 = 2$. The unit eigenvectors of A are the same $\boldsymbol{u}_1 = \boldsymbol{v}_1$ as for $A^{\mathrm{T}}A = AA^{\mathrm{T}}$ and $\boldsymbol{u}_2 = -\boldsymbol{v}_2$ (notice sign change because $\sigma_2 = -\lambda_2$).

13 Suppose the SVD of R is $R = U\Sigma V^{\mathrm{T}}$. Then multiply by Q. So the SVD of this A is $(QU)\Sigma V^{\mathrm{T}}$.

14 The smallest change in A is to set its smallest singular value σ_2 to zero.

15 (a) If A changes to $4A$, multiply Σ by 4. (b) $A^{\mathrm{T}} = V\Sigma^{\mathrm{T}}U^{\mathrm{T}}$. And if A^{-1} exists, it is square and equal to $(V^{\mathrm{T}})^{-1}\Sigma^{-1}U^{-1}$.

16 The singular values of $A+I$ are not σ_j+1. They come from eigenvalues of $(A+I)^{\mathrm{T}}(A+I)$.

17 This simulates the random walk used by *Google* on billions of sites to solve $A\boldsymbol{p}=\boldsymbol{p}$. It is like the power method of 9.3 except that it follows the links in one "walk" where the power method $\boldsymbol{p}_k = A^k\boldsymbol{p}_0$ converges to the average time at each site over all walks.

Problem Set 7.1, page 325

1 With $\boldsymbol{w}=\mathbf{0}$ linearity gives $T(\boldsymbol{v}+\mathbf{0})=T(\boldsymbol{v})+T(\mathbf{0})$. Thus $T(\mathbf{0})=\mathbf{0}$. With $c=-1$ linearity gives $T(-\mathbf{0})=-T(\mathbf{0})$. Thus $T(\mathbf{0})=\mathbf{0}$.

2 $T(c\boldsymbol{v}+d\boldsymbol{w})=cT(\boldsymbol{v})+dT(\boldsymbol{w})$; add $eT(\boldsymbol{u})$.

3 (d) is not linear.

4 (a) $S(T(\boldsymbol{v}))=\boldsymbol{v}$ (b) $S(T(\boldsymbol{v}_1)+T(\boldsymbol{v}_2))=S(T(\boldsymbol{v}_1))+S(T(\boldsymbol{v}_2))$.

5 Choose $\boldsymbol{v}=(1,1)$ and $\boldsymbol{w}=(-1,0)$. Then $T(\boldsymbol{v})+T(\boldsymbol{w})=\boldsymbol{v}+\boldsymbol{w}$ but $T(\boldsymbol{v}+\boldsymbol{w})=(0,0)$.

6 (b) and (c) are linear (d) satisfies $T(c\boldsymbol{v})=cT(\boldsymbol{v})$.

7 (a) $T(T(\boldsymbol{v}))=\boldsymbol{v}$ (b) $T(T(\boldsymbol{v}))=\boldsymbol{v}+(2,2)$ (c) $T(T(\boldsymbol{v}))=-\boldsymbol{v}$ (d) $T(T(\boldsymbol{v}))=T(\boldsymbol{v})$.

8 (a) Range $\mathbf{R}^2$, kernel $\{\mathbf{0}\}$ (b) Range $\mathbf{R}^2$, kernel $\{(0,0,v_3)\}$ (c) Range $\{\mathbf{0}\}$, kernel $\mathbf{R}^2$ (d) Range = multiples of $(1,1)$, kernel = multiples of $(1,-1)$.

9 $T(T(\boldsymbol{v}))=(v_3,v_1,v_2)$; $T^3(\boldsymbol{v})=\boldsymbol{v}$; $T^{100}(\boldsymbol{v})=T(\boldsymbol{v})$.

10 (a) $T(1,0)=\mathbf{0}$ (b) $(0,0,1)$ is not in the range (c) $T(0,1)=\mathbf{0}$.

11 $\mathbf{V}=\mathbf{R}^n$, $\mathbf{W}=\mathbf{R}^m$; the outputs fill the column space; $\boldsymbol{v}$ is in the kernel if $A\boldsymbol{v}=\mathbf{0}$.

12 $T(\boldsymbol{v})=(4,4)$; $(2,2)$; $(2,2)$; if $\boldsymbol{v}=(a,b)=b(1,1)+\frac{a-b}{2}(2,0)$ then $T(\boldsymbol{v})=b(2,2)+(0,0)$.

13 Associative gives $A(M_1+M_2)=AM_1+AM_2$. Distributive over c's gives $A(cM)=c(AM)$.

14 A is invertible. Multiply $AM=0$ and $AM=B$ by A^{-1} to get $M=0$ and $M=A^{-1}B$.

15 A is not invertible. $AM=I$ is impossible. $A\begin{bmatrix}2 & 2\\ -1 & -1\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$.

16 No matrix A gives $A\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$. To professors: The matrix space has dimension 4. Linear transformations come from 4 by 4 matrices. Those in Problems 13–15 were special.

17 (a) True (b) True (c) True (d) False.

18 $T(I)=0$ but $M=\begin{bmatrix}0 & b\\ 0 & 0\end{bmatrix}=T(M)$; these fill the range. $M=\begin{bmatrix}a & 0\\ c & d\end{bmatrix}$ in the kernel.

19 If $\boldsymbol{v}\neq 0$ is a column of B and $\boldsymbol{u}^{\mathrm{T}}\neq 0$ is a row of A, choose $M=\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$.

20 $T(T^{-1}(M))=M$ so $T^{-1}(M)=A^{-1}MB^{-1}$.

21 (a) Horizontal lines stay horizontal, vertical lines stay vertical (b) House squashes onto a line (c) Vertical lines stay vertical.

23 (a) $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ with $d > 0$ (b) $A = 3I$ (c) $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

24 (a) $ad-bc = 0$ (b) $ad-bc > 0$ (c) $|ad-bc| = 1$. If vectors to two corners transform to themselves then by linearity $T = I$. (Fails if one corner is $(0,0)$.)

25 Rotate the house by $180°$ and shift one unit to the right.

27 This emphasizes that circles are transformed to ellipses (figure in Section 6.7).

30 Squeezed by 10 in y direction; flattened onto $45°$ line; rotated by $45°$ and stretched by $\sqrt{2}$; flipped over and "skewed" so squares become parallelograms.

Problem Set 7.2, page 337

1 $S\boldsymbol{v}_1 = S\boldsymbol{v}_2 = \boldsymbol{0}$, $S\boldsymbol{v}_3 = 2\boldsymbol{v}_1$, $S\boldsymbol{v}_4 = 6\boldsymbol{v}_2$; $B = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

2 All functions $\boldsymbol{v}(x) = a + bx$; all vectors $(a, b, 0, 0)$.

3 $A^2 = B$ when $T^2 = S$ and output basis = input basis.

4 Third derivative has 6 in the $(1,4)$ position; fourth derivative of cubic is zero.

5 $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

6 $T(\boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3) = 2\boldsymbol{w}_1 + \boldsymbol{w}_2 + 2\boldsymbol{w}_3$; A times $(1,1,1)$ gives $(2,1,2)$.

7 $\boldsymbol{v} = c(\boldsymbol{v}_2 - \boldsymbol{v}_3)$ gives $T(\boldsymbol{v}) = \boldsymbol{0}$; nullspace is $(0, c, -c)$; solutions are $(1,0,0)$ + any $(0, c, -c)$.

8 $(1,0,0)$ is not in the column space; $\boldsymbol{w}_1$ is not in the range.

9 We don't know $T(\boldsymbol{w})$ unless the $\boldsymbol{w}$'s are the same as the $\boldsymbol{v}$'s. In that case the matrix is A^2.

10 Rank = 2 = dimension of the *range* of T.

11 $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$; for output $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ choose input $\boldsymbol{v} = \boldsymbol{v}_1 - \boldsymbol{v}_2$.

12 $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ so $T^{-1}(\boldsymbol{w}_1) = \boldsymbol{v}_1 - \boldsymbol{v}_2$, $T^{-1}(\boldsymbol{w}_2) = \boldsymbol{v}_2 - \boldsymbol{v}_3$, $T^{-1}(\boldsymbol{w}_3) = \boldsymbol{v}_3$; the only solution to $T(\boldsymbol{v}) = \boldsymbol{0}$ is $\boldsymbol{v} = \boldsymbol{0}$.

13 (c) is wrong because $\boldsymbol{w}_1$ is not generally in the input space.

14 (a) $T(\boldsymbol{v}_1) = \boldsymbol{v}_2$, $T(\boldsymbol{v}_2) = \boldsymbol{v}_1$ (b) $T(\boldsymbol{v}_1) = \boldsymbol{v}_1$, $T(\boldsymbol{v}_2) = \boldsymbol{0}$ (c) If $T^2 = I$ and $T^2 = T$ then $T = I$.

15 (a) $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$ = inverse of (a) (c) $A \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ must be $2A \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

16 (a) $M = \begin{bmatrix} r & s \\ t & u \end{bmatrix}$ (b) $N = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$ (c) $ad = bc$.

17 $MN = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -1 \\ -7 & 3 \end{bmatrix}$.

18 Permutation matrix; positive diagonal matrix.

19 $(a, b) = (\cos\theta, -\sin\theta)$. Minus sign from $Q^{-1} = Q^{\mathrm{T}}$.

20 $M = \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}$; $(a, b) = (5, -4)$ = first column of M^{-1}.

21 $\boldsymbol{w}_2(x) = 1 - x^2$; $\boldsymbol{w}_3(x) = \frac{1}{2}(x^2 - x)$; $\boldsymbol{y} = 4\boldsymbol{w}_1 + 5\boldsymbol{w}_2 + 6\boldsymbol{w}_3$.

22 $\boldsymbol{w}$'s to $\boldsymbol{v}$'s: $\begin{bmatrix} 0 & 1 & 0 \\ .5 & 0 & -.5 \\ .5 & -1 & .5 \end{bmatrix}$. $\boldsymbol{v}$'s to $\boldsymbol{w}$'s: inverse matrix $= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}$.

23 $\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$; Vandermonde determinant $= (b-a)(c-a)(c-b)$; a, b, c must be distinct.

24 The matrix M with these nine entries must be invertible.

25 $\boldsymbol{a}_2 = r_{12}\boldsymbol{q}_1 + r_{22}\boldsymbol{q}_2$ gives $\boldsymbol{a}_2$ as a combination of the $\boldsymbol{q}$'s. So the change of basis matrix is R.

26 Row 2 of A is ℓ_{21}(row 1 of U)$+\ell_{22}$(row 2 of U). The change of basis matrix is always *invertible*.

27 The matrix is Λ.

28 If T is not invertible then $T(\boldsymbol{v}_1), \ldots, T(\boldsymbol{v}_n)$ will not be a basis. Then we couldn't choose $\boldsymbol{w}_i = T(\boldsymbol{v}_i)$.

29 (a) $\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

30 $T(x, y) = (x, -y)$ and then $S(x, -y) = (-x, -y)$. Thus $ST = -I$.

31 $S(T(\boldsymbol{v})) = (-1, 2)$ but $S(\boldsymbol{v}) = (-2, 1)$ and $T(S(\boldsymbol{v})) = (1, -2)$.

32 $\begin{bmatrix} \cos 2(\theta - \alpha) & -\sin 2(\theta - \alpha) \\ \sin 2(\theta - \alpha) & \cos 2(\theta - \alpha) \end{bmatrix}$ rotates by $2(\theta - \alpha)$.

33 False, because the $\boldsymbol{v}$'s might not be linearly independent.

Problem Set 7.3, page 345

1 Multiply by $W^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$. Then $\boldsymbol{e} = \frac{1}{4}\boldsymbol{w}_1 + \frac{1}{4}\boldsymbol{w}_2 + \frac{1}{2}\boldsymbol{w}_3$ and $\boldsymbol{v} = \boldsymbol{w}_3 + \boldsymbol{w}_4$.

2 The last step writes 6, 6, 2, 2 as the overall average 4, 4, 4, 4 plus the difference 2, 2, −2, −2. Therefore $c_1 = 4$ and $c_2 = 2$ and $c_3 = 1$ and $c_4 = 1$.

3 The wavelet basis is $(1,1,1,1,1,1,1,1)$ and the long wavelet and two medium wavelets $(1,1,-1,-1,0,0,0,0)$ and $(0,0,0,0,1,1,-1,-1)$ and 4 short wavelets with a single pair $1,-1$.

4 $W_2^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $W_1^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$.

5 The Hadamard matrix H has orthogonal columns of length 2. So the inverse is $H^{\mathrm{T}}/4 = H/4$.

6 If $V\boldsymbol{b} = W\boldsymbol{c}$ then $\boldsymbol{b} = V^{-1}W\boldsymbol{c}$. The change of basis matrix is $V^{-1}W$.

7 The transpose of $WW^{-1} = I$ is $(W^{-1})^{\mathrm{T}}W^{\mathrm{T}} = I$. So the matrix W^{T} (which has the $\boldsymbol{w}$'s in its rows) is the inverse to the matrix that has the $\boldsymbol{w}^*$'s in its columns.

Problem Set 7.4, page 353

1 $A^{\mathrm{T}}A = \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}$ has $\lambda = 50$ and 0, $\boldsymbol{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\boldsymbol{v}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$; $\sigma_1 = \sqrt{50}$.

2 $AA^{\mathrm{T}} = \begin{bmatrix} 5 & 15 \\ 15 & 45 \end{bmatrix}$ has $\lambda = 50$ and 0, $\boldsymbol{u}_1 = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\boldsymbol{u}_2 = \frac{1}{\sqrt{10}}\begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

3 Orthonormal bases: $\boldsymbol{v}_1$ for row space, $\boldsymbol{v}_2$ for nullspace, $\boldsymbol{u}_1$ for column space, $\boldsymbol{u}_2$ for $\boldsymbol{N}(A^{\mathrm{T}})$.

4 The matrices with those four subspaces are multiples cA.

5 $A = QH = \frac{1}{\sqrt{50}}\begin{bmatrix} 7 & -1 \\ 1 & 7 \end{bmatrix} \frac{1}{\sqrt{50}}\begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}$. H is semidefinite because A is singular.

6 $A^+ = V\begin{bmatrix} 1/\sqrt{50} & 0 \\ 0 & 0 \end{bmatrix} U^{\mathrm{T}} = \frac{1}{50}\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$; $A^+A = \begin{bmatrix} .2 & .4 \\ .4 & .8 \end{bmatrix}$, $AA^+ = \begin{bmatrix} .1 & .3 \\ .3 & .9 \end{bmatrix}$.

7 $A^{\mathrm{T}}A = \begin{bmatrix} 10 & 8 \\ 8 & 10 \end{bmatrix}$ has $\lambda = 18$ and 2, $\boldsymbol{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\boldsymbol{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\sigma_1 = \sqrt{18}$ and $\sigma_2 = \sqrt{2}$.

8 $AA^{\mathrm{T}} = \begin{bmatrix} 18 & 0 \\ 0 & 2 \end{bmatrix}$ has $\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\boldsymbol{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

9 $\begin{bmatrix} \sigma_1\boldsymbol{u}_1 & \sigma_2\boldsymbol{u}_2 \end{bmatrix}\begin{bmatrix} \boldsymbol{v}_1^{\mathrm{T}} \\ \boldsymbol{v}_2^{\mathrm{T}} \end{bmatrix} = \sigma_1\boldsymbol{u}_1\boldsymbol{v}_1^{\mathrm{T}} + \sigma_2\boldsymbol{u}_2\boldsymbol{v}_2^{\mathrm{T}}$. In general this is $\sigma_1\boldsymbol{u}_1\boldsymbol{v}_1^{\mathrm{T}} + \cdots + \sigma_r\boldsymbol{u}_r\boldsymbol{v}_r^{\mathrm{T}}$.

10 $Q = UV^{\mathrm{T}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $K = \begin{bmatrix} \sqrt{18} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$.

11 A^+ is A^{-1} because A is invertible.

12 $A^{\mathrm{T}}A = \begin{bmatrix} 9 & 12 & 0 \\ 12 & 16 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ has $\lambda = 25,\ 0,\ 0$ and $\boldsymbol{v}_1 = \begin{bmatrix} .6 \\ .8 \\ 0 \end{bmatrix}$, $\boldsymbol{v}_2 = \begin{bmatrix} .8 \\ -.6 \\ 0 \end{bmatrix}$, $\boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

$AA^{\mathrm{T}} = [\,25\,]$ and $\sigma_1 = 5$.

13 $A = [\,1\,]\,[\,5 \quad 0 \quad 0\,]V^{\mathrm{T}}$ and $A^+ = V \begin{bmatrix} .2 \\ 0 \\ 0 \end{bmatrix} [\,1\,] = \begin{bmatrix} .12 \\ .16 \\ 0 \end{bmatrix}$; $AA^+ = [\,1\,]$; $A^+A = \begin{bmatrix} .36 & .48 & 0 \\ .48 & .64 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

14 Zero matrix; $\Sigma = 0$; $A^+ = 0$ is 3 by 2.

15 If $\det A = 0$ then $\operatorname{rank}(A) < n$; thus $\operatorname{rank}(A^+) < n$ and $\det A^+ = 0$.

16 A must be symmetric and positive definite.

17 (a) $A^{\mathrm{T}}A$ is singular (b) $A^{\mathrm{T}}A\boldsymbol{x}^+ = A^{\mathrm{T}}\boldsymbol{b}$ (c) $(I - AA^+)$ projects onto $\boldsymbol{N}(A^{\mathrm{T}})$.

18 $\boldsymbol{x}^+$ in the row space of A is perpendicular to $\widehat{\boldsymbol{x}} - \boldsymbol{x}^+$ in the nullspace of $A^{\mathrm{T}}A$ = nullspace of A. The right triangle has $c^2 = a^2 + b^2$.

19 $AA^+\boldsymbol{p} = \boldsymbol{p}$, $AA^+\boldsymbol{e} = \boldsymbol{0}$, $A^+A\boldsymbol{x}_r = \boldsymbol{x}_r$, $A^+A\boldsymbol{x}_n = \boldsymbol{0}$.

20 $A^+ = \frac{1}{5}[\,.6 \quad .8\,] = [\,.12 \quad .16\,]$ and $A^+A = [\,1\,]$ and $AA^+ = \begin{bmatrix} .36 & .48 \\ .48 & .64 \end{bmatrix}$.

21 L is determined by ℓ_{21}. Each eigenvector in S is determined by one number. The counts are $1+3$ for LU, $1+2+1$ for LDU, $1+3$ for QR, $1+2+1$ for $U\Sigma V^{\mathrm{T}}$, $2+2+0$ for $S\Lambda S^{-1}$.

22 The counts are $1+2+0$ because A is *symmetric.*

23 Column times row multiplication gives $A = U\Sigma V^{\mathrm{T}} = \sum \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^{\mathrm{T}}$ and also $A^+ = V\Sigma^+U^{\mathrm{T}} = \sum \sigma_i^{-1}\boldsymbol{v}_i\boldsymbol{u}_i^{\mathrm{T}}$. Multiplying A^+A and using orthogonality of each $\boldsymbol{u}_i$ to all other $\boldsymbol{u}_j$ leaves the projection matrix A^+A: $A^+A = \sum 1\boldsymbol{v}_i\boldsymbol{v}_i^{\mathrm{T}}$. Similarly $AA^+ = \sum 1\boldsymbol{u}_i\boldsymbol{u}_i^{\mathrm{T}}$ from $VV^{\mathrm{T}} = I$.

24 The columns of $\widehat{U}$ are a basis for the column space of A. So are the first r columns of U. Those r columns must have the form $\widehat{U}M_1$ for some r by r invertible matrix M_1. Similarly the columns of $\widehat{V}$ and the first r columns of V are bases for the row space of A. So $V = \widehat{V}M_2$.

Keep only the r by r invertible corner Σ_r of Σ (the rest is all zero). Then $A = U\Sigma V^{\mathrm{T}}$ has the required form $A = \widehat{U}M_1\Sigma_r M_2^{\mathrm{T}}\widehat{V}^{\mathrm{T}}$ with an invertible $M = M_1\Sigma_r M_2^{\mathrm{T}}$ in the middle.

Note The column space of $A = \widehat{U}M\widehat{V}^{\mathrm{T}}$ is certainly contained in the column space of $\widehat{U}$. They are the same space if $\operatorname{rank}(A) = r$. To verify that rank, look at $\widehat{U}^{\mathrm{T}}A\widehat{V} = (\widehat{U}^{\mathrm{T}}\widehat{U})M(\widehat{V}^{\mathrm{T}}\widehat{V})$ = product of invertible r by r matrices. So $r = \operatorname{rank}\,(\widehat{U}^{\mathrm{T}}A\widehat{V}) \le \operatorname{rank}\,(A) \le r$, and A has the desired column space (similarly the desired row space).

25 $\begin{bmatrix} 0 & A \\ A^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix} = \sigma \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix}$. That block matrix connects to $A^{\mathrm{T}}A$ and AA^{T}.

Problem Set 8.1, page ???

1 Det $A_0^{\mathrm{T}}C_0A_0$ is by direct calculation. Set $c_4 = 0$ to find det $A_1^{\mathrm{T}}C_1A_1 = c_1c_2c_3$.

2 $(A_1^{\mathrm{T}}C_1A_1)^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1^{-1} & & \\ & c_2^{-1} & \\ & & c_3^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} c_1^{-1}+c_2^{-1}+c_3^{-1} & c_2^{-1}+c_3^{-1} & c_3^{-1} \\ c_2^{-1}+c_3^{-1} & c_2^{-1}+c_3^{-1} & c_3^{-1} \\ c_3^{-1} & c_3^{-1} & c_3^{-1} \end{bmatrix}$

3 The rows of the free-free matrix in equation (9) add to $[0\ \ 0\ \ 0]$ so the right side needs $f_1 + f_2 + f_3 = 0$. For $\boldsymbol{f} = (-1, 0, 1)$ elimination gives $c_2u_1 - c_2u_2 = -1, c_3u_2 - c_3u_3 = -1$, and $0 = 0$. Then $\boldsymbol{u}_{\text{particular}} = (-c_2^{-1} - c_3^{-1}, -c_3^{-1}, 0)$. Add any multiple of $\boldsymbol{u}_{\text{nullspace}} = (1, 1, 1)$.

4 $\displaystyle\int -\frac{d}{dx}\left(c(x)\frac{du}{dx}\right) dx = \left[c(0)\frac{du}{dx}(0) - c(1)\frac{du}{dx}(1)\right] = 0$ so we need $\displaystyle\int f(x)\,dx = 0$.

5 $\displaystyle -\frac{dy}{dx} = f(x)$ gives $\displaystyle y(x) = C - \int_0^x f(t)\,dt$. Then $y(1) = 0$ gives $\displaystyle C = \int_0^1 f(t)\,dt$ and $y(x) =$ $\displaystyle\int_x^1 f(t)\,dt$. If $f(x) = 1$ then $y(x) = 1 - x$.

6 Multiply $A_1^{\mathrm{T}}C_1A_1$ as columns of A_1^{T} times c's times rows of A_1. The first "element matrix" $c_1E_1 = [1\ \ 0\ \ 0]^{\mathrm{T}}c_1[1\ \ 0\ \ 0]$ has c_1 in the top left corner.

7 For 5 springs and 4 masses, the 5 by 4 A has all $a_{ii} = 1$ and $a_{i+1,i} = -1$. With $C = \mathsf{diag}(c_1, c_2, c_3, c_4, c_5)$ we get $K = A^{\mathrm{T}}CA$, symmetric tridiagonal with $K_{ii} = c_i + c_{i+1}$ and $K_{i+1,i} = -c_{i+1}$. With $C = I$ this K is the $-1, 2, -1$ matrix and $K(2,3,3,2) = (1,1,1,1)$.

8 The solution to $-u'' = 1$ with $u(0) = u(1) = 0$ is $u(x) = \frac{1}{2}(x - x^2)$. At $x = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$ this $u(x)$ equals $\boldsymbol{u} = 2, 3, 3, 2$ (discrete solution in Problem 7) times $(\Delta x)^2 = 1/25$.

9 $-u'' = mg$ has complete solution $u(x) = A + Bx - \frac{1}{2}mgx^2$. From $u(0) = 0$ we get $A = 0$. From $u'(1) = 0$ we get $B = mg$. Then $u(x) = \frac{1}{2}mg(2x - x^2)$ at $x = \frac{1}{3}, \frac{2}{3}, \frac{3}{3}$ equals $mg/6, 4mg/9, mg/2$. This $u(x)$ is *not* proportional to the discrete $\boldsymbol{u}$ at the meshpoints.

10 The graphs of 100 points are "discrete parabolas" starting at $(0, 0)$: symmetric around 50 in the fixed-fixed case, ending with slope zero in the fixed-free case.

11 Forward vs. backward differences for du/dx have a big effect on the discrete $\boldsymbol{u}$, because that term has the large coefficient 10 (and with 100 or 1000 we would have a real boundary layer = near discontinuity at $x = 1$). The computed values are $\boldsymbol{u} = 0, .01, .03, .04, .05, .06, .07, .11, 0$ versus $\boldsymbol{u} = 0, .12, .24, .36, .46, .54, .55, .43, 0$. The MATLAB code is $E = \mathsf{diag}(\mathsf{ones}(6,1), 1)$; $K = 64 * (2 * \mathsf{eye}(7) - E - E')$; $D = 80 * (E - \mathsf{eye}(7))$; $(K + D)\backslash\mathsf{ones}(7,1)$, $(K - D')\backslash\mathsf{ones}(7,1)$.

Problem Set 8.2, page 366

1 $A = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$; nullspace contains $\begin{bmatrix} c \\ c \\ c \end{bmatrix}$; $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is not orthogonal to that nullspace.

2 $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{0}$ for $\boldsymbol{y} = (1, -1, 1)$; current $= 1$ along edge 1, edge 3, back on edge 2 (full loop).

3 $U = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$; tree from edges 1 and 2.

4 $A\boldsymbol{x} = \boldsymbol{b}$ is solvable for $\boldsymbol{b} = (1,1,0)$ and not solvable for $\boldsymbol{b} = (1,0,0)$; $\boldsymbol{b}$ must be orthogonal to $\boldsymbol{y} = (1,-1,1)$; $b_1 - b_2 + b_3 = 0$ is the third equation after elimination.

5 Kirchhoff's Current Law $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{f}$ is solvable for $\boldsymbol{f} = (1,-1,0)$ and not solvable for $\boldsymbol{f} = (1,0,0)$; $\boldsymbol{f}$ must be orthogonal to $(1,1,1)$ in the nullspace.

6 $A^{\mathrm{T}}A\boldsymbol{x} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} = \boldsymbol{f}$ produces $\boldsymbol{x} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} c \\ c \\ c \end{bmatrix}$; potentials 1, −1, 0 and currents $-A\boldsymbol{x} = 2,\ 1,\ -1$; $\boldsymbol{f}$ sends 3 units into node 1 and out from node 2.

7 $A^{\mathrm{T}} \begin{bmatrix} 1 & & \\ & 2 & \\ & & 2 \end{bmatrix} A = \begin{bmatrix} 3 & -1 & -2 \\ -1 & 3 & -2 \\ -2 & -2 & 4 \end{bmatrix}$; $\boldsymbol{f} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ yields $\boldsymbol{x} = \begin{bmatrix} 5/4 \\ 1 \\ 7/8 \end{bmatrix} + \begin{bmatrix} c \\ c \\ c \end{bmatrix}$; potentials $\frac{5}{4}$, 1, $\frac{7}{8}$ and currents $-CA\boldsymbol{x} = \frac{1}{4}, \frac{3}{4}, \frac{1}{4}$.

8 $A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$ leads to $\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\boldsymbol{y} = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$.

9 Elimination on $A\boldsymbol{x} = \boldsymbol{b}$ always leads to $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} = 0$ which is $-b_1 + b_2 - b_3 = 0$ and $b_3 - b_4 + b_5 = 0$ ($\boldsymbol{y}$'s from Problem 8 in the left nullspace). This is Kirchhoff's Voltage Law around the loops.

10 $U = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is the matrix that keeps edges 1, 2, 4; other trees from 1, 2, 5; 1, 3, 4; 1, 3, 5; 1, 4, 5; 2, 3, 4; 2, 3, 5; 2, 4, 5.

11 $A^{\mathrm{T}}A = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$ diagonal entry = number of edges into the node off-diagonal entry = −1 if nodes are connected.

12 (1) The nullspace and rank of $A^{\mathrm{T}}A$ and A are always the same (2) $A^{\mathrm{T}}A$ is always positive semidefinite because $\boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}A\boldsymbol{x} = \|A\boldsymbol{x}\|^2 \geq 0$. Not positive definite because rank is only 3 and $(1,1,1,1)$ is in the nullspace (3) Real eigenvalues all ≥ 0 because positive semidefinite.

13 $A^{\mathrm{T}}CA\boldsymbol{x} = \begin{bmatrix} 4 & -2 & -2 & 0 \\ -2 & 8 & -3 & -3 \\ -2 & -3 & 8 & -3 \\ 0 & -3 & -3 & 6 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$ gives potentials $\boldsymbol{x} = (\frac{5}{12}, \frac{1}{6}, \frac{1}{6}, 0)$ (grounded $x_4 = 0$ and solved 3 equations); $\boldsymbol{y} = -CA\boldsymbol{x} = (\frac{2}{3}, \frac{2}{3}, 0, \frac{1}{2}, \frac{1}{2})$.

14 $A^{\mathrm{T}}CA\boldsymbol{x} = \boldsymbol{0}$ for $\boldsymbol{x} = (c,c,c,c)$; then $\boldsymbol{f}$ must be orthogonal to $\boldsymbol{x}$.

15 $n - m + 1 = 7 - 7 + 1 = 1$ loop.

16 $5 - 7 + 3 = 1$; $5 - 8 + 4 = 1$.

17 (a) 8 independent columns (b) $\boldsymbol{f}$ must be orthogonal to the nullspace so $f_1 + \cdots + f_9 = 0$
(c) Each edge goes into 2 nodes, 12 edges make diagonal entries sum to 24.

18 Complete graph has $5 + 4 + 3 + 2 + 1 = 15$ edges; tree has 5 edges.

Problem Set 8.3, page 373

1 $\lambda = 1$ and $.75$; $(A - I)\boldsymbol{x} = \mathbf{0}$ gives $\boldsymbol{x} = (.6, .4)$.

2 $A = \begin{bmatrix} .6 & -1 \\ .4 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & .75 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -.4 & .6 \end{bmatrix}$;

A^k approaches $\begin{bmatrix} .6 & -1 \\ .4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -.4 & .6 \end{bmatrix} = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}$.

3 $\lambda = 1$ and $.8$, $\boldsymbol{x} = (1, 0)$; $\lambda = 1$ and $-.8$, $\boldsymbol{x} = (\frac{5}{9}, \frac{4}{9})$; $\lambda = 1, \frac{1}{4}$, and $\frac{1}{4}$, $\boldsymbol{x} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

4 A^{T} always has the eigenvector $(1, 1, \ldots, 1)$ for $\lambda = 1$.

5 The steady state is $(0, 0, 1) =$ all dead.

6 If $A\boldsymbol{x} = \lambda \boldsymbol{x}$, add components on both sides to find $s = \lambda s$. If $\lambda \neq 1$ the sum must be $s = 0$.

7 $\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} = \begin{bmatrix} .6 & -1 \\ .4 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & .5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -.4 & .6 \end{bmatrix}$; A^{16} has the same factors except now $(.5)^{16}$.

8 $(.5)^k \to 0$ gives $A^k \to A^\infty$; any $A = \begin{bmatrix} .6 + .4a & .6 - .6a \\ .4 - .4a & .4 + .6a \end{bmatrix}$ with $-\frac{2}{3} \le a \le 1$.

9 $\boldsymbol{u}_1 = (0, 0, 1, 0)$; $\boldsymbol{u}_2 = (0, 1, 0, 0)$; $\boldsymbol{u}_3 = (1, 0, 0, 0)$; $\boldsymbol{u}_4 = \boldsymbol{u}_0$. The eigenvalues $1, i, -1, -i$ are all on the unit circle. This Markov matrix contains zeros; a *positive* matrix has *one* largest eigenvalue.

10 M^2 is still nonnegative; $[1 \ \cdots \ 1]M = [1 \ \cdots \ 1]$ so multiply by M to find $[1 \ \cdots \ 1]M^2 = [1 \ \cdots \ 1] \Rightarrow$ columns of M^2 add to 1.

11 $\lambda = 1$ and $a + d - 1$ from the trace; steady state is a multiple of $\boldsymbol{x}_1 = (b, 1 - a)$.

12 Last row $.2, .3, .5$ makes $A = A^{\mathrm{T}}$; rows also add to 1 so $(1, \ldots, 1)$ is also an eigenvector of A.

13 B has $\lambda = 0$ and $-.5$ with $\boldsymbol{x}_1 = (.3, .2)$ and $\boldsymbol{x}_2 = (-1, 1)$; $e^{-.5t}$ approaches zero and the solution approaches $c_1 e^{0t} \boldsymbol{x}_1 = c_1 \boldsymbol{x}_1$.

14 Each column of $B = A - I$ adds to zero. Then $\lambda_1 = 0$ and $e^{0t} = 1$.

15 The eigenvector is $\boldsymbol{x} = (1, 1, 1)$ and $A\boldsymbol{x} = (.9, .9, .9)$.

16 $(I - A)(I + A + A^2 + \ldots) = I + A + A^2 + \ldots - (A + A^2 + A^3 + \ldots) = I$. This says that $I + A + A^2 + \ldots$ is $(I - A)^{-1}$. When $A = \begin{bmatrix} 0 & .5 \\ 1 & 0 \end{bmatrix}$, $A^2 = \frac{1}{2}I$, $A^3 = \frac{1}{2}A$, $A^4 = \frac{1}{4}I$ and the series adds to $\begin{bmatrix} 1 + \frac{1}{2} + \ldots & \frac{1}{2} + \frac{1}{4} + \ldots \\ 1 + \frac{1}{2} + \ldots & 1 + \frac{1}{2} + \ldots \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} = (I - A)^{-1}$.

17 $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 4 \\ .2 & 0 \end{bmatrix}$ have $\lambda_{\max} < 1$.

18 $p = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 130 \\ 32 \end{bmatrix}$; $I - \begin{bmatrix} .5 & 1 \\ .5 & 0 \end{bmatrix}$ has no inverse.

19 $\lambda = 1$ (Markov), 0 (singular), .2 (from trace). Steady state (.3, .3, .4) and (30, 30, 40).

20 *No*, A has an eigenvalue $\lambda = 1$ and $(I - A)^{-1}$ does not exist.

Problem Set 8.4, page 382

1 Feasible set = line segment from $(6, 0)$ to $(0, 3)$; minimum cost at $(6, 0)$, maximum at $(0, 3)$.

2 Feasible set is 4-sided with corners $(0, 0)$, $(6, 0)$, $(2, 2)$, $(0, 6)$. Minimize $2x - y$ at $(6, 0)$.

3 Only two corners $(4, 0, 0)$ and $(0, 2, 0)$; choose x_1 very negative, $x_2 = 0$, and $x_3 = x_1 - 4$.

4 From $(0, 0, 2)$ move to $\boldsymbol{x} = (0, 1, 1.5)$ with the constraint $x_1 + x_2 + 2x_3 = 4$. The new cost is $3(1) + 8(1.5) = \$15$ so $r = -1$ is the reduced cost. The simplex method also checks $\boldsymbol{x} = (1, 0, 1.5)$ with cost $5(1) + 8(1.5) = \$17$ so $r = 1$ (more expensive).

5 Cost = 20 at start $(4, 0, 0)$; keeping $x_1 + x_2 + 2x_3 = 4$ move to $(3, 1, 0)$ with cost 18 and $r = -2$; or move to $(2, 0, 1)$ with cost 17 and $r = -3$. Choose x_3 as entering variable and move to $(0, 0, 2)$ with cost 14. Another step to reach $(0, 4, 0)$ with minimum cost 12.

6 $\boldsymbol{c} = [3 \;\; 5 \;\; 7]$ has minimum cost 12 by the Ph.D. since $\boldsymbol{x} = (4, 0, 0)$ is minimizing. The dual problem maximizes $4y$ subject to $y \le 3$, $y \le 5$, $y \le 7$. Maximum = 12.

Problem Set 8.5, page 387

1 $\int_0^{2\pi} \cos(j+k)x \, dx = \left[\frac{\sin(j+k)x}{j+k}\right]_0^{2\pi} = 0$ and similarly $\int_0^{2\pi} \cos(j-k)x \, dx = 0$ (in the denominator notice $j - k \ne 0$). If $j = k$ then $\int_0^{2\pi} \cos^2 jx \, dx = \pi$.

2 $\int_{-1}^{1} (1)(x) \, dx = 0$, $\int_{-1}^{1} (1)(x^2 - \frac{1}{3}) \, dx = 0$, $\int_{-1}^{1} (x)(x^2 - \frac{1}{3}) \, dx = 0$. Then $2x^2 = 2(x^2 - \frac{1}{3}) + 0(x) + \frac{2}{3}(1)$.

3 $\boldsymbol{w} = (2, -1, 0, 0, \ldots)$ has $\|\boldsymbol{w}\| = \sqrt{5}$.

4 $\int_{-1}^{1} (1)(x^3 - cx) \, dx = 0$ and $\int_{-1}^{1} (x^2 - \frac{1}{3})(x^3 - cx) \, dx = 0$ for all c (integral of an odd function). Choose c so that $\int_{-1}^{1} x(x^3 - cx) \, dx = [\frac{1}{5}x^5 - \frac{c}{3}x^3]_{-1}^{1} = \frac{2}{5} - c\frac{2}{3} = 0$. Then $c = \frac{3}{5}$.

5 The integrals lead to $a_1 = 0$, $b_1 = 4/\pi$, $b_2 = 0$.

6 From equation (3) the a_k are zero and $b_k = 4/\pi k$. The square wave has $\|f\|^2 = 2\pi$. Then equation (6) is $2\pi = \pi(16/\pi^2)(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots)$ so this infinite series equals $\pi^2/8$.

8 $\|\boldsymbol{v}\|^2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$ so $\|\boldsymbol{v}\| = \sqrt{2}$; $\|\boldsymbol{v}\|^2 = 1 + a^2 + a^4 + \cdots = 1/(1 - a^2)$ so $\|\boldsymbol{v}\| = 1/\sqrt{1 - a^2}$; $\int_0^{2\pi} (1 + 2\sin x + \sin^2 x) \, dx = 2\pi + 0 + \pi$ so $\|f\| = \sqrt{3\pi}$.

9 (a) $f(x) = \frac{1}{2} + \frac{1}{2}$ (square wave) so a's are $\frac{1}{2}$, 0, 0, . . ., and b's are $2/\pi$, 0, $-2/3\pi$, 0, $2/5\pi$, . . .
(b) $a_0 = \int_0^{2\pi} x \, dx / 2\pi = \pi$, other $a_k = 0$, $b_k = -2/k$.

10 The integral from $-\pi$ to π or from 0 to 2π or from any a to $a+2\pi$ is over one complete period of the function. If $f(x)$ is odd (and periodic) then $\int_0^{2\pi} f(x)\,dx = \int_0^{\pi} f(x)\,dx + \int_{-\pi}^{0} f(x)\,dx$ and those integrals cancel.

11 $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$; $\cos(x+\frac{\pi}{3}) = \cos x \cos\frac{\pi}{3} - \sin x \sin\frac{\pi}{3} = \frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x$.

12 $\frac{d}{dx}\begin{bmatrix} 1 \\ \cos x \\ \sin x \\ \cos 2x \\ \sin 2x \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \cos x \\ \sin x \\ \cos 2x \\ \sin 2x \end{bmatrix}$.

13 $dy/dx = \cos x$ has $y = y_p + y_n = \sin x + C$.

Problem Set 8.6, page 392

1 (x,y,z) has homogeneous coordinates $(x,y,z,1)$ and also (cx,cy,cz,c) for any nonzero c.

2 For an affine transformation we need T (origin). Then $(x,y,z,1) \to xT(\boldsymbol{i}) + yT(\boldsymbol{j}) + zT(\boldsymbol{k}) + T(\boldsymbol{0})$.

3 $TT_1 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 1 & 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 0 & 2 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 1 & 6 & 8 & 1 \end{bmatrix}$ is translation along $(1,6,8)$.

4 $S = \begin{bmatrix} c & & & \\ & c & & \\ & & c & \\ & & & 1 \end{bmatrix}$, $ST = \begin{bmatrix} c & & & \\ & c & & \\ & & c & \\ 1 & 4 & 3 & 1 \end{bmatrix}$, $TS = \begin{bmatrix} c & & & \\ & c & & \\ & & c & \\ c & 4c & 3c & 1 \end{bmatrix}$, use $\boldsymbol{v}TS$.

5 $S = \begin{bmatrix} 1/8.5 & & \\ & 1/11 & \\ & & 1 \end{bmatrix}$ for a 1 by 1 square.

6 $\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 1 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & & & \\ & 2 & & \\ & & 2 & \\ & & & 1 \end{bmatrix} = \begin{bmatrix} 2 & & & \\ & 2 & & \\ & & 2 & \\ 2 & 2 & 4 & 1 \end{bmatrix}$.

9 $\boldsymbol{n} = (\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$ has $\|\boldsymbol{n}\| = 1$ and $P = I - \boldsymbol{n}\boldsymbol{n}^{\mathrm{T}} = \frac{1}{9}\begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}$.

10 Choose $(0,0,3)$ on the plane and multiply $T_-PT_+ = \frac{1}{9}\begin{bmatrix} 5 & -4 & -2 & 0 \\ -4 & 5 & -2 & 0 \\ -2 & -2 & 8 & 0 \\ 6 & 6 & 3 & 9 \end{bmatrix}$.

11 $(3,3,3)$ projects to $\frac{1}{3}(-1,-1,4)$ and $(3,3,3,1)$ projects to $(\frac{1}{3},\frac{1}{3},\frac{5}{3},1)$.

12 A parallelogram (or a line segment).

13 The projection of a cube is a hexagon.

14 $(3,3,3)(I-2\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}}) = (\frac{1}{3},\frac{1}{3},\frac{1}{3})\begin{bmatrix} 1 & -8 & -4 \\ -8 & 1 & -4 \\ -4 & -4 & 7 \end{bmatrix} = (-\frac{11}{3},-\frac{11}{3},-\frac{1}{3})$.

15 $(3,3,3,1) \to (3,3,0,1) \to (-\frac{7}{3},-\frac{7}{3},-\frac{8}{3},1) \to (-\frac{7}{3},-\frac{7}{3},\frac{1}{3},1)$.

16 $\boldsymbol{v} = (x,y,z,0)$ ending in 0; add a **vector** to a point.

17 Rescaled by $1/c$ because (x,y,z,c) is the same point as $(x/c,y/c,z/c,1)$.

Problem Set 9.1, page 402

1 Without exchange, pivots .001 and 1000; with exchange, pivots 1 and -1. When the pivot is larger than the entries below it, ℓ_{ij} = entry/pivot has $|\ell_{ij}| \le 1$. $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$.

2 $A^{-1} = \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix}$.

3 $A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 11/16 \\ 13/12 \\ 47/60 \end{bmatrix} = \begin{bmatrix} 1.833 \\ 1.083 \\ 0.783 \end{bmatrix}$ compared with $A \begin{bmatrix} 0 \\ 6 \\ -3.6 \end{bmatrix} = \begin{bmatrix} 1.80 \\ 1.10 \\ 0.78 \end{bmatrix}$. $\|\Delta \boldsymbol{b}\| < .04$ but $\|\Delta \boldsymbol{x}\| > 6$.

4 The largest $\|\boldsymbol{x}\| = \|A^{-1}\boldsymbol{b}\|$ is $1/\lambda_{\min}$; the largest error is $10^{-16}/\lambda_{\min}$.

5 Each row of U has at most w entries. Then w multiplications to substitute components of $\boldsymbol{x}$ (already known from below) and divide by the pivot. Total for n rows is less than wn.

6 L, U, and R need $\frac{1}{2}n^2$ multiplications to solve a linear system. Q needs n^2 to multiply the right side by $Q^{-1} = Q^{\mathrm{T}}$. So QR takes 1.5 times longer than LU to reach $\boldsymbol{x}$.

7 On column j of I, back substitution needs $\frac{1}{2}j^2$ multiplications (only the j by j upper left block is involved). Then $\frac{1}{2}(1^2+2^2+\cdots+n^2) \approx \frac{1}{2}(\frac{1}{3}n^3)$.

8 $\begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 2 \\ 0 & -1 \end{bmatrix} = U$ with $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $L = \begin{bmatrix} 1 & 0 \\ .5 & 1 \end{bmatrix}$; $A \to \begin{bmatrix} 2 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix} \to$
$\begin{bmatrix} 2 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = U$ with $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ .5 & -.5 & 1 \end{bmatrix}$.

9 The cofactors are $C_{13} = C_{31} = C_{24} = C_{42} = 1$ and $C_{14} = C_{41} = -1$.

10 With 16-digit floating point arithmetic the errors $\|\boldsymbol{x} - \boldsymbol{y}_{\text{computed}}\|$ for $\varepsilon = 10^{-3}, 10^{-6}, 10^{-9}, 10^{-12}, 10^{-15}$ are of order $10^{-16}, 10^{-11}, 10^{-7}, 10^{-4}, 10^{-3}$.

11 $\cos\theta = 1/\sqrt{10}$, $\sin\theta = -3/\sqrt{10}$, $R = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix} = \frac{1}{\sqrt{10}}\begin{bmatrix} 10 & 14 \\ 0 & 8 \end{bmatrix}$.

12 Eigenvalues 4 and 2. Put one of the unit eigenvectors in row 1 of Q: either

$Q = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $QAQ^{-1} = \begin{bmatrix} 2 & -4 \\ 0 & 4 \end{bmatrix}$ or

$Q = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$ and $QAQ^{-1} = \begin{bmatrix} 4 & -4 \\ 0 & 2 \end{bmatrix}$.

13 Changes in rows i and j; changes also in columns i and j.

14 $Q_{ij}A$ uses $4n$ multiplications (2 for each entry in rows i and j). By factoring out $\cos\theta$, the entries 1 and $\pm\tan\theta$ need only $2n$ multiplications, which leads to $\frac{2}{3}n^3$ for QR.

Problem Set 9.2, page 408

1 $\|A\| = 2$, $c = 2/.5 = 4$; $\|A\| = 3$, $c = 3/1 = 3$; $\|A\| = 2+\sqrt{2}$, $c = (2+\sqrt{2})/(2-\sqrt{2}) = 5.83$.

2 $\|A\| = 2$, $c = 1$; $\|A\| = \sqrt{2}$, $c =$ infinite (singular matrix); $\|A\| = \sqrt{2}$, $c = 1$.

3 For the first inequality replace $\boldsymbol{x}$ by $B\boldsymbol{x}$ in $\|A\boldsymbol{x}\| \le \|A\|\|\boldsymbol{x}\|$; the second inequality is just $\|B\boldsymbol{x}\| \le \|B\|\|\boldsymbol{x}\|$. Then $\|AB\| = \max(\|AB\boldsymbol{x}\|/\|\boldsymbol{x}\|) \le \|A\|\|B\|$.

4 Choose $B = A^{-1}$ and compute $\|I\| = 1$. Then $1 \le \|A\|\|A^{-1}\| = c(A)$.

5 If $\lambda_{\max} = \lambda_{\min} = 1$ then all $\lambda_i = 1$ and $A = SIS^{-1} = I$. The only matrices with $\|A\| = \|A^{-1}\| = 1$ are *orthogonal matrices.*

6 $\|A\| \le \|Q\|\|R\| = \|R\|$ and in reverse $\|R\| \le \|Q^{-1}\|\|A\| = \|A\|$.

7 The triangle inequality gives $\|A\boldsymbol{x} + B\boldsymbol{x}\| \le \|A\boldsymbol{x}\| + \|B\boldsymbol{x}\|$. Divide by $\|\boldsymbol{x}\|$ and take the maximum over all nonzero vectors to find $\|A + B\| \le \|A\| + \|B\|$.

8 If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ then $\|A\boldsymbol{x}\|/\|\boldsymbol{x}\| = |\lambda|$ for that particular vector $\boldsymbol{x}$. When we maximize the ratio over all vectors we get $\|A\| \ge |\lambda|$.

9 $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has $\rho(A) = 0$ and $\rho(B) = 0$ but $\rho(A + B) = 1$; also $AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ has $\rho(AB) = 1$; thus $\rho(A)$ is not a norm.

10 The condition number of A^{-1} is $\|A^{-1}\|\|(A^{-1})^{-1}\| = c(A)$. Since $A^{\mathrm{T}}A$ and AA^{T} have the same nonzero eigenvalues, A and A^{T} have the same norm.

11 $c(A) = (1.00005 + \sqrt{(1.00005)^2 - .0001})/(1.00005 - \sqrt{\qquad})$.

12 $\det(2A)$ is not $2\det A$; $\det(A + B)$ is not always less than $\det A + \det B$; taking $|\det A|$ does not help. The only reasonable property is $\det AB = (\det A)(\det B)$. The condition number should not change when A is multiplied by 10.

13 The residual $\boldsymbol{b} - A\boldsymbol{y} = (10^{-7}, 0)$ is much smaller than $\boldsymbol{b} - A\boldsymbol{z} = (.0013, .0016)$. But $\boldsymbol{z}$ is much closer to the solution than $\boldsymbol{y}$.

14 $\det A = 10^{-6}$ so $A^{-1} = \begin{bmatrix} 659{,}000 & -563{,}000 \\ -913{,}000 & 780{,}000 \end{bmatrix}$. Then $\|A\| > 1$, $\|A^{-1}\| > 10^6$, $c > 10^6$.

15 $\|\boldsymbol{x}\| = \sqrt{5}$, $\|\boldsymbol{x}\|_1 = 5$, $\|\boldsymbol{x}\|_\infty = 1$; $\|\boldsymbol{x}\| = 1$, $\|\boldsymbol{x}\|_1 = 2$, $\|\boldsymbol{x}\|_\infty = .7$.

16 $x_1^2 + \cdots + x_n^2$ is not smaller than $\max(x_i^2)$ and not larger than $x_1^2 + \cdots + x_n^2 + 2|x_1||x_2| + \cdots = \|\boldsymbol{x}\|_1^2$. Certainly $x_1^2 + \cdots + x_n^2 \leq n \max(x_i^2)$ so $\|\boldsymbol{x}\| \leq \sqrt{n}\|\boldsymbol{x}\|_\infty$. Choose $\boldsymbol{y} = (\operatorname{sign} x_1, \operatorname{sign} x_2, \ldots, \operatorname{sign} x_n)$ to get $\boldsymbol{x} \cdot \boldsymbol{y} = \|\boldsymbol{x}\|_1$. By Schwarz this is at most $\|\boldsymbol{x}\|\|\boldsymbol{y}\| = \sqrt{n}\|\boldsymbol{x}\|$. Choose $\boldsymbol{x} = (1, 1, \ldots, 1)$ for maximum ratios $\sqrt{n}$.

17 The largest component $|(x + y)_i| = \|\boldsymbol{x} + \boldsymbol{y}\|_\infty$ is not larger than $|x_i| + |y_i| \leq \|\boldsymbol{x}\|_\infty + \|\boldsymbol{y}\|_\infty$. The sum of absolute values $|(x + y)_i|$ is not larger than the sum of $|x_i| + |y_i|$. Therefore $\|\boldsymbol{x} + \boldsymbol{y}\|_1 \leq \|\boldsymbol{x}\|_1 + \|\boldsymbol{y}\|_1$.

18 $|x_1| + 2|x_2|$ is a norm; $\min |x_i|$ is not a norm; $\|\boldsymbol{x}\| + \|\boldsymbol{x}\|_\infty$ is a norm; $\|A\boldsymbol{x}\|$ is a norm provided A is invertible (otherwise a nonzero vector has norm zero; for rectangular A we require independent columns).

Problem Set 9.3, page 417

1 $S = I$ and $T = I - A$ and $S^{-1}T = I - A$.

2 If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ then $(I - A)\boldsymbol{x} = (1 - \lambda)\boldsymbol{x}$. Real eigenvalues of $B = I - A$ have $|1 - \lambda| < 1$ provided λ is between 0 and 2.

3 This matrix A has $I - A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ which has $|\lambda| = 2$.

4 Always $\|AB\| \leq \|A\|\|B\|$. Choose $A = B$ to find $\|B^2\| \leq \|B\|^2$. Then choose $A = B^2$ to find $\|B^3\| \leq \|B^2\|\|B\| \leq \|B\|^3$. Continue (or use induction). Since $\|B\| \geq \max |\lambda(B)|$ it is no surprise that $\|B\| < 1$ gives convergence.

5 $A\boldsymbol{x} = \boldsymbol{0}$ gives $(S - T)\boldsymbol{x} = \boldsymbol{0}$. Then $S\boldsymbol{x} = T\boldsymbol{x}$ and $S^{-1}T\boldsymbol{x} = \boldsymbol{x}$. Then $\lambda = 1$ means that the errors do not approach zero.

6 Jacobi has $S^{-1}T = \frac{1}{3}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ with $|\lambda|_{\max} = \frac{1}{3}$.

7 Gauss-Seidel has $S^{-1}T = \begin{bmatrix} 0 & \frac{1}{3} \\ 0 & \frac{1}{9} \end{bmatrix}$ with $|\lambda|_{\max} = \frac{1}{9} = (|\lambda|_{\max} \text{ for Jacobi})^2$.

8 Jacobi has $S^{-1}T = \begin{bmatrix} a & \\ & d \end{bmatrix}^{-1} \begin{bmatrix} 0 & -b \\ -c & 0 \end{bmatrix} = \begin{bmatrix} 0 & -b/a \\ -c/d & 0 \end{bmatrix}$ with $|\lambda| = |bc/ad|^{1/2}$. Gauss-Seidel has $S^{-1}T = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} 0 & -b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -b/a \\ 0 & -bc/ad \end{bmatrix}$ with $|\lambda| = |bc/ad|$.

9 Set the trace $2 - 2\omega + \frac{1}{4}\omega^2$ equal to $(\omega - 1) + (\omega - 1)$ to find $\omega_{\text{opt}} = 4(2 - \sqrt{3}) \approx 1.07$. The eigenvalues $\omega - 1$ are about .07.

11 If the iteration gives all $x_i^{\text{new}} = x_i^{\text{old}}$ then the quantity in parentheses is zero, which means $A\boldsymbol{x} = \boldsymbol{b}$. For Jacobi change the whole right side to x^{old}.

13 $\boldsymbol{u}_k/\lambda_1^k = c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2(\lambda_2/\lambda_1)^k + \cdots + c_n\boldsymbol{x}_n(\lambda_n/\lambda_1)^k \to c_1\boldsymbol{x}_1$ if all ratios $|\lambda_i/\lambda_1| < 1$. The largest ratio controls, when k is large. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has $|\lambda_2| = |\lambda_1|$ and no convergence.

14 The eigenvectors of A and also A^{-1} are $\boldsymbol{x}_1 = (.75, .25)$ and $\boldsymbol{x}_2 = (1, -1)$. The inverse power method converges to a multiple of $\boldsymbol{x}_2$.

15 The jth component of $A\boldsymbol{x}_1$ is $2\sin\frac{j\pi}{n+1} - \sin\frac{(j-1)\pi}{n+1} - \sin\frac{(j+1)\pi}{n+1}$. The last two terms, using $\sin(a+b) = \sin a\cos b + \cos a\sin b$, combine into $-2\sin\frac{j\pi}{n+1}\cos\frac{\pi}{n+1}$. The eigenvalue is $\lambda_1 = 2 - 2\cos\frac{\pi}{n+1}$.

16 $\boldsymbol{u}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\boldsymbol{u}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\boldsymbol{u}_2 = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$, $\boldsymbol{u}_3 = \begin{bmatrix} 14 \\ -13 \end{bmatrix}$ is converging to the eigenvector direction $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ with $\lambda_{\max} = 3$.

17 $A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ gives $\boldsymbol{u}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\boldsymbol{u}_1 = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\boldsymbol{u}_2 = \frac{1}{9}\begin{bmatrix} 5 \\ 4 \end{bmatrix}$, $\boldsymbol{u}_3 = \frac{1}{27}\begin{bmatrix} 14 \\ 13 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

18 $R = Q^{\mathrm{T}}A = \begin{bmatrix} 1 & \cos\theta\sin\theta \\ 0 & -\sin^2\theta \end{bmatrix}$ and $A_1 = RQ = \begin{bmatrix} \cos\theta(1+\sin^2\theta) & -\sin^3\theta \\ -\sin^3\theta & -\cos\theta\sin^2\theta \end{bmatrix}$.

19 If A is orthogonal then $Q = A$ and $R = I$. Therefore $A_1 = RQ = A$ again.

20 If $A - cI = QR$ then $A_1 = RQ + cI = Q^{-1}(QR + cI)Q = Q^{-1}AQ$. No change in eigenvalues from A to A_1.

21 Multiply $A\boldsymbol{q}_j = b_{j-1}\boldsymbol{q}_{j-1} + a_j\boldsymbol{q}_j + b_j\boldsymbol{q}_{j+1}$ by $\boldsymbol{q}_j^{\mathrm{T}}$ to find $\boldsymbol{q}_j^{\mathrm{T}}A\boldsymbol{q}_j = a_j$ (because the $\boldsymbol{q}$'s are orthonormal). The matrix form (multiplying by columns) is $AQ = QT$ where T is *tridiagonal.* Its entries are the a's and b's.

22 Theoretically the $\boldsymbol{q}$'s are orthonormal. In reality this algorithm is not very stable. We must stop every few steps to reorthogonalize.

23 If A is symmetric then $A_1 = Q^{-1}AQ = Q^{\mathrm{T}}AQ$ is also symmetric. $A_1 = RQ = R(QR)R^{-1} = RAR^{-1}$ has R and R^{-1} upper triangular, so A_1 cannot have nonzeros on a lower diagonal than A. If A is tridiagonal and symmetric then (by using symmetry for the upper part of A_1) the matrix $A_1 = RAR^{-1}$ is also tridiagonal.

24 The proof of $|\lambda| < 1$ when every absolute row sum < 1 uses $|\sum a_{ij}x_j| \le \sum |a_{ij}||x_i| < |x_i|$. (Note $|x_i| \ge |x_j|$.) The Gershgorin circle theorem (very useful) is proved after its statement.

25 The maximum row sums give all $|\lambda| \le .9$ and $|\lambda| \le 3$. The circles around diagonal entries give tighter bounds. The circle $|\lambda - .2| \le .7$ contains the other circles $|\lambda - .3| \le .5$ and $|\lambda - .1| \le .6$ and all three eigenvalues. The circle $|\lambda - 2| \le 2$ contains the circle $|\lambda - 2| \le 1$ and all three eigenvalues $2 + \sqrt{2}, 2$, and $2 - \sqrt{2}$.

26 The circles $|\lambda - a_{ii}| \le r_i$ don't include $\lambda = 0$ (so A is invertible!) when $a_{ii} > r_i$.

27 From the last line of code, $\boldsymbol{q}_2$ is in the direction of $\boldsymbol{v} = A\boldsymbol{q}_1 - h_{11}\boldsymbol{q}_1 = A\boldsymbol{q}_1 - (\boldsymbol{q}_1^{\mathrm{T}} A\boldsymbol{q}_1)\boldsymbol{q}_1$. The dot product with $\boldsymbol{q}_1$ is zero. This is Gram-Schmidt with $A\boldsymbol{q}_1$ as the second input vector.

28 $\boldsymbol{r}_1 = \boldsymbol{b} - \alpha_1 A\boldsymbol{b} = \boldsymbol{b} - (\boldsymbol{b}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{b}^{\mathrm{T}}A\boldsymbol{b})A\boldsymbol{b}$ is orthogonal to $\boldsymbol{r}_0 = \boldsymbol{b}$: *the residuals* $\boldsymbol{r} = \boldsymbol{b} - A\boldsymbol{x}$ *are orthogonal at each step.* To show that $\boldsymbol{p}_1$ is orthogonal to $A\boldsymbol{p}_0 = A\boldsymbol{b}$, simplify $\boldsymbol{p}_1$ to $c\boldsymbol{P}_1$: $\boldsymbol{P}_1 = \|A\boldsymbol{b}\|^2\boldsymbol{b} - (\boldsymbol{b}^{\mathrm{T}}A\boldsymbol{b})A\boldsymbol{b}$ and $c = \boldsymbol{b}^{\mathrm{T}}\boldsymbol{b}/(\boldsymbol{b}^{\mathrm{T}}A\boldsymbol{b})^2$. Certainly $(A\boldsymbol{b})^{\mathrm{T}}\boldsymbol{P}_1 = 0$ because $A^{\mathrm{T}} = A$. (That simplification put α_1 into $\boldsymbol{p}_1 = \boldsymbol{b} - \alpha_1 A\boldsymbol{b} + (\boldsymbol{b}^{\mathrm{T}}\boldsymbol{b} - 2\alpha_1\boldsymbol{b}^{\mathrm{T}}A\boldsymbol{b} + \alpha_1^2\|A\boldsymbol{b}\|^2)\boldsymbol{b}/\boldsymbol{b}^{\mathrm{T}}\boldsymbol{b}$. For a good discussion see *Numerical Linear Algebra* by Trefethen and Bau.)

Problem Set 10.1, page 427

1 Sums $4,\ -2+2i,\ 2\cos\theta$; products $5,\ -2i,\ 1$.

2 In polar form these are $\sqrt{5}e^{i\theta},\ 5e^{2i\theta},\ \frac{1}{\sqrt{5}}e^{-i\theta},\ \sqrt{5}$.

3 Absolute values $r = 10,\ 100,\ \frac{1}{10},\ 100$; angles $\theta,\ 2\theta,\ -\theta,\ -2\theta$.

4 $|z \times w| = 6,\ \ |z+w| \le 5,\ \ |z/w| = \frac{2}{3},\ \ |z-w| \le 5$.

5 $a + ib = \frac{\sqrt{3}}{2} + \frac{1}{2}i,\ \frac{1}{2} + \frac{\sqrt{3}}{2}i,\ i,\ -\frac{1}{2} + \frac{\sqrt{3}}{2}i;\ \ w^{12} = 1$.

6 $1/z$ has absolute value $1/r$ and angle $-\theta$; $\frac{1}{r}e^{-i\theta}$ times $re^{i\theta} = 1$.

7 $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} ac - bd \\ bc + ad \end{bmatrix}$ $\begin{matrix}\text{real part} \\ \text{imaginary part}\end{matrix}$

8 $\begin{bmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{bmatrix}\begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{bmatrix}$.

9 $2+i;\ \ (2+i)(1+i) = 1+3i;\ \ e^{-i\pi/2} = -i;\ \ e^{-i\pi} = -1;\ \ \frac{1-i}{1+i} = -i;\ \ (-i)^{103} = (-i)^3 = i$.

10 $z + \overline{z}$ is real; $z - \overline{z}$ is pure imaginary; $z\overline{z}$ is positive; $z/\overline{z}$ has absolute value 1.

11 If $a_{ij} = i - j$ then $\det(A - \lambda I) = -\lambda^3 - 6\lambda = 0$ gives $\lambda = 0,\ \sqrt{6}i,\ -\sqrt{6}i$ (the conjugate of $\sqrt{6}i$).

12 (a) When $a = b = d = 1$ the square root becomes $\sqrt{4c}$; λ is complex if $c < 0$ (b) $\lambda = 0$ and $\lambda = a + d$ when $ad = bc$ (c) the λ's can be real and different.

13 Complex λ's when $(a+d)^2 < 4(ad - bc)$; write $(a+d)^2 - 4(ad-bc)$ as $(a-d)^2 + 4bc$ which is positive when $bc > 0$.

14 $\det(P - \lambda I) = \lambda^4 - 1 = 0$ has $\lambda = 1,\ -1,\ i,\ -i$ with eigenvectors $(1,1,1,1)$ and $(1,-1,1,-1)$ and $(1,i,-1,-i)$ and $(1,-i,-1,i)$ = columns of Fourier matrix.

15 $\det(P_6 - \lambda I) = \lambda^6 - 1 = 0$ when $\lambda = 1,\ w,\ w^2,\ w^3,\ w^4,\ w^5$ with $w = e^{2\pi i/6}$ as in Figure 10.3.

16 The block matrix has real eigenvalues; so $i\lambda$ is real and λ is pure imaginary.

17 (a) $2e^{i\pi/3},\ \ 4e^{2i\pi/3}$ (b) $e^{2i\theta},\ \ e^{4i\theta}$
(c) $73^{3\pi i/2},\ \ 49e^{3\pi i}(= -49),\ \ \sqrt{50}e^{-\pi i/4},\ \ 50e^{-\pi i/2}$.

18 $r = 1$, angle $\frac{\pi}{2} - \theta$; multiply by $e^{i\theta}$ to get $e^{i\pi/2} = i$.

19 $a + ib = 1,\ i,\ -1,\ -i,\ \pm\frac{1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}}$.

20 $1,\ \ e^{2\pi i/3},\ \ e^{4\pi i/3};\ \ -1,\ \ e^{\pi i/3},\ \ e^{-\pi i/3};\ \ 1$.

21 $\cos 3\theta = \text{Re}(\cos\,\theta + i\sin\theta)^3 = \cos^3\theta - 3\cos\theta\sin^2\theta$; $\sin 3\theta = \text{Im}(\cos\,\theta + i\sin\theta)^3 = 3\cos^2\theta\sin\theta - \sin^3\theta$.

22 If $\overline{z} = 1/z$ then $|z|^2 = 1$ and z is any point $e^{i\theta}$ on the unit circle.

23 (a) e^i is at angle $\theta = 1$ on the unit circle; $|i^e| = 1^e = 1$ (c) There are infinitely many candidates $i^e = e^{i(\pi/2 + 2\pi n)e}$.

24 (a) Unit circle (b) Spiral in to $e^{-2\pi}$ (c) Circle continuing around to angle $\theta = 2\pi^2$.

Problem Set 10.2, page 436

1 $\|\boldsymbol{u}\| = \sqrt{9} = 3$, $\|\boldsymbol{v}\| = \sqrt{3}$, $\boldsymbol{u}^{\mathrm{H}}\boldsymbol{v} = 3i + 2$, $\boldsymbol{v}^{\mathrm{H}}\boldsymbol{u} = -3i + 2$ (conjugate of $\boldsymbol{u}^{\mathrm{H}}\boldsymbol{v}$).

2 $A^{\mathrm{H}}A = \begin{bmatrix} 2 & 0 & 1+i \\ 0 & 2 & 1+i \\ 1-i & 1-i & 2 \end{bmatrix}$ and $AA^{\mathrm{H}} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ are Hermitian matrices.

3 $\boldsymbol{z}$ = multiple of $(1+i, 1+i, -2)$; $A\boldsymbol{z} = \boldsymbol{0}$ gives $\boldsymbol{z}^{\mathrm{H}}A^{\mathrm{H}} = \boldsymbol{0}^{\mathrm{H}}$ so $\boldsymbol{z}$ (not $\overline{\boldsymbol{z}}$!) is orthogonal to all columns of A^{H} (using complex inner product $\boldsymbol{z}^{\mathrm{H}}$ times column).

4 The four fundamental subspaces are $\boldsymbol{C}(A)$, $\boldsymbol{N}(A)$, $\boldsymbol{C}(A^{\mathrm{H}})$, $\boldsymbol{N}(A^{\mathrm{H}})$.

5 (a) $(A^{\mathrm{H}}A)^{\mathrm{H}} = A^{\mathrm{H}}A^{\mathrm{HH}} = A^{\mathrm{H}}A$ again (b) If $A^{\mathrm{H}}A\boldsymbol{z} = \boldsymbol{0}$ then $(\boldsymbol{z}^{\mathrm{H}}A^{\mathrm{H}})(A\boldsymbol{z}) = 0$. This is $\|A\boldsymbol{z}\|^2 = 0$ so $A\boldsymbol{z} = \boldsymbol{0}$. The nullspaces of A and $A^{\mathrm{H}}A$ are the ***same***. $A^{\mathrm{H}}A$ is invertible when $\boldsymbol{N}(A) = \{\boldsymbol{0}\}$.

6 (a) False: $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (b) True: $-i$ is not an eigenvalue if $A = A^{\mathrm{H}}$ (c) False.

7 cA is still Hermitian for real c; $(iA)^{\mathrm{H}} = -iA^{\mathrm{H}} = -iA$ is skew-Hermitian.

8 Orthogonal, invertible, unitary, factorizable into QR.

9 $P^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $P^3 = I$, $P^{100} = P^{99}P = P$; λ = cube roots of $1 = 1$, $e^{2\pi i/3}$, $e^{4\pi i/3}$.

10 $(1,1,1)$, $(1, e^{2\pi i/3}, e^{4\pi i/3})$, $(1, e^{4\pi i/3}, e^{2\pi i/3})$ are orthogonal (complex inner product!) because P is an orthogonal matrix—and therefore unitary.

11 $C = \begin{bmatrix} 2 & 5 & 4 \\ 4 & 2 & 5 \\ 5 & 4 & 2 \end{bmatrix} = 2 + 5P + 4P^2$ has $\lambda = 2+5+4 = 11$, $2 + 5e^{2\pi i/3} + 4e^{4\pi i/3}$, $2 + 5e^{4\pi i/3} + 4e^{8\pi i/3}$.

12 If $U^{\mathrm{H}}U = I$ then $U^{-1}(U^{\mathrm{H}})^{-1} = U^{-1}(U^{-1})^{\mathrm{H}} = I$ so U^{-1} is also unitary.
Also $(UV)^{\mathrm{H}}(UV) = V^{\mathrm{H}}U^{\mathrm{H}}UV = V^{\mathrm{H}}V = I$ so UV is unitary.

13 The determinant is the product of the eigenvalues (all real).

14 $(\boldsymbol{z}^{\mathrm{H}}A^{\mathrm{H}})(A\boldsymbol{z}) = \|A\boldsymbol{z}\|^2$ is positive unless $A\boldsymbol{z} = \boldsymbol{0}$; with independent columns this means $\boldsymbol{z} = \boldsymbol{0}$; so $A^{\mathrm{H}}A$ is positive definite.

15 $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & -1+i \\ 1+i & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1-i \\ -1-i & 1 \end{bmatrix}$.

16 $K = (iA^{\mathrm{T}} \text{ in Problem 15}) = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & -1-i \\ 1-i & 1 \end{bmatrix}\begin{bmatrix} 2i & 0 \\ 0 & -i \end{bmatrix}\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ -1+i & 1 \end{bmatrix};$

λ's are imaginary.

17 $Q = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}\begin{bmatrix} \cos\theta + i\sin\theta & 0 \\ 0 & \cos\theta - i\sin\theta \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$ has $|\lambda| = 1$.

18 $V = \frac{1}{L}\begin{bmatrix} 1+\sqrt{3} & -1+i \\ 1+i & 1+\sqrt{3} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\frac{1}{L}\begin{bmatrix} 1+\sqrt{3} & 1-i \\ -1-i & 1+\sqrt{3} \end{bmatrix}$ with $L^2 = 6 + 2\sqrt{3}$ has $|\lambda| = 1$.

$V = V^{\mathrm{H}}$ gives real λ, trace zero gives $\lambda = 1, -1$.

19 The $\boldsymbol{v}$'s are columns of a unitary matrix U. Then $\boldsymbol{z} = UU^{\mathrm{H}}\boldsymbol{z} =$ (multiply by columns) $= \boldsymbol{v}_1(\boldsymbol{v}_1^{\mathrm{H}}\boldsymbol{z}) + \cdots + \boldsymbol{v}_n(\boldsymbol{v}_n^{\mathrm{H}}\boldsymbol{z})$.

20 Don't multiply e^{-ix} times e^{ix}; conjugate the first, then $\int_0^{2\pi} e^{2ix}\,dx = [e^{2ix}/2i]_0^{2\pi} = 0$.

21 $\boldsymbol{z} = (1, i, -2)$ completes an orthogonal basis for $\mathbf{C}^3$.

22 $R + iS = (R + iS)^{\mathrm{H}} = R^{\mathrm{T}} - iS^{\mathrm{T}}$; R is symmetric but S is skew-symmetric.

23 $\mathbf{C}^n$ has dimension n; the columns of any unitary matrix are a basis: $(i, 0, \ldots, 0), \ldots, (0, \ldots, 0, i)$

24 $[\,1\,]$ and $[\,-1\,]$; any $[\,e^{i\theta}\,]$; $\begin{bmatrix} a & b+ic \\ b-ic & d \end{bmatrix}$; $\begin{bmatrix} w & e^{i\phi}\overline{z} \\ -z & e^{i\phi}\overline{w} \end{bmatrix}$ with $|w|^2 + |z|^2 = 1$.

25 Eigenvalues of A^{H} are complex conjugates of eigenvalues of A: $\det(A - \lambda I) = 0$ gives $\det(A^{\mathrm{H}} - \overline{\lambda}I) = 0$.

26 $(I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{H}})^{\mathrm{H}} = I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{H}}$; $(I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{H}})^2 = I - 4\boldsymbol{u}\boldsymbol{u}^{\mathrm{H}} + 4\boldsymbol{u}(\boldsymbol{u}^{\mathrm{H}}\boldsymbol{u})\boldsymbol{u}^{\mathrm{H}} = I$; the matrix $\boldsymbol{u}\boldsymbol{u}^{\mathrm{H}}$ projects onto the line through $\boldsymbol{u}$.

27 Unitary means $U^{\mathrm{H}}U = I$ or $(A^{\mathrm{T}} - iB^{\mathrm{T}})(A + iB) = (A^{\mathrm{T}}A + B^{\mathrm{T}}B) + i(A^{\mathrm{T}}B - B^{\mathrm{T}}A) = I$. Then $A^{\mathrm{T}}A + B^{\mathrm{T}}B = I$ and $A^{\mathrm{T}}B - B^{\mathrm{T}}A = 0$ which makes the block matrix orthogonal.

28 We are given $A + iB = (A + iB)^{\mathrm{H}} = A^{\mathrm{T}} - iB^{\mathrm{T}}$. Then $A = A^{\mathrm{T}}$ and $B = -B^{\mathrm{T}}$.

29 $AA^{-1} = I$ gives $(A^{-1})^{\mathrm{H}}A^{\mathrm{H}} = I$. Therefore $(A^{-1})^{\mathrm{H}} = (A^{\mathrm{H}})^{-1} = A^{-1}$ and A^{-1} is Hermitian.

30 $A = \begin{bmatrix} 1-i & 1-i \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\frac{1}{6}\begin{bmatrix} 2+2i & -2 \\ 1+i & 2 \end{bmatrix} = S\Lambda S^{-1}$.

Problem Set 10.3, page 444

1 Equation (3) is correct using $i^2 = -1$ in the last two rows and three columns.

2 $$F^{-1} = \begin{bmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 1 & & \\ 1 & i^2 & & \\ & & 1 & 1 \\ & & 1 & i^2 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & & 1 & \\ & 1 & & 1 \\ 1 & & -1 & \\ & -i & & i \end{bmatrix} = \frac{1}{4}F^{\mathrm{H}}.$$

3 $$F = \begin{bmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & & \\ 1 & i^2 & & \\ & & 1 & 1 \\ & & 1 & i^2 \end{bmatrix}\begin{bmatrix} 1 & & 1 & \\ & 1 & & 1 \\ 1 & & -1 & \\ & i & & -i \end{bmatrix}.$$

4 $D = \begin{bmatrix} 1 & & \\ & e^{2\pi i/6} & \\ & & e^{4\pi i/6} \end{bmatrix}$ and $F_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{4\pi i/3} \\ 1 & e^{4\pi i/3} & e^{2\pi i/3} \end{bmatrix}$.

5 $F^{-1}\boldsymbol{w} = \boldsymbol{v}$ and $F^{-1}\boldsymbol{v} = \frac{1}{4}\boldsymbol{w}$.

6 $(F_4)^2 = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 \end{bmatrix}$ and $(F_4)^4 = 16I$.

7 $\boldsymbol{c} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} = F\boldsymbol{c}; \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} \to \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}$.

8 $\boldsymbol{c} \to (1,1,1,1,0,0,0,0) \to (4,0,0,0,0,0,0,0) \to (4,0,0,0,4,0,0,0)$ which is $F_8\boldsymbol{c}$. The second vector becomes $(0,0,0,0,1,1,1,1) \to (0,0,0,0,4,0,0,0) \to$ $(4,0,0,0,-4,0,0,0)$.

9 If $w^{64} = 1$ then w^2 is a 32nd root of 1 and $\sqrt{w}$ is a 128th root of 1.

10 For every integer n, the nth roots of 1 add to zero.

11 The eigenvalues of P are 1, i, $i^2 = -1$, and $i^3 = -i$.

12 $\Lambda = \text{diag}(1, i, i^2, i^3)$; $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and P^{T} lead to $\lambda^3 - 1 = 0$.

13 $e_1 = c_0 + c_1 + c_2 + c_3$ and $e_2 = c_0 + c_1 i + c_2 i^2 + c_3 i^3$; E contains the four eigenvalues of C.

14 Eigenvalues $e_1 = 2-1-1 = 0$, $e_2 = 2-i-i^3 = 2$, $e_3 = 2-(-1)-(-1) = 4$, $e_4 = 2-i^3-i^9 = 2$. Check trace $0+2+4+2 = 8$.

15 Diagonal E needs n multiplications, Fourier matrix F and F^{-1} need $\frac{1}{2}n\log_2 n$ multiplications each by the **FFT**. Total much less than the ordinary n^2.

16 $(c_0+c_2)+(c_1+c_3)$; then $(c_0-c_2)+i(c_1-c_3)$; then $(c_0+c_2)-(c_1+c_3)$; then $(c_0-c_2)-i(c_1-c_3)$. These steps are the **FFT**!